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Nonlinear Motion Analysis of a Two-Link Arm Using First Integrals

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Abstract

In this paper we discuss the nonlinear motion of a conservative two-link arm using first integrals, which includes one integral constant. In the analysis of the motion, the constant plays important role. First, we give some discussions on the free motion by focusing on the integral constant. As the result, the free motion can be classified into two types – the one is oscillation and the other is rotation. Second, we discuss the forced motion of the arm actuated only at the second joint. We take the first integral in a more general form, and show that the forced motion of the second link can be expressed as a variation of the integral constant. Also, the characteristic of the forced motion actuated by arbitrary constant torques is discussed.

1. Introduction

The first integrals are the solution of the equations of motion, and the integral constants included in them plays important role in the motion analysis. If n independent first integrals are found for a system of n first order differential equations, the system of differential equations can be solved. Unfortunately, these first integrals are not always found, but we can discuss the motion characteristic by investigating the obtained some first integrals. There is some study for finding the first integrals of a nonlinear dynamical systems and its application. For example, Kowalski and Steeb discuss the problem of finding symmetries and first integrals for a nonlinear dynamical systems using Hilbert space approach[1]. Sarlet and some authors discuss the approach for finding the first integrals of some nonlinear dynamical equations[2,3,4]. Moreover, the motion analysis of a cart pendulum using the first integrals is studied[5].

In this paper we present the motion analysis of a conservative two-link arm by focusing on the constant of the first integrals. First, we derive the equations of motion using lagrangians and rewrite it by introducing some dimensionless quantities. And we show that the one first integral can be obtained for the system. Second, we discuss the free motion and the forced motion actuated at the second joint based on the obtained first integral. We show that the free motion can be classified into two types by investigating the integral constant. Also, we show that the forced motion of the second link can be expressed schematically as a variation of the integral constant. Finally, the characteristic of the forced motion actuated at the second joint is discussed.

2. The Equations of Motion and the First Integral

The two-link arm which have planar motion is modeled in Fig.1, where
θ is angle of the first joint,
ϕ is angle of the second joint related to the first link,
$m_1(m_2)$ is mass of the link 1(link 2),
$l_1(l_2)$ is length of the link 1(link 2), and
$T_1(T_2)$ is actuated torque at the first joint(second joint).
The kinetic energy of the system, $\tilde{E}_k$, can be expressed as

$$\tilde{E}_k = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}^2 + \frac{1}{2}m_2\left\{l_2^2(\dot{\theta}+\dot{\phi})^2 + 2l_1l_2\dot{\theta}(\dot{\theta}+\dot{\phi})\cos\phi\right\}. \quad (1)$$

The equations of motion of the arm can be written as follows using lagrangians:

$$\{(m_1+m_2)l_1^2 + m_2l_2^2 + 2m_2l_1l_2\cos\phi\}\ddot{\theta} + (m_2l_2^2 + m_2l_1l_2\cos\phi)\ddot{\phi} - m_2l_1l_2\dot{\phi}(2\dot{\theta}+\dot{\phi})\sin\phi = T_1, \quad (2)$$

$$(m_2l_2^2 + m_2l_1l_2\cos\phi)\ddot{\theta} + m_2l_2^2\ddot{\phi} + m_2l_1l_2\dot{\theta}^2\sin\phi = T_2. \quad (3)$$

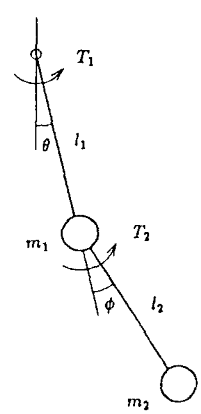


Fig.1, Modelling of the two-link arm.

