Characteristics of Optimal Solutions in Kinematic Resolutions of Redundancy

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ABSTRACT

The inverse kinematic solutions for redundant manipulators using the optimality augmented resolution schemes have been used without investigating the characteristics of the optimal solutions. The questions with this kind of resolution methods are answered in this paper, that is (i) the characteristics of solutions, (ii) of algorithmic singularities, (iii) their dimensionality, and (iv) the invariance of the characteristics during resolutions. 3-DOF planar redundant robot is analyzed when the inverse kinematic method is applied with the manipulability as an example.

1 Introduction

A robotic system is considered to have kinematic redundancy if the dimension of its configuration space is greater than that of its task space. From application viewpoint, prerequisite is the inverse kinematic problem of specifying the displacements of the joint variables algorithmically corresponding to the position[1], of the hand. That kind of algorithm is termed as the inverse kinematic resolution scheme for redundant manipulators.

The basic idea to resolve the redundancy is to add enough internal or external constraints which are either explicit or implicit. The constraints can be local or instantaneous in nature, or more global and can consider the dynamics of robot or not. The constraints help to make the system well-posed which is originally ill-posed. Once a specific algorithm and an initial setting are chosen, there should be a unique joint-space path for each hand-space trajectory. In other words the arm should always reproduce the same joint path in case of the same situation.

Suppose the manipulator has n degrees of freedom at the joints and operates in an m-dimensional space (where $m \leq 6$). Let $\boldsymbol{Q}$ be the n-dimensional configuration space (or joint space), $\boldsymbol{W}$ the m-dimensional operational space (or task space), and $\boldsymbol{f} : \boldsymbol{Q} \rightarrow \boldsymbol{W}$ the forward kinematic function. We assume the manipulator be redundant, *i.e.* $n > m$.

If $\boldsymbol{q} = (q_1, q_2, \cdots, q_n)^T$ is a configuration and $\boldsymbol{p} = (p_1, p_2, \cdots, p_m)^T$ a position, the relation can be described as follows:

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) \quad (1)$$

$$\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \quad (2)$$

where $\boldsymbol{J}(\boldsymbol{q})$ is the Jacobian matrix of partial derivatives of $\boldsymbol{f}$, evaluated at the current configuration $\boldsymbol{q}$. Since $n > m$, $\boldsymbol{J}$ is a rectangular matrix having m rows and n columns. Eq. (1) describes the relation in position level and Eq. (2) shows the linearized or instantaneous relation in rate level between two spaces. We can get a general solution to Eq. (2) as an algebraic equation of the form:

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^+\dot{\boldsymbol{p}} + (\boldsymbol{I} - \boldsymbol{J}^+\boldsymbol{J})\boldsymbol{v}. \quad (3)$$

where $\boldsymbol{J}^+$ is the pseudoinverse matrix for $\boldsymbol{J}(\boldsymbol{q})$, and the second term represents the null space vector as the projection of any vector function, say $\boldsymbol{v}(\boldsymbol{q}, t)$, chosen to fulfill some desired requirements onto the null space of $\boldsymbol{J}$. Eq. (3) is the key equation for one group of kinematic resolution scheme, called as *resolved motion rate control*(RMRC) [1]. The resulting $\dot{\boldsymbol{q}}$ of Eq. (3) is integrated to give joint-space path corresponding to the desired operational space trajectory.

The other resolution scheme incorporates the null space element implicitely in position level [2, 3], while RMRC represents it in rate level in more explicit way. Suppose some requirement on the configuration of the arm yields a system of holonomic constraints of the form:

$$\mu_1(\boldsymbol{q}) = \mu_2(\boldsymbol{q}) = \cdots = \mu_{n-m}(\boldsymbol{q}) = 0. \quad (4)$$

This determines an m-dimensional surface in $\Re^n$ [4], to which the joint-space paths will be restricted. These

[1] The position just denotes the position and orientation of the hand from now on.

equations together with the forward kinematic relation can be solved. Let $J_a = \partial\mu/\partial q$ be the $(n-m)\times n$ matrix of partial derivatives of the μ's, and let

$$\Phi = \Phi(q) = \begin{pmatrix} J \\ J_a \end{pmatrix} \tag{5}$$

be the $n \times n$ extended Jacobian matrix function which is invertible. Hence the resolved path is obtained from this scheme.

