# A Regularity Condition for Asymptotic Tracking in Discrete-Time Nonlinear Systems

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**Abstract**

A well-defined relative degree, which is one of the basic assumptions in adaptive control or nonlinear synthesis problems, is addressed. It is shown that this is essentially a necessary condition for asymptotic tracking in discrete-time nonlinear systems. To show this, tracking problems are defined, and a local linear input-output behavior of a discrete-time system is introduced in relation to a well-defined relative degree. It is then shown that if a plant is invertible and accessible from the origin and a compensator solves the local asymptotic tracking problem, then the plant necessarily has a well-defined relative degree at the origin.

## 1 Introduction

A well-defined relative degree is a starting point of most adaptive control or synthesis problems in nonlinear systems [10]. Grizzle et al. [6] have recently shown that a well-defined relative degree and the minimum phase property are necessary conditions for asymptotic tracking with internal stability in continuous-time nonlinear systems. In this work, we study relative degrees and tracking problems to show that a well-defined relative degree is also a necessary condition for asymptotic tracking as in the case of discrete-time nonlinear systems. In discrete-time, due to the status of the currently available results on accessibility, we will need a few more assumptions that were used in [11]; in particular, the invertibility of the transition maps of both the plant and the closed-loop system will be required [11].

The outline of this paper is as follows. We start with the definition of a relative degree in Section 2. The local asymptotic tracking and local exact tracking problems are defined and a relation between them is established in Section 3. In Section 4, the linear input-output behavior of a discrete-time system is defined and a sufficient condition for when a discrete-time nonlinear system has a well-defined relative degree is obtained. It is also shown that if a compensator solves the local exact tracking problems, then the closed-loop system has a local linear input-output behavior. Finally, in Section 5, we show that a well-defined relative degree at the origin is a necessary condition for asymptotic tracking in discrete-time systems. More precisely, it is proven that, for a single-input single-output system, if the plant is invertible and accessible from the origin and a compensator solves the local asymptotic tracking problem, then the plant necessarily has a well-defined relative degree at the origin.

## 2 Relative Degree of a System

The notion of relative degree plays a key role in many synthesis problems for nonlinear systems. The relative degree of a discrete-time system is defined as follows. Consider a discrete-time system

$$
\begin{aligned}
x(k+1) &= f(x(k),u(k)), \quad x \in X \subset \mathbf{R}^n, u \in U \subset \mathbf{R}^m, \\
y(k) &= h(x(k),u(k)), \quad y \in \mathbf{R}^m,
\end{aligned} \tag{2.1}
$$

Define $\hat{h}\circ\hat{f}^0(x,u) = h(x,u)$, and by induction $\hat{h}\circ\hat{f}^{i+1}(x,u) = \hat{h}\circ\hat{f}^i(f(x,u),0), i \geq 0$. Let $(\hat{h}\circ\hat{f}^i)_j$ denote the $j$-th component of $\hat{h}\circ\hat{f}^i$.

**Definition 2.1** *[15] (A) A discrete-time SISO system is said to have a* relative degree *equal to zero if there exists a point* $x^0 \in X$ *such that* $\frac{\partial}{\partial u}(\hat{h}\circ\hat{f}^0(x,u))_{x=x^0} \neq 0$, *and to have a* relative degree $r$ *if there exists a point* $x^0 \in X$ *such that*

1. $\frac{\partial}{\partial u}(\hat{h}\circ\hat{f}^k(x,u))_{x=x^0} \equiv 0$, *for all* $0 \leq k \leq r-1$;
2. $\frac{\partial}{\partial u}(\hat{h}\circ\hat{f}^r(x,u))_{x=x^0} \neq 0$, *for some* $u \in U$.

