

Computational Solution for the Problem of a Stochastic Optimal Switching Control

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Abstract

In this paper, we consider the problem of a stochastic optimal switching control, which can be applied to the control of a system with uncertain demand such as a control problem of a power plant. The dynamic programming method is applied for the formulation of the optimal control problem. We solve the system of Quasi-Variational Inequalities(QVI) using an algorithm which involves the finite difference approximation and contraction mapping method. A mathematical example of the optimal switching control is constructed. The actual performance of the algorithm is also tested through the solution of the constructed example.

1 Introduction

We consider a problem of optimal switching control, which can be applied for the control of a system with uncertain demand such as a control of a power plant.

Assume a system operating with m number of units. Each unit can be operated on various levels, say "on", "off", or "half". Then the status of units can be represented by the following set.

$$u_i = \{u_i^1, u_i^2, \dots, u_i^m\} \text{ for } i = 1, 2, \dots$$

Let

$$U = \{(u_1, \tau_1), (u_2, \tau_2), \dots\}$$

Now the problem is to choose the set u_i and time τ_i to switch in order to satisfy uncertain demand of the system. We call (u_i, τ_i) a control policy from the system theoretic point of view.

For the problem of a power plant control since the electricity demand is randomly fluctuating, it can be modelled as a stochastic process. Let $y_{i,x}^u(s)$ be a difference between the output and the demand of the system at time s evolved from (t, x) . The Itô stochastic differential equation for $y_{i,x}^u(s)$ takes the following form provided m_i^u, σ_{ij}^u satisfy the so called Itô conditions [7].

$$d(y_{i,x}^u)_i(s) = m_i^u(s, y_{i,x}^u(s))ds + \sum_{j=1}^n \sigma_{ij}^u(s, y_{i,x}^u(s))dw_j(s) \quad (1)$$

$$\text{for } i = 1, 2, \dots, m$$

where

$$y_{i,x}^u(t) = x$$

dw_j : white noise

The cost functional for the optimal switching control takes the following form which consists of the continuous cost and the discrete cost due to a switching from one control to another.

$$J(t, x, u; U) =$$

$$E_{t,x,u} \left\{ \int_t^T f^{u(s)}(s, y_{i,x}^{u(s)}(s)) \exp\left(-\int_t^s c^{u(\sigma)}(\sigma, y_{i,x}^{u(\sigma)}(\sigma))d\sigma\right) ds \right. \\ \left. + \sum_{i=1}^{N_T} S(u_{i-1}, u_i) \exp\left(-\int_{\tau_{i-1}}^{\tau_i} c^{u_{i-1}}(\sigma, y_{i,x}^{u_{i-1}}(\sigma))d\sigma\right) \right\} \quad (2)$$

2 Optimality Condition

Let $v^u(t, x) = \inf_U J(t, x, u; U)$. Now applying the method of dynamic programming[4,7] for this optimal switching control problem yields the following condition. Itô's rule[7,8] is also used in the course of deriving the optimality condition. We call this optimality condition a parabolic Quasi-Variational Inequalities (QVI). The term "Quasi" comes from the fact that $B^u v$ is an implicit obstacle [5].

$$\max \left\{ \frac{\partial v^u(t, x)}{\partial t} + L^u(t, x)v^u(t, x) - f^u(t, x), \right. \\ \left. v^u(t, x) - B^u v(t, x) \right\} = 0 \quad (3) \\ \text{for } (t, x) \in (0, T) \times \Omega \\ v^u(t, x) = 0 \text{ for } (t, x) \in (0, T) \times \partial\Omega \\ v^u(0, x) = \phi(x) \text{ for } x \in \Omega \\ \Omega \subseteq \mathbb{R}^n$$

where

$$L^u(t, x) = -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sigma_{ik}^u(t, x) \sigma_{jk}^u(t, x) \frac{\partial^2}{\partial x_i \partial x_j} \\ - \sum_{i=1}^n m_i^u(t, x) \frac{\partial}{\partial x_i} + c^u(t, x)I$$

$$B^u v(t, x) = \min_{u^* \in U \setminus u} \{S(u, u^*) + v^{u^*}(t, x)\}$$

$c^u(t, x)$: a discount factor

3 Algorithm for the parabolic QVI

The algorithm[2] described in the following is based on the finite difference approximation and the contraction mapping principle[6]. Finite difference discretization of the parabolic QVI(3) in both space and time yields the following discrete QVI.