The holonomic constraints $\mu(q) = 0$ which determine the surfaces on which the joint paths are to lie, would normally be a result of optimizing some criterion. Suppose $h : Q \rightarrow \Re$ is a real-valued objective function of the configuration which is to be optimized relative to each position. The system can be driven dynamically toward such optima by the gradient projection method suggested in RMRC where $v = \pm\kappa\nabla h$ [1]. Another way to trace the exact optima is described as following. For each position $p \in W$, let

$$F_p = \{q \in Q | f(q) = p\} \tag{6}$$

be the fiber over p, *i.e.* the set of all inverse kinematic solutions. This fiber was characterized with algebraic topological tools and named as the self-motion manifold by Burdick [5] We look for relative or constrained optima of h, restricted to F_p. It was shown that if h attains a constrained optimum relative to the fiber at a point q^*, then $(I - J^+J)\nabla h = 0$ at q^* [6]. It can also be shown that as parameters p vary, the sets of such special configurations satisfy $n-m$ equations of the form $\mu_i(q) = 0$, forming an m-dimensional surface in Q. This kind of resolution method is termed as the *extended Jacobian method*(EJM) [6] in this paper from now on.

2 Problems on EJM

The characteristics of the solutions induced from EJM have not received much attention. The characteristics[2] of equilibrium solution cannot be identified with the conventional EJM. The RMRC has internal gradient mechanism to maximize or minimize some measure appropriately if we want it to be maximized or minimized, respectively. However, EJM just follows exact optima starting from exact optima and we must have an idea of the characteristics of the equilibrium.

The remark alluded to above requires that the initial configuration should satisfy the necessary condition and kinematic constraint. Unfortunately, general measures for multiple redundancy can have four different types of constrained extremum, *i.e.* maximum, minimum, saddle, and degeneracy. The conventional EJM, however, are not able to confirm the characteristic (or type) of the candidate equilibrium, because only necessary condition for constrained extremum is used to get solutions. Moreover, it is not clear whether the characteristics of a sequence of equilibrium points can be changed during the resolution with EJM.

The algorithmic singularity of EJM is unavoidable with the specific method itself. Also the exact characteristic of it has not been discussed, which makes it impossible to analyze and synthesize the control scheme related to the algorithmic singularities. Only a limited number of literatures have treated the problems on algorithmic singularities. They could just explain algorithmic singularities in a mathematical sence that they occur in case of the linear dependency between the row space of Jacobian and the partial derivatives matrix of constraints [3, 6] or that, at algorithmically singular configurations, the minimum eigenvalue of the extended Jacobian matrix is zero [7]. Cho *et al.* tried to devise the sufficient condition [8], but their result was not satisfactory in a sence that it is difficult to apply to multiple redundancy and requires high computational burden, and moreover it did not explain the relevance to the previous results [3, 6].

This paper addresses the problem of identifying the characteristics of the solution from EJM with analytical scrutiny. In section 3, a set of simple algebraic equations are proposed which describe the necessary and sufficient condition for local constrained extremum. Using these equations, results on the characteristics and the dimensionality of the algorithmic singularity are presented in section 4 with a planar 3-DOF redundant manipulator example in section 5. Section 6 will conclude this paper.

3 Characteristics of Optimal Solutions

In this section, the sufficient condition for the constrained extremum of the measure is proposed to identify the characteristics of the candidate solutions from the necessary condition. The sufficient condition of local extremum is a set of simple algebraic equations, which can be offered *offline*. With this sufficient condition, the characteristics of the algorithmic singularity will be analyzed in the next section.

3.1 Necessary and Sufficient Condition for Local Maximum

To maximize locally the performance measure, $h(q)$, satisfying kinematic equation, $F(q) = f(q) - p = 0$, involves the classical application of constrained optimization of the equality constraint using the method of *Lagrange multipliers* [2, 9]. According to the Lagrange theorem, the necessary condition needed for local extremum is:

$$N^T\nabla h = 0. \tag{7}$$

From the above equation, $\nabla h = -J^T\lambda$ at the critical point without any loss of generality.

[2] By the characteristics of equilibrium solution we mean whether the solution is (relative) minimum, maximum, saddle, or degeneracy.