And, we introduce following dimensionless quantities:

$$\tau = \omega t, \quad \omega^2 = \frac{g}{l_2}, \quad \kappa = \frac{m_1 + m_2}{m_2}, \quad \lambda = \frac{l_1}{l_2},$$

$$\alpha = \frac{T_1}{m_2 g l_2}, \quad \beta = \frac{T_2}{m_2 g l_2},$$

where g is acceleration of gravity. Using these dimensionless quantities, the equations of motion (2) and (3) can be rewritten as follows:

$$\begin{aligned}(1 + \kappa\lambda^2 + 2\lambda\cos\phi)\ddot{\theta} + (1 + \lambda\cos\phi)\ddot{\phi} \\ -\lambda\dot{\phi}(2\dot{\theta} + \dot{\phi})\sin\phi &= \alpha, \quad (4)\\ (1 + \lambda\cos\phi)\ddot{\theta} + \ddot{\phi} + \lambda\dot{\theta}^2\sin\phi &= \beta, \quad (5)\end{aligned}$$

where

$$\dot{\theta} = \frac{d\theta}{d\tau}, \quad \ddot{\theta} = \frac{d^2\theta}{d\tau^2}, \quad \dot{\phi} = \frac{d\phi}{d\tau}, \quad \ddot{\phi} = \frac{d^2\phi}{d\tau^2}.$$

In the same manner, the energy equation (1) have dimensionless form as

$$E_k = \frac{1}{2}\kappa\lambda^2\dot{\theta}^2 + \frac{1}{2}(\dot{\theta} + \dot{\phi})^2 + \lambda\dot{\theta}(\dot{\theta} + \dot{\phi})\cos\phi, \quad (6)$$

where

$$E_k \equiv \frac{\tilde{E}_k}{m_2 g l_2}.$$

Solving the equation (4) and (5) for $\ddot{\theta}$ and $\ddot{\phi}$ leads to

$$\begin{aligned}\ddot{\theta} &= \frac{1}{\lambda^2(\kappa - \cos^2\phi)}\{\alpha - (1 + \lambda\cos\phi)\beta \\ &\quad + \{(\dot{\theta} + \dot{\phi})^2 + \lambda\dot{\theta}^2\cos\phi\}\lambda\sin\phi\}, \quad (7)\\ \ddot{\phi} &= \frac{1}{\lambda^2(\kappa - \cos^2\phi)}\{(1 + \kappa\lambda^2 + 2\lambda\cos\phi)\beta \\ &\quad -(1 + \lambda\cos\phi)\alpha - \{\kappa\lambda^2\dot{\theta}^2 + (\dot{\theta} + \dot{\phi})^2 \\ &\quad + 2\lambda\dot{\theta}(\dot{\theta} + \dot{\phi})\cos\phi + \lambda\dot{\phi}^2\cos\phi\}\lambda\sin\phi\}. \quad (8)\end{aligned}$$

Since the first three terms in the parentheses {} of the equation (8) equal two times of the kinetic energy (6), we have

$$\begin{aligned}\ddot{\phi} &= \frac{1}{\lambda^2(\kappa - \cos^2\phi)}\{(1 + \kappa\lambda^2 + 2\lambda\cos\phi)\beta \\ &\quad -(1 + \lambda\cos\phi)\alpha - (2E_k + \lambda\dot{\phi}^2\cos\phi)\lambda\sin\phi\}.\end{aligned} \quad (9)$$

To find the first integral, we consider the following auxiliary equation from the equation (9):

$$\ddot{\phi} = -\frac{\dot{\phi}^2\sin\phi\cos\phi}{\kappa - \cos^2\phi}. \quad (10)$$

Integrating (10), we have

$$\dot{\phi} = \frac{c_1}{\sqrt{\kappa - \cos^2\phi}}, \quad (11)$$

where c_1 is a constant. Defining the angular velocity of the second link of the arm at $\phi = 0$ as $\dot{\phi}_*$, then the angular velocity of the second link at a arbitrary angle can be represented as

$$\dot{\phi} = \sqrt{\frac{\kappa - 1}{\kappa - \cos^2\phi}}\dot{\phi}_*. \quad (12)$$

Next, we consider the equation (9) including the neglected terms. Assuming that $\dot{\phi}_*$ is the function of time τ, the equation (12) can be expressed by

$$\dot{\phi}_*(\tau) = \sqrt{\frac{\kappa - \cos^2\phi}{\kappa - 1}}\dot{\phi}. \quad (13)$$

Differentiating and arranging (13), we have

$$\begin{aligned}\frac{d\dot{\phi}_*(\tau)}{d\tau} &= \frac{1}{\lambda\dot{\phi}_*(\kappa - 1)} \\ &\quad \left\{\frac{(1 + \kappa\lambda^2 + 2\lambda\cos\phi)\beta - (1 + \lambda\cos\phi)\alpha}{\lambda}\right. \\ &\quad \left. -2E_k\sin\phi\right\}\dot{\phi}. \quad (14)\end{aligned}$$