*(B) A multivariable discrete-time system is said to have a* vector relative degree $\{r_1,\cdots,r_m\}$ *if there exist a set of integers* $\{r_1,\cdots,r_m\}$ *and a point* $x^0 \in X$ *such that*

1. $\frac{\partial}{\partial u}[(\hat{h}\circ\hat{f}^k(x,u))_i]_{x=x^0} \equiv 0$, *for all* $1 \leq i \leq m$, *for all* $0 \leq k \leq r_i - 1$;
2. *the decoupling matrix*

$$
A(x,u) = \begin{bmatrix} \frac{\partial}{\partial u}[(\hat{h}\circ\hat{f}^{r_1}(x,u))_1] \\ \cdots \\ \frac{\partial}{\partial u}[(\hat{h}\circ\hat{f}^{r_m}(x,u))_m] \end{bmatrix}
$$

*is nonsingular at $x = x^0$ for some $u \in U$.*

*A (vector) relative degree is said to be* well-defined *at the points $(x^0)$ where the above holds.*

The following will help tie together the notions of a relative degree, a well defined relative degree at a point, and the relative degree of a system's linearization. Consider a SISO nonlinear system of the form (2.1), and suppose that the origin is an equilibrium point of the system. Let $r_N$ denote the (possibly not well-defined) relative degree of the nonlinear system. Let the Jacobian linearization of the system around the origin be given by

$$\begin{aligned} z(k+1) &= Az(k) + bu(k), \\ \eta(k) &= cz(k) + du(k), \end{aligned}$$

where

$$A = (\frac{\partial f}{\partial x})_{(0,0)}, \quad b = (\frac{\partial f}{\partial u})_{(0,0)}, \quad c = (\frac{\partial h}{\partial x})_{(0,0)}, \quad d = (\frac{\partial h}{\partial u})_{(0,0)}.$$

Let $r_L$ denote the relative degree of its Jacobian linearization, respectively. Then the following is an immediate consequence of the chain rule.

**Lemma 2.2** *With the above notations, it is true that $r_N \leq r_L$. Furthermore, a SISO nonlinear system has a well-defined relative degree at zero if and only if $r_N = r_L$.*

**Proof:** If we use the above definition of $\hat{h} \circ \hat{f}^k$, then it follows from the chain rule that

$$\begin{aligned} \frac{\partial}{\partial u}(\hat{h} \circ \hat{f}^0)_{(0,0)} &= d, \\ \frac{\partial}{\partial u}(\hat{h} \circ \hat{f}^k)_{(0,0)} &= cA^{k-1}b, \quad k \geq 1. \end{aligned}$$

Thus it follows that $r_N \leq r_L$ because it may hold that $\frac{\partial}{\partial u}(\hat{h} \circ \hat{f}^k) \neq 0$ for some $x_0 \neq 0$, for some $k < r - 1$. From this, it is obvious that $r_N = r_L$ is equivalent to the SISO nonlinear system having a well-defined relative degree at zero. □

Now we define a relative degree of a compensator in the case of a SISO system. Consider a SISO plant $P$ of the form (2.1) and a compensator $C$

$$C: \begin{aligned} \xi(k+1) &= c(x(k), \xi(k), v(k)), \\ u(k) &= d(x(k), \xi(k), v(k)). \end{aligned}$$

If we define $\eta := \begin{pmatrix} x \\ \cdot\cdot \\ \xi \end{pmatrix}$, then the closed-loop system is given by

$$\begin{aligned} \eta(k+1) &= F(\eta(k), v(k)), \\ u(k) &= D(\eta(k), v(k)), \\ y(k) &= H(\eta(k), v(k)), \end{aligned}$$

where

$$\begin{aligned} F(\eta, v) &= \begin{bmatrix} f(x, d(x, \xi, v)) \\ c(x, \xi, v) \end{bmatrix}, \\ D(\eta, v) &= d(x, \xi, v) \\ H(\eta, v) &= h(x, d(x, \xi, v)). \end{aligned}$$

The relative degree of the compensator $C$ is defined on the basis of the closed-loop system $P \circ C$, viewing $v$ as the input and $u$ as the output. That is, define $\hat{D} \circ \hat{F}^0(\eta, v) = D(\eta, v)$, and $\hat{D} \circ \hat{F}^{i+1}(\eta, v) = \hat{D} \circ \hat{F}^i(F(\eta, v), 0), i \geq 0$.