Note that the introduction of the Lagrange multiplier, although it makes the derivation of the necessary condition simple, makes the practical utilization difficult due to the increase of the dimension of the variable. Therefore, the elimination of the multiplier is executed by projecting the necessary condition onto the null space of the Jacobian. During the following derivation of the sufficient condition, the configurations are assumed to satisfy the necessary condition already, so that the multiplier does not appear in the formulation. Cho[8] derived the sufficient condition for the one degree of redundancy case by introducing the pseudoinverse of the Jacobian to eliminate λ that is in the form of $\lambda^T = -(\nabla h)^T J^+$. The pseudoinverse of Jacobian is one method but high computational expense is incurred.

The above result however does not guarantee that *minimizing a function subject to constraints is equivalent to minimizing the related Lagrangian function.* In fact, the Lagrangian function has a saddle point at (q^*, λ^*) resulted from the necessary condition [9]. Thus, the sign definiteness of the Hessian of Lagrangian function does not indicate the sufficiency of the candidate extremum configuration as in the unconstrained problem. The sufficient condition for constrained extremum relative to the fiber with inverse kinematic language is given as follows [10]:

THEOREM 3.1 [3] *Let q^* be a local extremum satisfying*

$$N^T \nabla h\Big|_{q^*} = 0. \tag{8}$$

Then q^* is a local maximum (or minimum) if and only if the $(n-m)$-by-$(n-m)$ matrix defined by

$$A(q) = N^T HhN + (\nabla h)^T \frac{\partial N}{\partial q} N \tag{9}$$

whose (i,j) element is

$$A_{i,j}(q) = n_i^T Hhn_j + (\nabla h)^T \frac{\partial n_i}{\partial q} n_j, \tag{10}$$

is negative (or positive) definite at q^* where

$$N = \det(J_m) \times \begin{bmatrix} J_m^{-1} J_{n-m} \\ -I_{n-m} \end{bmatrix}, \tag{11}$$

when

$$J = [J_m \; J_{n-m}]. \tag{12}$$

The hessian of h, Hh is a symmetric matrix whose (i,j) element is given as:

$$Hh_{i,j} = \frac{\partial^2 h}{\partial q_i \partial q_j}. \tag{13}$$

When the degree of redunduncy, $r = n - m$, is more than one, the sufficient condition consists of r equations, each of which is the determinant of each principal minor. In case of one redundancy there is only one sufficient condition such as

$$n^T Hhn + (\nabla h)^T \frac{\partial n}{\partial q} n\Big|_{q^*} < 0. \tag{14}$$

From the above theorem one can determine the characteristics of the equilibrium configuration by simply calculating the $(n-m)$ algebraic equations which can be provided *offline*. If the measure is to be maximized, all the sufficient conditions should take the negative values at that time. The analysis on the characteristics of optimal solutions will be performed in detail in the next section.

4 Algorithmic singularity

In the previous section, two characteristics, *i.e.* maximum and minimum, can be identified with the proposed condition for any degree of redundancy with any measure. Two other characteristics are left unknown, one of which is the saddle, and the other is the degeneracy. In unconstrained problem, the degenerate critical point means the critical point where the Hessian is singular. Saddle critical points occur when the Hessian is sign-indefinite there. This section will be focused on these two types of critical points and characterize algorithmic singularities.

One might think that the algorithmic singularity comes from the indefiniteness of the matrix of $N^T(\partial^2 L/\partial q^2)N$ but it is not true. It will be shown to occur in degenerate critical point, *i.e.* if and only if the matrix A is singular. When the redundancy is equal to one, the sign-indefiniteness of the matrix A implies the degeneracy, because in this case the matrix A is just a scalar. In the multiple redundancy case, however, the candidate non-degenerate solutions are classified into three groups, *i.e.* the minimum, the maximum, and the saddle depending on whether the matrix A is positive definite, negative definite, or sign-indefinite. Sign-indefinite candidate configurations need not to be degenerate configurations in general.

As a matter of fact, the algorithmic singularity is located at the intersection of maximum, minimum, and/or saddle loci in EJM, since any of the eigenvalues must change the sign and this cause the matrix A to be singular. In other words, the extremum is undecidable at algorithmic singularities. Because of the specific way of generating the matrix A, the singularity can be identified with the following lemma.