From this equation, we obtain

$$\begin{aligned}\dot{\phi}_* d\dot{\phi}_* &= \frac{1}{\lambda(\kappa - 1)} \\ &\quad \left\{\frac{(1 + \kappa\lambda^2 + 2\lambda\cos\phi)\beta - (1 + \lambda\cos\phi)\alpha}{\lambda}\right. \\ &\quad \left. -2E_k\sin\phi\right\}d\phi. \quad (15)\end{aligned}$$

The equation (15) is the prior form of the first integral will be discussed in this paper.

3. The Characteristic of the Free Motion

We take the first integral of the equations of the free motion, and show that the free motion can be classified into two types by investigating the integral constant. Furthermore, we show that the angular velocity of the system defined at a specified angle determine which motion will be appear.

From the equation (15), we take the first integral. Since the kinetic energy is a constant in the free motion, let the kinetic energy be E_{k_c}. Also, since the actuated torque α and β are 0, the equation (15) simplified as

$$\dot{\phi}_* d\dot{\phi}_* = -\frac{2E_{k_c}\sin\phi}{\lambda(\kappa - 1)}d\phi. \quad (16)$$

Integrating (16), we have

$$\frac{1}{2}\dot{\phi}_*(\phi)^2 = \frac{2E_{k_c}\cos\phi}{\lambda(\kappa - 1)} + p, \quad (17)$$

where p is a constant. Defining the constant p at $\phi = 0$, it can be represented as

$$p = \frac{1}{2}\dot{\phi}_*^2 - \frac{2E_{k_c}}{\lambda(\kappa - 1)}. \quad (18)$$

The equation (17) expresses the free motion of the second link of the arm. Since the left side of the equation (17) is positive, we have

$$\frac{2E_{k_c}\cos\phi}{\lambda(\kappa - 1)} + p \geq 0. \quad (19)$$

In the equation (19), we can consider the following two conditions:

$$(a)\quad \frac{2E_{k_c}}{\lambda(\kappa-1)} \geq p, \qquad (b)\quad \frac{2E_{k_c}}{\lambda(\kappa-1)} < p. \qquad (20)$$

Under the condition (a), the equation (19) is satisfied on a constrained range of ϕ. In this case, the second link of the arm oscillates between two points defined by the solutions of $\frac{2E_{k_c}\cos\phi}{\lambda(\kappa-1)} + p = 0$. On the other hand, under the condition (b), the equation (19) is satisfied on all range of ϕ. In this case, the second link of the arm rotates continuously. When the condition (a) is satisfied, we call it the oscillation mode, and when the condition (b) is satisfied, we call it the rotation mode. Using (18), the conditions of the equation (20) can be rewritten by

$$(a)\quad \frac{\dot{\phi}_*^2}{4} \leq \frac{2E_{k_c}}{\lambda(\kappa-1)}, \qquad (b)\quad \frac{\dot{\phi}_*^2}{4} > \frac{2E_{k_c}}{\lambda(\kappa-1)}. \qquad (21)$$

From the equation (21), we consider the following equation as the boundary of the classified two modes:

$$\frac{2E_{k_c}}{\lambda(\kappa-1)} - \frac{1}{4}\dot{\phi}_*^2 = 0. \qquad (22)$$

Letting $\dot{\theta}$ be $\dot{\theta}_*$ at $\phi = 0$ and rewritting (22) as quadratic form, we get

$$a_{11}\dot{\theta}_*^2 + a_{22}\dot{\phi}_*^2 + 2a_{12}\dot{\theta}_*\dot{\phi}_* = 0, \qquad (23)$$

where

$$\begin{aligned} a_{11} &= 4(\kappa\lambda^2 + 2\kappa + 1), \\ a_{22} &= 4 - \kappa\lambda + \lambda, \\ a_{12} &= 4(1+\kappa). \end{aligned}$$

From the equation (23), we get the following:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = -4\lambda(\kappa-1)(\kappa\lambda^2 - 2\lambda + 1) < 0.$$