$$\max \left\{ \frac{v_{i,k}^u - v_{i,k-1}^u}{t_s} + \sum_{j=1}^N D_{ij,k}^u v_{j,k}^u - f_{j,k}^u, v_{i,k}^u - (B^u v)_{i,k} \right\} = 0 \quad (4)$$

where

$$\begin{aligned} k &= 1, 2, \dots, n_k \quad (n_k : \# \text{ of time steps}) \\ i &= 1, 2, \dots, N \quad (N : \# \text{ of grid points in } U) \\ t_s &= \text{time step} \\ D_{ij,k}^u &\stackrel{\text{def}}{=} D_{ij}^u(k t_s) \end{aligned}$$

Matrix $D_{ij,k}^u$ can be obtained from the finite difference discretization of the operator $L^u(t, x)v^u(t, x)$ [9]. The following assumptions [2] are required for the existence and uniqueness of the discrete QVI above.

$$D_{ii}^u(t) > 0, \forall t \in [0, T]$$

$$D_{ij}^u(t) \leq 0 \text{ for } i \neq j$$

$$\sum_{j=1}^N D_{ij}^u(t) > 0 \text{ or } \sum_{j=1}^N D_{ij}^u(t) \forall i \text{ and } t \in [0, T] \quad (5)$$

There is no set of controls u_1, u_2, \dots, u_m such that

$$S(u_1, u_2) = S(u_2, u_3) = \dots = S(u_m, u_1) = 0. \quad (6)$$

The discrete QVI(4) can be modified to a form of a fixed point iteration[2,9].

$$v_{i,k}^u = (Qv)_{i,k}^u : k = 1, 2, \dots, n_k : i = 1, 2, \dots, N \quad (7)$$

where

$$v_{i,0}^u = \phi_i$$

$$(Qv)_{i,k}^u = \min \left\{ \sum_{j=1}^N H_{ij,k}^u v_{j,k}^u + \tilde{H}_{ii,k-1}^u v_{i,k-1}^u + \tilde{f}_{i,k}^u, (B^u v)_{i,k} \right\}$$

$$(B^u v)_{i,k} = \min_{u^* \in U \setminus u} \{S(u, u^*) + v_{i,k}^{u^*}\}$$

$$H_{ij,k}^u = \begin{cases} \frac{-d_{ij,k}^u}{t_s + d_{ii,k}^u}, & i \neq j \\ 0, & i = j \end{cases}$$

$$\tilde{H}_{ii,k}^u = \begin{cases} \frac{t_s}{t_s + d_{ii,k}^u}, & i = j \\ 0, & i \neq j \end{cases}$$

$$\tilde{f}_{i,k}^u = \frac{f_{i,k}^u}{t_s + d_{ii,k}^u}$$

This iterative scheme achieves stability and convergence under any arbitrary space and time discretization. This follows from the properties of contraction mapping principle and discretization scheme employed, and mathematical proofs regarding the speed of convergence etc. can be found in [2].

3.1 Procedure for the iterative scheme

1. Set the initial conditions $(v_{i,0})$, and the initial guess $\{(v_{i,k}^u)^{(0)}\}$.
2. Specify the time step t_s .
3. For $\tilde{n} = 1, 2, \dots$
4. For each $u \in U$
5. For $k = 1, 2, \dots, n_k$
- 5.1 Discretize $L^u v$ to obtain $D_{ij,k}^u$.
- 5.2 Compute $H_{ij,k}^u, \tilde{H}_{ij,k}^u$ and $\tilde{f}_{i,k}^u$.
6. For $i = 1, 2, \dots, N$
- 6.1 Compute $(B^u v)_{i,k}$
- 6.2 Compute $\text{Temp}^u \stackrel{\text{def}}{=} \sum_{j=1}^N H_{ij,k}^u v_{j,k}^u + \tilde{H}_{ii,k-1}^u v_{i,k-1}^u + \tilde{f}_{i,k}^u$
7. Determine $M\text{temp} \stackrel{\text{def}}{=} \min_u \{\text{Temp}^u, (B^u v)_{i,k}\}$
8. Store the corresponding u as the optimal control.
9. Store the solution : $(v_{i,k}^u)^{(\tilde{n})} = M\text{temp}$
10. Compute $\text{Error}_i^u(i) = |(v_{i,k}^u)^{(\tilde{n})} - (v_{i,k-1}^u)^{(\tilde{n}-1)}|$
11. Next i
12. Compute : $\text{Error}_k^u(k) = \max_i \{\text{Error}_i^u(i)\}$
13. Next k
14. Compute : $\text{Error}^u = \max_k \{\text{Error}_k^u(k)\}$
15. If $\text{Error}^u < \text{Tolerance}$, stop the iteration.
16. Next u
17. Next \tilde{n}