LEMMA 4.1 *Suppose q^* is a joint configuration satisfying*

$$N^T \nabla h\Big|_{q^*} = 0. \tag{15}$$

Then the algorithmic singularity occurs at q^* if and only if $A(q)$ whose (i,j) element is given by Eq. (10) *is singular.*

When J_a is partial derivative matrix of $N^T \nabla h$, $J_a N$ was shown to be equal to A in Eq. (9) [10]. Thus $J_a N$ can

[3] All the proofs of the theorems in this paper can be referred in Park *et al.* [10].

also be used as a sufficient condition equivalently. That is, rank deficiency of $J_a N$ implies the degenerate critical point in constrained extremum relative to the fiber.

Lemma 4.1 specifies the number of equations to be satisfied for the algorithmic singularity to occur. It requires r equations for the necessary condition of Eq. (15) and one equation for the sufficient condition of det $A = 0$, thus $(r+1)$ equations in total. To determine the dimension of the *algorithmic singularity locus* which is defined as the set of the algorithmic singular configurations in joint space, one must check the following lemma.

LEMMA **4.2** *Let $h(q)$ be the measure to resolve the redundancy and be the function of all n variables, $q_1, \cdots, q_n$. Then the algorithmic singularity locus in the configuration space has the dimension equal to or less than $n - (r+1) = m - 1$.*

Similar result was suggested independently in [7] based on the fact that in algorithmic singularity the minimum eigenvalue of the numerically obtained extended Jacobian should be zero and hence constitute one constraint equation. The above lemma states that the dimensionality of algorithmic singularity locus is not dependent on the degree of redundancy but on the number of task coordinates. For example, with regard to planar task, algorithmic singularity locus has always one dimension and it consists of a few curves each of which can be parametrized by one parameter. In particular, the measure based on the manipulator Jacobian in the planar case is usually independent of q_1 and the dimension of the algorithmic singularity locus is $m-2$ in $q_2, \cdots, q_n$ plane. The algorithmic singularity locus projected onto $q_2, \cdots, q_n$ plane is only a set of discrete points. However, the number of the discrete algorithmic singularity locus is closely related to the selected measure and kinematic structure chosen.

One should be aware that the resolution scheme based on exact optimization cannot pass from the minimum locus to the maximum one, because it meets the algorithmic singularity thereof. This fact clarifies the last curious question: Is it possible to change the characteristic of the equilibrium configuration with EJM? The answer is no. Thus, the initial equilibrium configuration determines the characteristics of the total resolved path. The characteristic of the initial configuration is very important by this assertion. This kind of problem can not be identified with the conventional EJM, but the proposed condition can be used effectively to avoid this problem.

5 Numerical Example

In this example, the chosen mechanism is the 3-R planar redundant manipulator, where l_1 is 3 units, l_2, 2.5 units, and l_3, 2 units. To resolve the redundancy with the inverse kinematic method, the manipulability-like measure $H = det(JJ^T)$ is to be maximized so as not to encounter kinematic singularity. The locus of the necessary condition is independent of the task of the manipulator and is shown in Fig. 1(a). The sufficient condition, $A(q_2, q_3)$, is also independent of q_1 where

$$A(q_2, q_3) = n^T H h n + (\nabla h)^T \frac{\partial n}{\partial q} n = 0.$$

One can identify the algorithmic singularity in $q_2 - q_3$ plane as the intersection of the two loci graphically [8] or by numerically solving the above equation with the necessary condition. Note that usually graphical representation of algorithmic singularity is restricted to 3 DOF planar redundant manipulator with the measure based on the Jacobian. Even for this 3-DOF case, the accurate plots of two conditions are rather difficult to analyze. The closed equations from the proposed conditions can overcome the shortcomings of graphical solution and is applicable to general situations compared to graphical solution.

The loci are depicted in Fig. 1(b)[4], where this figure shows the overlapped diagram of two loci. The configurations numerically obtained in $q_2 - q_3$ plane, where the algorithmic singularity occurs, are summarized in Table 1. The dimension of the algorithmic singularity locus is one, where the parameter can be chosen as q_1, and thus the locus in the operational space is the circle centered at the base. There are ten configurations where the algorithmic singularity can occur, however, the five configurations in the Table 1 are enough to describe the algorithmic singularity due to the symmetry with respect to the base. The column radius in the Table 1 denotes the radius of the image of each algorithmic singularity locus in operational space. Note that the configurations whose forward kinematic values reach the base are always the algorithmic singularity when the manipulability measure is maximized.