Therefore, the equation (23) expresses crossed two lines, and it is rewritten as

$$(\dot{\theta}_* + a\dot{\phi}_*)(\dot{\theta}_* + b\dot{\phi}_*) = 0.$$

From this, the two lines which classify the modes of the motion are determined by

$$\dot{\theta}_* + a\dot{\phi}_* = 0, \qquad \dot{\theta}_* + b\dot{\phi}_* = 0, \qquad (24)$$

where

$$a = \frac{4(1+\lambda) + \sqrt{4\lambda(\kappa-1)(\kappa\lambda^2 - 2\lambda + 1)}}{4(\kappa\lambda^2 + 2\lambda + 1)}, \qquad (25)$$

$$b = \frac{4(1+\lambda) - \sqrt{4\lambda(\kappa-1)(\kappa\lambda^2 - 2\lambda + 1)}}{4(\kappa\lambda^2 + 2\lambda + 1)}. \qquad (26)$$

The classification of the modes are shown in Fig. 2 for some physical parameters. The hatched regions indicate rotation mode, the others are oscillation mode. If the value of κ becomes large, the rotation mode will be more wide. And, if the value of λ becomes large, the rotation mode will exist around the value of $\dot{\theta}_* = 0$. Also the examples of two types of the motion are illustrated in Fig.3.

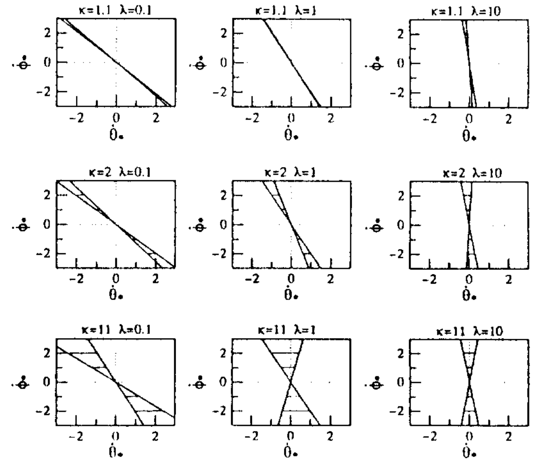
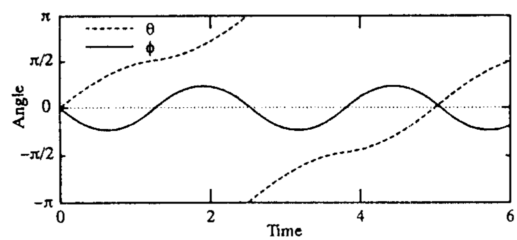
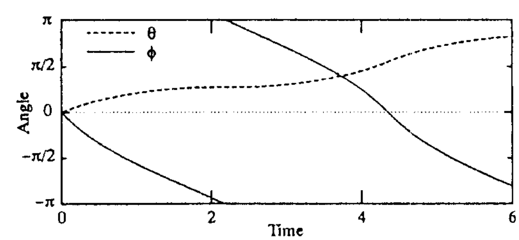


Fig.2, The classification of the modes for some physical parameters κ and λ.



(a), Oscillation mode ($\dot{\theta}_* = 2$, $\dot{\phi}_* = -2$.)



(b), Rotation mode ($\dot{\theta}_* = 1$, $\dot{\phi}_* = -2.5$.)

Fig.3, The examples of two types of the motion in case of $\kappa = 2$ and $\lambda = 1$.

4. The Forced Motion Actuated at the Second Joint

We take the first integral in a more general form, and show that the forced motion can be expressed as a variation of the integral constant. Using the variation of the constant, we can easily obtain an appropriate control sequence so as to move the second link of the arm along the specified path.

From the equation (15), letting α be 0, it becomes

$$\dot{\phi}_* d\dot{\phi}_* = \frac{1}{\lambda(\kappa-1)} \left\{ \frac{(1+\kappa\lambda^2+2\lambda\cos\phi)\beta}{\lambda} - 2E_k \sin\phi \right\} d\phi. \quad (27)$$

Integrating (27), we can obtain a variation of $\dot{\phi}_*$. But the integration of (27) is impossible since the kinetic energy E_k is the function of $\phi, \dot{\phi}$, and $\dot{\theta}$. Differentiating (6), we have

$$\begin{aligned} \frac{dE_k}{dt} &= (1+\kappa\lambda^2+2\lambda\cos\phi)\ddot{\theta}\dot{\theta} \\ &+(1+\lambda\cos\phi)\ddot{\phi}\dot{\theta} - \lambda\dot{\phi}(\dot{\theta}+\dot{\phi})\sin\phi\dot{\theta} \\ &+(1+\lambda\cos\phi)\ddot{\theta}\dot{\phi} + \ddot{\phi}\dot{\phi}. \end{aligned} \quad (28)$$