4 Mathematical Example of an Optimal Switching Control

As an example of a switching control we consider a system consists of two units, and each unit is assumed to be operated at one of two possible levels, say "on" and "off". Then U takes

$$U = \{u_1, u_2, u_3, u_4\}$$

The control problem is to determine the optimal control u_i and the time τ_i to switch control in order to satisfy the demand over the finite time horizon. For simplicity we consider the case of only two controls, i.e.

$$U = \{u_1, u_2\} \stackrel{\text{def}}{=} \{1, 2\}$$

Let the domain Ω be a rectangle such that

$$\Omega = \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < 1\}$$

Then in Ω , there is a part where $u = 1$ is the only optimal control. Let this part of Ω be G_1^1 . Likewise G_2^2 is a part of Ω where $u = 2$ is the optimal control, and G_1^{12} is for both $u = 1$ and $u = 2$ are optimal. The domain G_2^{12} can be written as $G_1^{12} = G_1^1 \cap G_2^2$. Then we can define the part of domain where $u = 1$ or $u = 2$ is the only optimal control in the following way.

$$\tilde{G}_1^1 = G_1^1 \setminus G_1^{12}, \quad \tilde{G}_2^2 = G_2^2 \setminus G_1^{12}$$

For the case of two controls the QVI takes the following form.

$$\max \left\{ \frac{\partial v^1}{\partial t} + L^1 v^1 - f^1, v^1 - B^1 v \right\} = 0$$

$$\max\left\{\frac{\partial v^2}{\partial t} + L^2 v^2 - f^2, v^2 - B^2 v\right\} = 0 \quad (8)$$

where

$$B^1 v = S(1, 2) + v^2, \quad B^2 v = S(2, 1) + v^1$$

In view of the QVI(8), the following conditions should hold for the corresponding part of the domain.

In the part \tilde{G}_i^1 :

$$\begin{aligned} \frac{\partial v^1}{\partial t} + L^1 v^1 - f^1 &= 0, \quad v^1 < S(1, 2) + v^2 \\ \frac{\partial v^2}{\partial t} + L^2 v^2 - f^2 &< 0, \quad v^2 = S(2, 1) + v^1 \end{aligned} \quad (9)$$

In the part \tilde{G}_i^2 :

$$\begin{aligned} \frac{\partial v^1}{\partial t} + L^1 v^1 - f^1 &< 0, \quad v^1 = S(1, 2) + v^2 \\ \frac{\partial v^2}{\partial t} + L^2 v^2 - f^2 &= 0, \quad v^2 < S(2, 1) + v^1 \end{aligned} \quad (10)$$

In the part G_i^{12} :

$$\begin{aligned} \frac{\partial v^1}{\partial t} + L^1 v^1 - f^1 &= 0, \quad v^1 \leq S(1, 2) + v^2 \\ \frac{\partial v^2}{\partial t} + L^2 v^2 - f^2 &= 0, \quad v^2 \leq S(2, 1) + v^1 \end{aligned} \quad (11)$$

On $\partial\Omega$, $v^1 = v^2 = 0$.

Thus if

$$S(1, 2) > 0, \quad S(2, 1) > 0$$

then

$$v^1 < S(1, 2) + v^2 \text{ and } v^2 < S(2, 1) + v^1.$$

Therefore, on $\partial\Omega$

$$\begin{aligned} \frac{\partial v^1}{\partial t} + L^1 v^1 - f^1 &= 0 \\ \frac{\partial v^2}{\partial t} + L^2 v^2 - f^2 &= 0 \end{aligned} \quad (12)$$

This implies both $u = 1$ and $u = 2$ are optimal on the boundary of the domain. Since this is the case for part G_i^{12} , the boundary has to be a subset of G_i^{12} . Therefore the domain has to take the form in the figure 1 below.

Based on the domain in Figure 1 an example can be constructed in the following way. We define v^1 on the part \tilde{G}_i^1 . Then v^2 , f^1 and f^2 can be determined by using the QVI(9).