When one wants to analyze the trajectory with the task considered, the self-motion manifold should be overlapped on the loci of neccessary condition to find the possible optimal paths to resolve the redundancy. Consider the task of the circular trajectory described by:

$$x = (-r cos(2\pi t) + c_x, -r sin(2\pi t)). \qquad (16)$$

where r is the radius of the circle to be carried out and c_x is the x-axis position of the center of the circle. The manipulator is commanded to trace the circle in 1.0 second maximizing the manipulability measure. When $r =$ 1.5 *units* and $c_x = 3.5$ *units*, the self-motion manifold at the initial configuration $x = 2.0$ *units* has six intersections denoted as (P1),...,(P6), with the loci of the necessary condition of Fig. 1(a), as shown in Fig. 2. These points only can be used as the initial state for the inverse kinematic methods, because the manipulator should trace the locus of the necessary condition from the corresponding chosen initial equilibrium configuration.

[4] The plot of locus near (0,0) is not accurate, because the locus should intersect at the origin.

The initial value of measure and sufficient condition and the configurations are summarized in Table 2. If one checks the sufficient condition, the results show that only (P1), (P2), and (P3) in Fig. 2 can make the manipulability to have a maximum. In this case, (P4), (P5), and (P6) are not suitable as an initial configuration, since the paths from these configurations will encounter singularities. Note that the minimum configuration, say (P6), can have larger value of manipulability than that of maximum one, (P3).

The inverse kinematic method simulates[5] the resolution of redundancy, and the resolved paths from each configuration are described also in Fig. 2, where these are projected onto $q_2 - q_3$ plane. If one checks Fig. 2, one can understand that the paths resolved from (P2) and (P3) will meet the algorithmic singularities at (d) and (e) in Fig. 1(b), respectively. It is thus only possible to complete the task when resolved from the configuration (P1) in Fig. 2, which has the smallest sufficient condition.

The values of manipulability and sufficient condition during resolution are shown in Fig. 3(a) and (b).

6 Conclusion

This paper discussed the characteristics of the resolved paths with the optimality augmented method like EJM and solved some problems on EJM. The necessary and sufficient condition for local extremum relative to the fiber was formulated for general n-DOF redundant manipulators. In particular, the conditions consist of simple algebraic equations that can be provided offline. These conditions can be used to analyze the solutions with EJM. The planar 3-DOF redundant manipulator was analyzed when the inverse kinematic method is used with the manipulability measure as an example.

The contents of this paper can be summarized as following. The characteristic of the equilibrium solution with EJM can be identified exactly with the proposed condition, $\boldsymbol{A}$. Although it is difficult to predict whether the resolved path will meet algorithmic singularity or not, the above condition can effectively give informations for the selection of proper initial condition for the given performance measure. The characteristic of initial equilibrium configuration is invariant during resolution with EJM when started from that configuration and unless singularities are encountered. The algorithmic singularity is located at the degenerate critical point, where $\det \boldsymbol{A} = \det \boldsymbol{J}_a \boldsymbol{N} = 0$. The algorithmic singularity locus in configuration space has $m-1$ dimension without regard to the measure to be used.

[5]Numerical simulations is based on the Broyden's method, also known as the Quasi Newton method, which is one way for solving the simulataneous nonlinear equations.

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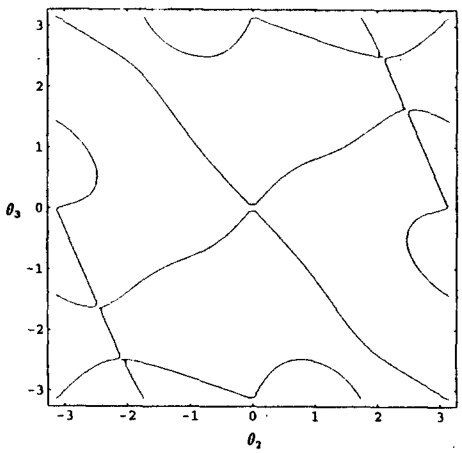
Table 1: The configurations of algorithmic singularities