Considering $\alpha = 0$ in the equation (4), we can obtain as follows from the equation (4),(5) and (28):

$$\frac{dE_k}{dt} = \beta\dot{\phi}. \quad (29)$$

Integrating (29), we have

$$E_k = \beta\phi + c_k, \quad (30)$$

where c_k is a constant, and it is the initial kinetic energy which the arm had before the torque is actuated.

The two-link arm system is changed into a discrete time system. The control input is taken as discrete values and the nondimensional sampling time is chosen appropriately. The control action is started from the time τ_0 applying a constant torque, and switched by other value at the time $\tau_1, \tau_2, ..., \tau_n$. Here ϕ_i and $E_{k_i} (i = 0, 1, ..., n)$ are the angle of the second joint and the kinetic energy of the system respectively at those times. Also, let the interval $\tau_{i-1} \leq \tau < \tau_i$ be control interval i, and β_i is a constant torque at that interval. The kinetic energy of the control interval i is defined as follows:

$$\begin{aligned} E_k &= \beta_i(\phi - \phi_{i-1}) + \beta_{i-1}(\phi_{i-1} - \phi_{i-2}) \\ &+ \cdots + \beta_1(\phi_1 - \phi_0) + E_{k_0} \\ &= \beta_i\phi + \xi_{k_i}, \end{aligned} \quad (31)$$

where $i = 1, 2, ..., n$, and ξ_{k_i} is

$$\xi_{k_i} = -\beta_i\phi_{i-1} + E_{k_{i-1}}. \quad (32)$$

Substituting (31) into (27), we have

$$\dot{\phi}_*(\phi)d\dot{\phi}_* = \frac{1}{\lambda(\kappa-1)} \left\{ \frac{(1+\kappa\lambda^2+2\lambda\cos\phi)\beta_i}{\lambda} - (\beta_i\phi + \xi_{k_i})\sin\phi \right\} d\phi. \quad (33)$$

Integrating (33), we obtain

$$\frac{1}{2}\dot{\phi}_*(\phi)^2 = \frac{(1+\kappa\lambda^2+2\lambda\cos\phi)\beta_i\phi + 2\lambda\xi_{k_i}\cos\phi}{\lambda^2(\kappa-1)} + c_{t_i}. \quad (34)$$

In the equation (34), since $\dot{\phi}_*$ expresses the forced motion actuated by the torque β_i, we call it the trajectory parameter. Also c_{t_i} is a constant and determined at the angle ϕ_{i-1} in the equation (34):

$$c_{t_i} = \frac{1}{2}\dot{\phi}_*(\phi_{i-1})^2 - \frac{(1+\kappa\lambda^2+2\lambda\cos\phi_{i-1})\beta_i\phi_{i-1} + 2\lambda\xi_{k_i}\cos\phi_{i-1}}{\lambda^2(\kappa-1)}. \quad (35)$$

Here ξ_{k_i} and c_{t_i} are determined by (32) and (35) respectively, and then the motion of the second link can be obtained by the equation (34). Fig.4 shows the plots of the equation (34) for $\beta_i = 1, 0, -1$ in case of $\kappa = 2$ and $\lambda = 1$. The plots represent the forced motion started from the point O with $\phi_0 = 0$ and $E_{k_0} = 0$ actuated by a constant torque, and then switched by other constant torque at the point A,B,C and D respectively.

Here we show an example of the control action using the Fig.4. The motion control considered in this example is to move the second link from the initial point O to the final point Z, and makes the angular velocity of the second joint to be 0 at the final point Z. In the beginning, the second link is moved from the point O to the point A applying $\beta_1 = 1$. At the point A, the torque is changed into $\beta_2 = -1$, then the second link moves to the point Z. At the point Z, the angular velocity of the second joint to be 0, and then the second link stops applying $\beta_2 = 0$ at that point. The control action mentioned above is listed in Table 1, and the motion of the second link versus τ is shown in Fig.5. It shows that the objective of the control was successfully achieved.