**Definition 2.3** *A compensator $C$ is said to have a* well-defined relative degree *equal to zero at $\eta^0$ if $\frac{\partial}{\partial v}(\hat{D} \circ \hat{F}^0(\eta, v))_{\eta=\eta^0} \neq 0$, and to have a* well-defined relative degree *$r$ at $\eta^0$ if*

1. *$\frac{\partial}{\partial v}(\hat{D} \circ \hat{F}^k(\eta, v))_{\eta=\eta^0} \equiv 0$, for all $0 \leq k \leq r-1$,*
2. *$\frac{\partial}{\partial v}(\hat{D} \circ \hat{F}^r(\eta, v))_{\eta,\eta^0} \neq 0$.*

With this definition, the following lemma is a straightforward result.

**Lemma 2.4** *If the closed-loop system $P \circ C$ has a well-defined relative degree at zero, so do also the plant $P$ and the compensator $C$.*

**Proof:** Let $y^{P \circ C}$ denote the output of the closed-loop system $P \circ C$, and let $n_{P \circ C}$ denote the relative degree of the closed-loop system $P \circ C$ so that

$$\frac{\partial}{\partial v(0)}[y(n_{P \circ C})]_{\eta=0} \neq 0.$$

Then by the chain rule, there must exist integers $n_P \leq n_{P \circ C}$ and $n_C \leq n_{P \circ C}$ such that

1. $\frac{\partial}{\partial u(0)}[h(x(n_P), u(n_P))]_{x=0} \neq 0$ and $\frac{\partial}{\partial v(0)}[d(x(n_C), \xi(n_C), v(n_C))] \neq 0$;
2. $n_{P \circ C} = n_P + n_C$

That is, $n_P$ and $n_C$ are the relative degrees of the plant $P$ and the compensator $C$, respectively. The same holds for the Jacobian linearizations; that is, $n^L_{P \circ C} = n^L_P + n^L_C$. By the hypothesis that $P \circ C$ has a well-defined relative degree at zero, and Lemma 2.2, $n_{P \circ C} = n^L_{P \circ C}$, so that, $n_P + n_C = n^L_P + n^L_C$. Since $n_P \leq n^L_P$ and $n_C \leq n^L_C$, it follows that $n_P = n^L_P$ and $n_C = n^L_C$. □

## 3 Tracking Problems

In this Section, the local asymptotic and local exact tracking problems are defined. In a similar way to that of [6], we prove that if any compensator solves the local asymptotic tracking problem, then it also resolves the local exact tracking problem. This is a key result to the development of later sections.

Consider a plant $P$ of the form (2.1) and suppose that $f(0,0) = 0, h(0,0) = 0$, and the entries of $f$ and $h$ are analytic functions. Let the class of desired output trajectories be given by

$$\sup_{k \geq 0} ||y_d(k)|| < \epsilon, \tag{3.2}$$

for a certain $\epsilon > 0$ and let $Y_d(k) = [y_d(k)^T, y_d((k+1)^T, \cdots, y_d(k+N-1)]^T]^T$, where $^T$ denotes transpose.

The compensator $Q$ used to control the plant $P$ is assumed to be a dynamic state-feedback compensator, with inputs $x$ and $Y_d$ and output $u$, of the following general form:

$$Q: \begin{aligned} z(k+1) &= a(x(k), z(k), Y_d(k)), \\ u(k) &= \alpha(x(k), z(k), Y_d(k)), \end{aligned} \tag{3.3}$$

where the state $z$ belongs to some simply connected open subset $Z$ of $\mathbb{R}^q$, the two functions $a$ and $\alpha$ are assumed to be analytic on $X \times Z \times \mathbb{R}^{Nm}$. In the sequel, the solution of (2.1) corresponding to the input u and initial state $x(0) = x_0$ will be denoted by $x(k, x_0, u)$; the corresponding output will be denoted by $y(k, x_0, u)$.

