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Some Bounds on the Solution of the Continuous Algebraic Riccati Equation

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Abstract

Some upper bounds for the solution of the continuous algebraic Riccati equation are presented. These consist of bounds for summations of eigenvalues, products of eigenvalues, individual eigenvalues and the minimum eigenvalue of the solution matrix. Among these bounds, the first three are the first results for the upper bound of each case, while bounds for the minimum eigenvalue supplement the existing ones and require no side conditions for their validities.

1. Introduction

The Riccati equation has been widely used in modern engineering theory. For the algebraic Riccati equation, there are many numerical algorithms to obtain the solution. However, we sometimes need only an estimate of the solution which can be obtained without heavy computational burdens. Bounds for some scalar quantities (especially the eigenvalues) of the solution matrix provide such estimates of the solution. They can be used as approximations of the solution or initial guesses for the numerical solution algorithms.

Bounds for the solution of the Riccati equation were first proposed in [1]. Since then, various types of bounds have been reported - See [2] for a summary. Several scalar quantities of a matrix were employed to represent the 'size' of the solution matrix, such as the maximum eigenvalue, the minimum eigenvalue, the trace, the determinant, etc. In recent years, some literature proposed bounds for summations and products of the eigenvalues which include multiple different bounds.

In this paper, we present some new bounds for the solution of the continuous algebraic Riccati equation. A set of upper bounds for summations of eigenvalues, products of eigenvalues, individual eigenvalues and the minimum eigenvalue are derived. The first three bounds are believed to be the first results for the upper bound of each case. To the best of authors' knowledge, only lower bounds have been obtained for summations of eigenvalues and individual eigenvalues [5] - [7], and no results for products of eigenvalues. For the minimum eigenvalue, the derived upper bounds supplement the existing ones and require no side conditions for their validities.

2. Main Results

Let us consider the continuous algebraic Riccati equation (CARE)

$$A'P + PA - PRP + Q = 0, \qquad (1)$$

where A, P, Q, R $\in R^{n\times n}$ and A' is the transpose of A. Matrices Q and R are symmetric positive-semidefinite. It is well known that there exists a unique symmetric positive-definite solution P assuming that $A - RP$ is asymtotically stable.

We define several notations as follows. All eigenvalues $\lambda_i(X)$, where $i = 1, 2, \ldots, n$, of a matrix X are arranged such that their real parts are nonincreasing, i.e.,

$$Re\lambda_1(X) \geq Re\lambda_2(X) \geq \ldots \geq Re\lambda_n(X).$$

The same order applies to the singular values $\sigma_i(X) = [\lambda_i(X'X)]^{1/2}$. Notations $\sum_1^k$ and $\prod_1^k$ denote summations and products, respectively, from $i = 1$ to $i = k$, where the index k runs from 1 to n. To give a concise representation, the following scalar function is used:

$$f(a,b,c) := \frac{-a + \sqrt{(a^2 + bc)}}{b} = \frac{c}{a + \sqrt{(a^2 + bc)}}.$$

Theorem 1 *Let P be the solution of the CARE (1) when R is a positive definite matrix. Then the following inequalities are satisfied for* $k = 1, 2, \ldots, n$:

$$\sum_1^k \lambda_i(P) \le f(-\sigma_1(A), \lambda_n(R)/k, \sum_1^k \lambda_i(Q)). \quad (2)$$

Proof: Rewrite (1) as

$$A'P + PA + Q = PRP \quad (3)$$

and take summations of singular values to obtain

$$\sum_1^k \sigma_i(A'P + PA + Q) = \sum_1^k \sigma_i(PRP). \quad (4)$$

In order to proceed, we need the following inequalities which can be found in [8]:

$$\sum_1^k \sigma_i(X + Y) \le \sum_1^k \sigma_i(X) + \sum_1^k \sigma_i(Y), \quad (5)$$

$$\sum_1^k \sigma_i(XY) \le \sum_1^k \sigma_i(X)\sigma_i(Y). \quad (6)$$

Apply (5) and (6) to the left-hand side of (4) to get

$$2\sum_1^k \sigma_i(A)\sigma_i(P) + \sum_1^k \sigma_i(Q) \ge \sum_1^k \sigma_i(PRP). \quad (7)$$

Recalling that $\sigma_i(X) = \lambda_i(X)$ for $X = X' \ge 0$, we can obtain

$$2\sum_1^k \sigma_i(A)\lambda_i(P) + \sum_1^k \lambda_i(Q) \ge \sum_1^k \lambda_i(PRP). \quad (8)$$

Using the following inequality [8] for $n \times n$ symmetric positive-semidefinite matrices X and Y,

$$\sum_1^k \lambda_i(XY) \ge \sum_1^k \lambda_i(X)\lambda_{n-i+1}(Y) \quad (9)$$

and the identity $\lambda_i(PRP) = \lambda_i(P^2R)$, we obtain

$$2\sum_1^k \sigma_i(A)\lambda_i(P) + \sum_1^k \lambda_i(Q) \ge \sum_1^k \lambda_i^2(P)\lambda_{n-i+1}(R). \quad (10)$$

Bound (10) as

$$2\sigma_1(A)\sum_1^k \lambda_i(P) + \sum_1^k \lambda_i(Q) \ge \lambda_n(R)\sum_1^k \lambda_i^2(P). \quad (11)$$

With the Chebyshev's Inequality [8]

$$\sum_1^k \lambda_i^2(P) \ge [\sum_1^k \lambda_i(P)]^2/k, \quad (12)$$

we can obtain

$$\lambda_n(R)[\sum_1^k \lambda_i(P)]^2/k - 2\sigma_1(A)\sum_1^k \lambda_i(P) - \sum_1^k \lambda_i(Q) \le 0. \quad (13)$$

From (13), (2) follows directly.