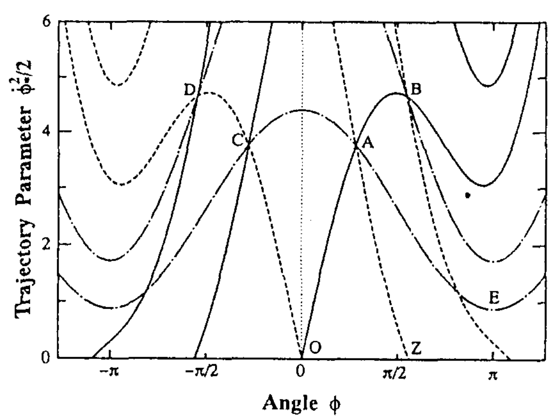


Fig.4, The variations of the trajectory parameter $\dot{\phi}_*$ in case of $\kappa = 2$ and $\lambda = 1$, where the solid line is $\beta_i = 1$, the dotted line is $\beta_i = -1$, and the dotted chain is $\beta_i = 0$.

Also, we can consider following control action. For example, applying $\beta_2 = 0$ instead of $\beta_2 = -1$ at the point A, the second link moves towards the point E along the dotted chain. Also at the point Z, if the torque is held by $\beta_2 = -1$, the second link begins to go reversely towards the point A. In this example a simple control action is considered relatively. However we can discuss the general case switched more times by an arbitrary torques. As long as the actuated torque and switched point are given, we can take the motion trajectory schematically by the similar manner in the above example.

Table 1, The control sequence of the example.

point	angle ϕ	time τ	torque β_i
O	0.0	0.0	1
A	0.883830	0.6430	−1
Z	1.763320	1.6850	0

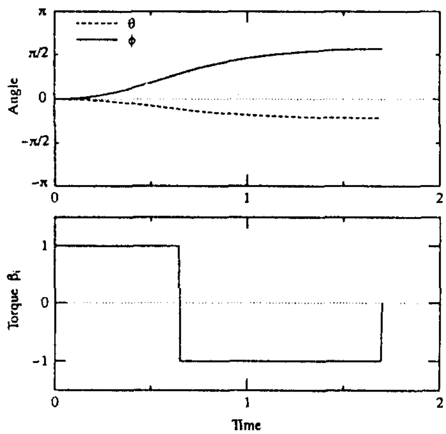


Fig.5, The motion of the second link and the control action designed in Fig. 4.

Next, we discuss the characteristic of the forced motion actuated by arbitrary constant torques including 0.

Theorem 1 Assume that the initial kinetic energy is zero, i.e., $E_{k_0} = 0$. Then, if the angular velocity of the second joint is zero, the two-link arm will be stopped. At that time, the angle ϕ is determined by the condition of $E_k = 0$ as follows:

$$\begin{aligned}\phi &= \frac{1}{\beta_i}\{\beta_{i-1}(\phi_{i-2}-\phi_{i-1})+\beta_{i-2}(\phi_{i-3}-\phi_{i-2}) \\ &\quad +\cdots+\beta_1(\phi_0-\phi_1)\}+\phi_{i-1}. \qquad (36)\end{aligned}$$

Proof. The variation of the trajectory parameter in control interval i can be expressed by the equation (32), (34), and (35) as follows (Refer to Appendix.):

$$\begin{aligned}\frac{1}{2}\dot{\phi}_*(\phi)^2 &= \frac{1}{\lambda^2(\kappa-1)}\{(1+\kappa\lambda^2+2\lambda\cos\phi)\beta_i\phi \\ &\quad +2\lambda(-\beta_i\phi_{i-1}+E_{k_{i-1}})\cos\phi\}+c_i, \\ &= \frac{1+\kappa\lambda^2}{\lambda^2(\kappa-1)}(E_k-E_{k_0}) \\ &\quad +\frac{2}{\lambda(\kappa-1)}\{E_k\cos\phi-E_{k_0}\cos\phi_0\} \\ &\quad +\frac{1}{2}\dot{\phi}_*(\phi_0)^2. \qquad (37)\end{aligned}$$