Namely,

$$\begin{aligned} v^2 &= S(2, 1) + v^1 \\ f^1 &= \frac{\partial v^1}{\partial t} + L^1 v^1 \\ f^2 &> \frac{\partial v^2}{\partial t} + L^2 v^2 \end{aligned} \quad (13)$$

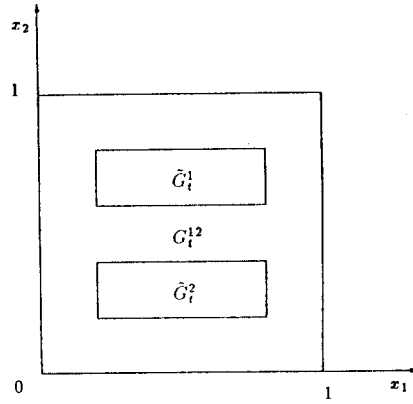


Figure 1 : Partition of Domain (Ω)

Similarly on \tilde{G}_i^2 , we define v^2 .

Then

$$\begin{aligned} v^1 &= S(1, 2) + v^2 \\ f^1 &> \frac{\partial v^1}{\partial t} + L^1 v^1 \\ f^2 &= \frac{\partial v^2}{\partial t} + L^2 v^2 \end{aligned} \quad (14)$$

The functions (v^1, v^2, f^1, f^2) on G_i^{12} can also be determined similarly[9]. Now we introduce a function $\tilde{v}(t, x_1, x_2)$ which will be used for determining the function above.

$$\tilde{v}(t, x_1, x_2) = \exp(\gamma t)\phi(x_1, x_2) + \lambda \sin(t)\xi(x_1, x_2) \quad (15)$$

where

$$\phi(x_1, x_2) = x_1(x_1 - 1)x_2(x_2 - 1)$$

$$\xi(x_1, x_2) = \{x_1(x_1 - 1)\}^\alpha \{x_2(x_2 - 1)\}^\beta$$

In order to solve the QVIs above numerically we still have to know the generator $L^u v^u$ which can be defined as in the below.

Type 1 :

$$\begin{aligned} L^u v^u(t, x_1, x_2) = \\ - (a_{11}^u + \epsilon) \frac{\partial^2 v^u}{\partial x_1^2} - 2a_{12}^u \frac{\partial^2 v^u}{\partial x_1 \partial x_2} - (a_{22}^u + \epsilon) \frac{\partial^2 v^u}{\partial x_2^2} + cv^u \end{aligned} \quad (16)$$

for $u = 1, 2$

Coefficients a_{ij}^u can be chosen such that the matrix $[a_{ij}^u]$ is positive definite. One such candidate of coefficients is:

$$\begin{aligned} a_{11}^1 &= 1, \quad a_{12}^1 = \frac{1}{4}, \quad a_{22}^1 = 1 \\ a_{11}^2 &= \frac{1}{2}, \quad a_{12}^2 = \frac{1}{20}, \quad a_{22}^2 = 1 \end{aligned}$$

Type II :

$$L^u v^u(t, x_1, x_2) = - (a_{11}^u + \epsilon) \frac{\partial^2 v^u}{\partial x_1^2} - (a_{22}^u + \epsilon) \frac{\partial^2 v^u}{\partial x_2^2} + b_1^u \frac{\partial v^u}{\partial x_1} + b_2^u \frac{\partial v^u}{\partial x_2} + c v^u \quad (17)$$

where

$$\begin{aligned} a_{11}^u &= \frac{1}{2}(\sigma_{11}^u)^2, \quad a_{22}^u = \frac{1}{2}(\sigma_{22}^u)^2, \\ b_1^u &= -x_2, \quad b_2^u = -\{(2l^u - 1)x_1 - \eta_0^u x_2\} \\ \sigma_{11}^u &= 0, \quad \sigma_{22}^u = -\delta_0^u(x_1 + \eta_0^u x_2) \\ &\text{for } u = 1, 2 \end{aligned}$$

5 Computational Results and Discussions

The constructed example of an optimal switching control problem in previous section is solved numerically using the algorithm(7), and the corresponding exact solution is also computed for comparisons. In order to evaluate the performance of the algorithm and also to check the computational results, two types of errors are introduced[9].

$$\text{Absolute error} : \max_{i,t} |v_{i,t}^{\hat{n}} - v_{i,t}|$$

$$\text{Relative error} : \max_{i,t} |v_{i,t}^{\hat{n}} - v_{i,t}^{\hat{n}-1}|$$

$v_{i,t}^{\hat{n}}$ and $v_{i,t}$ represent numerical solution at \hat{n}_{th} iteration obtained from the algorithm(7) and the exact solution(15) respectively.