Q.E.D.

When $k = 1$ and $k = n$, (2) gives bounds for the maximum eigenvalue and the trace of P. The bound for the maximum eigenvalue is equal to the existing bound and the trace bound is weaker than the existing one [2].

With some well known inequalities, we can derive the following two corollaries that propose upper bounds for the individual eigenvalues and products of the eigenvalues of P.

Corollary 1 *Let P be the solution of the CARE (1) when R is a positive definite matrix. Then the following inequalities are satisfied for* $k = 1, 2, \ldots, n$:

$$\lambda_k(P) \le f(-\sigma_1(A), \lambda_n(R), \sum_1^k \lambda_i(Q)/k). \quad (14)$$

Proof: By applying an inequality $\sum_1^k \lambda_i(P) \ge k\lambda_k(P)$ to (2), (14) follows.

Corollary 2 *Let P be the solution of the CARE (1) when R is a positive definite matrix. Then the following inequalities are satisfied for* $k = 1, 2, \ldots, n$:

$$\prod_1^k \lambda_i(P) \le \left\{ f(-\sigma_1(A), \lambda_n(R), \sum_1^k \lambda_i(Q)/k) \right\}^k. \quad (15)$$

Proof: Application of the arithmetic geometric mean inequality [8] to (2) gives (15).

It is noticed that the above three bounds are valid under the assumption that R is positive definite. The following theorem gives upper bounds for the minimum eigenvalue that are applicable even when R is singular.

Theorem 2 *Let P satisfy the CARE (1). Then the following inequalities are satisfied for* $k = 1, 2, \ldots, n$:

$$\lambda_n(P) \le f(-\sum_1^k \sigma_i(A), \sum_1^k \lambda_i(R), \sum_1^k \lambda_i(Q)). \quad (16)$$

Proof: Rewrite (1) as

$$P^{-1}A' + AP^{-1} + P^{-1}QP^{-1} = R \quad (17)$$

and take summations of singular values to obtain

$$\sum_1^k \sigma_i(P^{-1}A' + AP^{-1} + P^{-1}QP^{-1}) = \sum_1^k \sigma_i(R). \quad (18)$$

Using (5), (6) and the following inequality [5] for $n \times n$ symmetric positive-semidefinite matrices X and Y,

$$\sum_1^k \lambda_i(XY) \leq \sum_1^k \lambda_i(X)\lambda_i(Y), \quad (19)$$

we obtain

$$2\sum_1^k \sigma_i(A)\lambda_i(P^{-1}) + \sum_1^k \lambda_i^2(P^{-1})\lambda_i(Q) \geq \sum_1^k \lambda_i(R). \quad (20)$$

Keeping in mind that $\lambda_i(P^{-1}) = \lambda_{n-i+1}^{-1}(P)$, take bound of (20) as

$$2\sum_1^k \sigma_i(A)/\lambda_n(P) + \sum_1^k \lambda_i(Q)/\lambda_n^2(P) \geq \sum_1^k \lambda_i(R). \quad (21)$$

Then (16) follows immediately from (21).

Q.E.D.

Several upper bounds for the minimum eigenvalue have been reported so far [2] - [4]. However, when R is singular, only the following bound in [4] is valid.

$$\lambda_n(P) \leq f(-tr(A), tr(R), tr(Q)). \quad (22)$$

As the index k runs from 1 to n, our result (16) gives n different bounds. Although it is not possible in general to state which is tighter, bounds for the extremal cases, when $k = 1$ and $k = n$, have some computational advantages. When $k = n$, (16) is equivalent to (22), while the one for $k = 1$ gives

$$\lambda_n(P) \leq f(-\sigma_1(A), \lambda_1(R), \lambda_1(Q)). \quad (23)$$

The following short example shows that (23) gives a better result than (22) in some cases.

Example: Let

$$A = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the minimum eigenvalue of the solution P of the CARE (1) is 3.3354. From (23), we obtain $\lambda_n(P) \leq 5.4813$, whereas (22) yields $\lambda_n(P) \leq 8.1231$.

3. Conclusions

In this paper, some upper bounds for the solution of the continuous algebraic Riccati equation were presented. Among the derived results, upper bounds for summations of eigenvalues, products of eigenvalues, individual eigenvalues are new results. The other results, upper bounds for the minimum eigenvalue, are applicable even when R is singular and supplement the existing bounds. These facts are illustrated by an example.

Research is required to obtain lower bounds for products of eigenvalues of the algebraic Riccati equations which have not been yet derived explicitly.

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