**Definition 3.1** *Consider a plant P of the form (2.1).*

*(A) The* local asymptotic tracking problem *is to find a compensator Q of the form (3.3), with $a(0,0,0) = 0$, $\alpha(0,0,0) = 0$, and $\epsilon > 0$ such that there exists an open neighborhood $\mathcal{O}_1$ of the origin in $X \times Z$ such that, for all $y_d(k)$ satisfying (3.2),*

1. $\lim_{k\to\infty} y^{PoQ}(k, x_0, z_0, Y_d(k)) - y_d(k) = 0$, *for all* $(x_0, z_0) \in \mathcal{O}_1$;

2. *the equilibrium (0,0) of the unforced system*

$$\begin{aligned} x(k+1) &= f(x(k), \alpha(x(k), z(k), 0)), \\ z(k+1) &= a(x(k), z(k), 0) \end{aligned} \tag{3.4}$$

*is asymptotically stable.*

*(B) The* local exact tracking problem *from the origin is to find a compensator Q of the form (3.3), with $a(0,0,0) = 0$, $\alpha(0,0,0) = 0$, and $\epsilon > 0$ such that for all $y_d(k)$ in the class (3.2) satisfying $Y_d(0) = 0$,*

$$y^{PoQ}(k, 0, 0, Y_d(k)) - y_d(k) = 0, \ \ \forall k \geq 0.$$

Now we establish a relation between the local asymptotic tracking and local exact tracking problems.

**Proposition 3.2** *Consider a plant of the form (2.1). Suppose that a compensator Q of the form (3.3) solves the local asymptotic tracking problem. Then the same compensator Q solves the local exact tracking problem from the origin.*

**Proof:** The proof is similar to that of Theorem 3.1 in [6]; it is omitted for brevity.

## 4 Local Linear Input-Output Behavior of a Discrete-Time Nonlinear System

In this Section a local linear input-output behavior of a system is introduced. Based on the definitions, a necessary and sufficient condition for linear input-output behavior is obtained.

**Definition 4.1** *(A) A discrete-time system is said to exhibit* a local linear input-output behavior *in a neighborhood of the origin if there exist matrices (A, B, C, D) of appropriate dimensions such that, for every $T < \infty$, there exist $\epsilon > 0, \eta > 0, w(k, x_0)$ such that*

$$\begin{aligned} & y(k, x(0), u(0), \cdots, u(k)) \\ = \ & w(k, x_0) + \sum_{j=0}^{k-1} CA^{k-1-j}Bu(j) + Du(k), \ \textit{for all}\ 0 \leq k \leq T, \end{aligned}$$

*for all $x(0)$ such that $||x(0)|| < \epsilon$, and for all controls such that $||u(k)|| < \eta, 0 \leq k \leq T$.*

*(B) A discrete-time system is said to exhibit* a local linear input-output behavior from the origin *if (A) holds for $x(0) = 0$ with $w(k, 0) = 0$.*

We seek now to obtain a necessary and sufficient condition for a system to have a local linear input-output behavior. First, consider an analytic system of the form (2.1). Define $h_u(x) := h(x, u), f_u(x) := f(x, u)$, and let $\hat{h} \circ \hat{f}^i(x, u)$ be defined as in Section 5.2. Let $B_\epsilon(0)$ denote the open ball centered at the origin with radius $\epsilon$.

**Lemma 4.2** *Consider an analytic system of the form (2.1) with $f(0,0) = 0$ and $h(0,0) = 0$. The system exhibits a local linear input-output behavior in a neighborhood of the origin if and only if there exist matrices (A, B, C, D) of appropriate dimensions such that, for every $T < \infty$, there exist $\epsilon_1 > 0, \eta_1 > 0$ such that*

$$\frac{\partial \hat{h} \circ \hat{f}^k(x,u)}{\partial u} = \begin{cases} D, & k = 0, \\ CA^{k-1}B, & 1 \leq k \leq T, \end{cases}$$

*for all $x \in B_{\epsilon_1}(0)$ and for all controls $u \in B_{\eta_1}(0)$*

**Proof:**
Necessity: Since $\hat{h} \circ \hat{f}^0(x, u) = h(x, u) = y(0, x, u(0))|_{u(0)=u}$,

$$\frac{\partial \hat{h} \circ \hat{f}^0(x,u)}{\partial u} = D, \ \ \forall x \in B_{\epsilon_1}(0), \ \forall u \in B_{\eta_1}(0).$$

Also, it holds that

$$\hat{h} \circ \hat{f}^k(x, u) = y(k, x, u(0), u(1), \cdots, u(k))|_{u(0)=u, u(1)=\cdots=u(k)=0}.$$

Therefore, the necessity holds.