If the equation (37) is 0, the angular velocity of the second joint will be 0 also. Rewritting the right side of the equation (37), we have

$$\begin{aligned}&\frac{E_k}{\lambda(\kappa-1)}\left(\frac{1+\kappa\lambda^2}{\lambda}+2\cos\phi\right) \\ &-\frac{E_{k_0}}{\lambda(\kappa-1)}\left(\frac{1+\kappa\lambda^2}{\lambda}+2\cos\phi_0\right)+\frac{1}{2}\dot{\phi}_*(\phi_0)^2 \\ &\quad =\frac{E_k}{\lambda(\kappa-1)}\left\{2+\frac{\kappa}{\lambda}(\lambda-\frac{1}{\kappa})^2+\frac{1}{\lambda}(1-\frac{1}{\kappa})+2\cos\phi\right\} \\ &\quad\quad -\frac{E_{k_0}}{\lambda(\kappa-1)}\left\{2+\frac{\kappa}{\lambda}(\lambda-\frac{1}{\kappa})^2+\frac{1}{\lambda}(1-\frac{1}{\kappa})+2\cos\phi_0\right\} \\ &\quad\quad +\frac{1}{2}\dot{\phi}_*(\phi_0)^2. \qquad (38)\end{aligned}$$

Since the condition $E_{k_0} = 0$ and the physical parameters are $\kappa > 1$ and $\lambda > 0$, the equation (38) cannot be 0 without $E_k = 0$. Therefore it is proven that the two-link arm stops when the angular velocity of the second joint is 0.

Theorem 2 Assume that the initial kinetic energy is zero, i.e., $E_{k_0} = 0$. Then, a free motion occurred after a forced motion is always rotation.

Proof. Consider the free motion in control interval i. Since the torque $\beta_i = 0$, the variation of the trajectory parameter can be expressed as from the equation (37):

$$\begin{aligned}\frac{1}{2}\dot{\phi}_*(\phi)^2 &= \frac{2E_{k_{i-1}}\cos\phi}{\lambda(\kappa-1)} \\ &\quad +\frac{(1+\kappa\lambda^2)(E_{k_{i-1}}-E_{k_0})-2\lambda E_{k_0}\cos\phi_o}{\lambda^2(\kappa-1)} \\ &\quad +\frac{1}{2}\dot{\phi}_*(\phi_0)^2. \qquad (39)\end{aligned}$$

By the same manner of the equation (19) and the followings from that, we investigate the sign of the following equation:

$$\begin{aligned}&\frac{2E_{k_{i-1}}}{\lambda(\kappa-1)}-\frac{(1+\kappa\lambda^2)(E_{k_{i-1}}-E_{k_0})-2\lambda E_{k_0}\cos\phi_o}{\lambda^2(\kappa-1)} \\ &-\frac{1}{2}\dot{\phi}_*(\phi_0)^2 \\ &\quad =\frac{E_{k_{i-1}}}{\lambda(\kappa-1)}\left\{-\frac{\kappa}{\lambda}(\lambda-\frac{1}{\kappa})^2-\frac{1}{\lambda}(1-\frac{1}{\kappa})\right\} \\ &\quad\quad +\frac{(1+\kappa\lambda^2+2\lambda\cos\phi_0)E_{k_0}}{\lambda^2(\kappa-1)}-\frac{1}{2}\dot{\phi}_*(\phi_0)^2. \\ &\qquad (40)\end{aligned}$$

Since the condition $E_{k_0} = 0$ and the physical parameters are $\kappa > 1$ and $\lambda > 0$, the equation (40) is always less than 0. Thus we can say that the type of the free motion is rotation.

Also in case of the initial kinetic energy $E_{k_0} \neq 0$, the types of the free motion is determined by the equation (40). If the dynamical condition of the system makes the equation (40) to be less than 0, the rotating motion will be appears. On the other hand, if the condition makes the equation (40) greater than 0, the oscillating motion will be appears.

5. Conclusions

In this paper we discussed the motion of a conservative two-link arm using first integrals. The theoretical analysis of the free motion and the forced motion actuated at the second joint are discussed by focusing on the integral constant. In this paper, since one first integral is obtained for the system, we cannot discuss the motion of the first link, unless we use the numerical computation. However, we were able to analyze the relative motion of the two-link arm adopted in this paper. The proposed analytic approach can be applied to a similar nonlinear dynamical systems which have the first integrals.