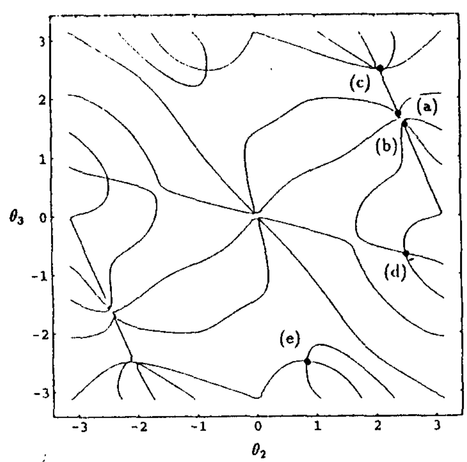
configuration	$q_2(°)$	$q_3(°)$	radius
(a)	138.5904	97.1808	0.0000
(b)	141.6125	92.7837	0.1442
(c)	124.5862	142.9171	1.4950
(d)	142.8637	-37.6906	3.4734
(e)	46.9843	-142.4039	4.5195

Table 2: The states of each initial configurations

configuration	$q_1(°)$	$q_2(°)$	$q_3(°)$	$H(q)$	$A(q)$
(P1)	-87.1742	111.7847	77.5352	75.0395	-4860.58
(P2)	-83.4304	179.8777	-82.0847	61.0392	-3428.24
(P3)	-8.8531	112.6675	155.6028	20.5785	-897.94
(P4)	-21.6863	94.1577	147.1470	18.7790	1373.58
(P5)	12.3080	150.8068	153.8209	13.9617	1024.89
(P6)	-126.9744	161.2385	-4.6209	27.0532	2251.39



(a) necessary condition only



(b) necessary and sufficient conditions

Figure 1: The loci of the necessary and sufficient conditions for the manipulability measure

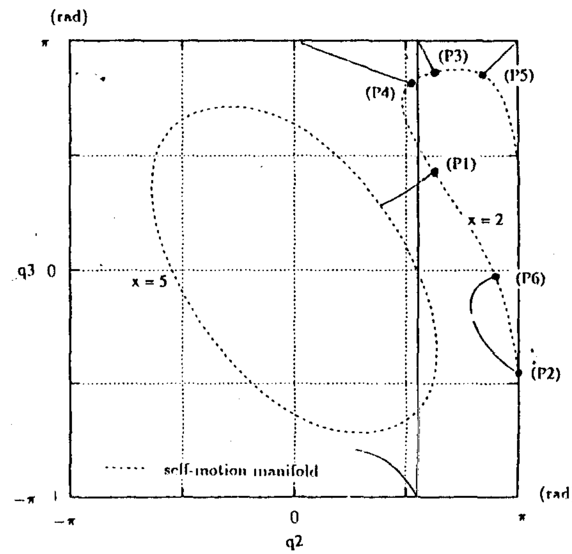
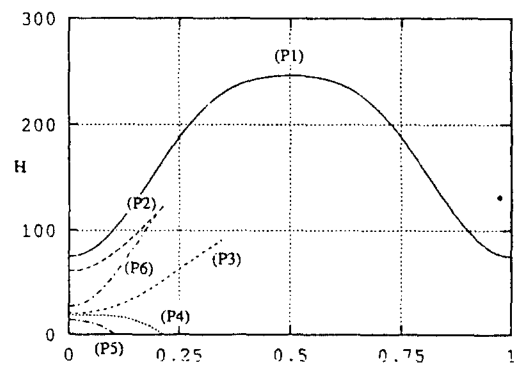
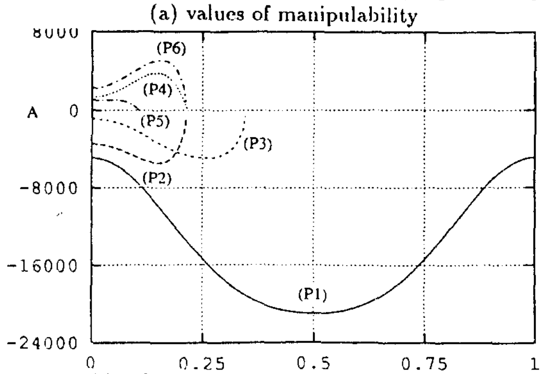


Figure 2: the resolved joint trajectories when the manipulability is used



(a) values of manipulability



(b) values of sufficient condition

Figure 3: Values of (a) the manipulability and (b) the sufficient condition during resolution