Table 1 shows that the steady state error(absolute error) decreases as the number of grid points increases. This is expected in the sense that the approximate solution converges to the exact solution of the Taylor series approximation as the mesh size approaches to zero. It is also observed that the solution is sensitive to the magnitude of the switching costs, i.e. error is approximately the same order of the switching costs. On the map of optimal controls(Figure 2), there are four different sets involved at every time t , which are not necessarily disjoint. Number "1" in the map represents the part starting with control "1" and staying with it which is optimal. Number "12" is the case of switching from control "1" to control "2". Similar interpretations can also be deduced for the other cases. Computational results are obtained for both types(16,17) of the operator $L^u v^u$, and the exactly same phenomena have been observed for both cases. In view of the discussion above and by comparing the approximate map(Figure 2) with the a prior specified map(Figure 1), it can also be concluded that the approximate map(Figure 2) obtained is acceptable.

6 Conclusions and Recommendations

The problem of a stochastic optimal switching control is studied by constructing a simple mathematical example. The constructed example is solved using the algorithm[2] based on the finite difference discretization and contraction mapping method. Performance of the algorithm is also tested by comparing the numerical solution with the exact one. It turns out that the algorithm yields a satisfactory solution in view

of the absolute error and the map of optimal controls. Since the constructed example works properly, the methodology involved can also be extended for the system with more controls. Through the computer implementation of this kind of mathematical example, useful properties on the algorithm itself and the switching control system can be obtained.

Table 1 : Computational Results (Type 1)

Switching Costs	# of Grid	Steady State Error (absolute)	
		u = 1	u = 2
S(1, 2) = 0.001	6 x 6	0.1705E - 01	0.1680E - 01
S(2, 1) = 0.005	8 x 8	0.6356E - 02	0.5734E - 02
	10 x 10	0.3022E - 02	0.2903E - 02
S(1, 2) = 0.0001	6 x 6	0.1606E - 02	0.1580E - 02
	8 x 8	0.5749E - 03	0.5126E - 03
	10 x 10	0.3316E - 03	0.3316E - 03

Note; For 6 x 6; $t_s = 0.02$, $n_k = 10$

For 8 x 8; $t_s = 0.01$, $n_k = 20$

For 10 x 10; $t_s = 0.008$, $n_k = 25$

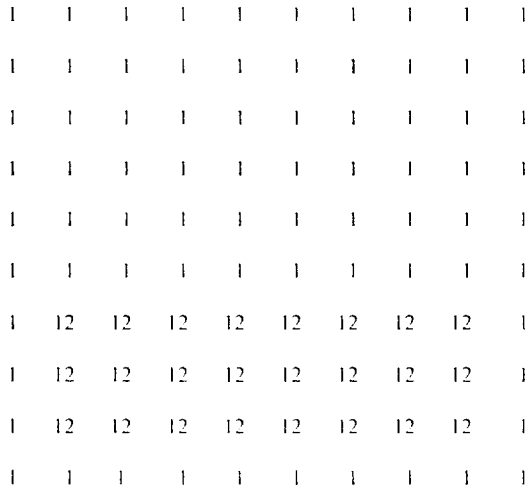
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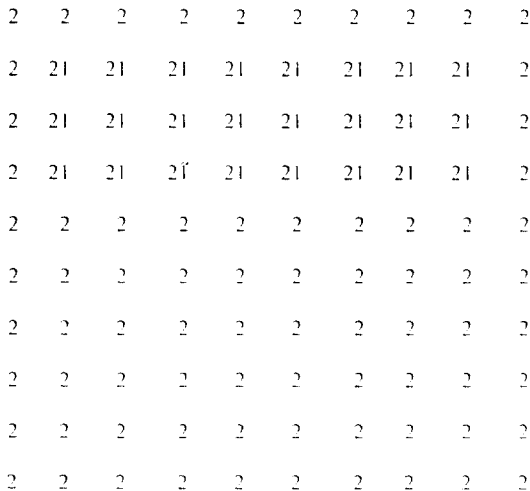
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<Map evolved from u = 1>



<Map evolved from u = 2>

Figure 2 : Map of Optimal Controls (10 x 10)