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Appendix

The variation of the trajectory parameter in control interval i can be expanded and arranged by the equation (32), (34), and (35) as follows:

$$
\begin{aligned}
\frac{1}{2}\dot{\phi}_*(\phi)^2 &= \frac{1}{\lambda^2(\kappa-1)}\{(1+\kappa\lambda^2+2\lambda\cos\phi)\beta_i\phi \\
&\quad +2\lambda(-\beta_i\phi_{i-1}+E_{k_{i-1}})\cos\phi\} + c_{t_i} \\
&= \frac{1}{\lambda^2(\kappa-1)}\{(1+\kappa\lambda^2+2\lambda\cos\phi)\beta_i\phi \\
&\quad +2\lambda(-\beta_i\phi_{i-1}+E_{k_{i-1}})\cos\phi \\
&\quad +(1+\kappa\lambda^2+2\lambda\cos\phi_{i-1})\beta_{i-1}\phi_{i-1} \\
&\quad +2\lambda(-\beta_{i-1}\phi_{i-2}+E_{k_{i-2}})\cos\phi_{i-1} \\
&\quad -(1+\kappa\lambda^2+2\lambda\cos\phi_{i-1})\beta_i\phi_{i-1} \\
&\quad -2\lambda(-\beta_i\phi_{i-1}+E_{k_{i-1}})\cos\phi_{i-1} \\
&\quad +(1+\kappa\lambda^2+2\lambda\cos\phi_{i-2})\beta_{i-2}\phi_{i-2} \\
&\quad +2\lambda(-\beta_{i-2}\phi_{i-3}+E_{k_{i-3}})\cos\phi_{i-2} \\
&\quad -(1+\kappa\lambda^2+2\lambda\cos\phi_{i-2})\beta_{i-1}\phi_{i-2} \\
&\quad -2\lambda(-\beta_{i-1}\phi_{i-2}+E_{k_{i-2}})\cos\phi_{i-2} \\
&\quad +\cdots \\
&\quad +(1+\kappa\lambda^2+2\lambda\cos\phi_1)\beta_1\phi_1 \\
&\quad +2\lambda(-\beta_1\phi_0+E_{k_0})\cos\phi_1 \\
&\quad -(1+\kappa\lambda^2+2\lambda\cos\phi_1)\beta_2\phi_1 \\
&\quad -2\lambda(-\beta_2\phi_1+E_{k_1})\cos\phi_1 \\
&\quad -(1+\kappa\lambda^2+2\lambda\cos\phi_0)\beta_1\phi_0 \\
&\quad -2\lambda(-\beta_1\phi_0+E_{k_0})\cos\phi_0\} \\
&\quad +\frac{1}{2}\dot{\phi}_*(\phi_o)^2 \\
&= \frac{1+\kappa\lambda^2}{\lambda^2(\kappa-1)}\{\beta_i(\phi-\phi_{i-1}) \\
&\quad +\beta_{i-1}(\phi_{i-1}-\phi_{i-2}) \\
&\quad +\cdots+\beta_1(\phi_1-\phi_0)\} \\
&\quad +\frac{2}{\lambda(\kappa-1)}\{E_{k_{i-1}}(\cos\phi-\cos\phi_{i-1}) \\
&\quad +E_{k_{i-2}}(\cos\phi_{i-1}-\cos\phi_{i-2}) \\
&\quad +\cdots+E_{k_0}(\cos\phi_1-\cos\phi_0)\} \\
&\quad +\frac{2}{\lambda(\kappa-1)}\{\beta_i(\phi-\phi_{i-1})\cos\phi \\
&\quad +\beta_{i-1}(\phi_{i-1}-\phi_{i-2})\cos\phi_{i-1} \\
&\quad +\cdots+\beta_1(\phi_1-\phi_0)\cos\phi_1\} \\
&\quad +\frac{1}{2}\dot{\phi}_*(\phi_0)^2 \\
&= \frac{1+\kappa\lambda^2}{\lambda^2(\kappa-1)}(E_k-E_{k_0}) \\
&\quad +\frac{2}{\lambda(\kappa-1)}\{E_k\cos\phi-E_{k_0}\cos\phi_0\} \\
&\quad +\frac{1}{2}\dot{\phi}_*(\phi_0)^2
\end{aligned}
\tag{A.1}
$$