Sufficiency: Since $\frac{\partial h(x,u)}{\partial u} = D$, there exists an analytic function $\phi^1(x)$ such that

$$h(x, u) = \phi^1(x) + Du, \ \ \forall x \in B_{\epsilon_1}(0), \ \forall u \in B_{\eta_1}(0).$$

Since $y(k, x(k), u(k))|_{u(k)=0} = \phi^1(x(k)) = h(x(k), 0)$, it follows that

$$\begin{aligned} & y(k, x(k), u(k)) \\ = \ & h_0(x(k)) + Du(k), \ \forall x(k) \in B_{\epsilon_1}(0), \ \forall u(k) \in B_{\eta_1}(0). \end{aligned}$$

Thus, it holds that, by the continuity of $f$, there exist $0 < \epsilon_2 \leq \epsilon_1$ and $0 < \eta_2 \leq \eta_1$ such that

$$\begin{gathered} y(k, x(k), u(k)) = h_0(f(x(k-1), u(k-1))) + Du(k), \\ \forall x(k-1) \in B_{\epsilon_2}(0), \ \forall u(k-1), u(k) \in B_{\eta_2}(0). \end{gathered}$$

In a similar way, since $\frac{\partial h(f(x,u),0)}{\partial u} = CB$, there exists an analytic function $\phi^2(x)$ such that

$$h_0(f(x, u)) = \phi^2(x) + CBu, \ \ \forall x \in B_{\epsilon_1}(0), \ \forall u \in B_{\eta_1}(0).$$

Since $y(k, x(k-1), u(k-1), u(k))|_{u(k-1)=u(k)=0} = \phi^2(x(k-1)) = h(f(x(k-1), 0), 0)$, it follows that

$$\begin{aligned} & y(k, x(k-1), u(k-1), u(k)) \\ = \ & h_0 \circ f_0(x(k-1)) + CBu(k-1) + Du(k), \\ & \forall x(k-1) \in B_{\epsilon_2}(0), \ \forall u(k-1), u(k) \in B_{\eta_2}(0). \end{aligned}$$

Continuing in this way, we can show that there exist $\epsilon_k > 0$ and $\eta_k > 0$ such that

$$\begin{aligned} & y(k, x(0), u(0), u(1), \cdots, u(k)) \\ = \ & h_0 \circ f_0^k(x(0)) + \sum_{j=0}^{k-1} CA^{k-1-j}Bu(j) + Du(k), \\ & \forall x(0) \in B_{\epsilon_k}(0), \ \forall u(l) \in B_{\eta_k}(0), 0 \leq l \leq T. \square \end{aligned}$$

**Corollary 4.3** *If a SISO system of the form (2.1) exhibits a local linear input-output behavior in a neighborhood of the origin and the corresponding linear system $(A, B, C, D)$ has a relative degree, then the system has a well-defined relative degree at zero.*

The following lemma establishes the connection between the local exact tracking and the local linear input-output behavior.

**Lemma 4.4** *Consider a SISO system $P$ of the form (2.1) and suppose that a compensator $Q$ of the form (3.3) solves the local exact tracking problem from the origin. Then with the following compensator $\bar{Q}$:*

$$\begin{aligned}
\xi_1(k+1) &= \xi_2(k), \\
\xi_2(k+1) &= \xi_3(k), \\
&\vdots \\
\xi_N(k+1) &= \xi_1(k+1) + v(k), \\
z(k+1) &= a(x(k), z(k), \xi_1(k), \xi_2(k), \cdots, \xi_N(k)), \\
u(k) &= \alpha(x(k), z(k), \xi_1(k), \xi_2(k), \cdots, \xi_N(k)),
\end{aligned}$$

*the closed-loop system $P \circ \bar{Q}$ exhibits a local linear input-output behavior from the origin.*

**Proof:** Note that if we let $v(k) = y_d(k+N) - y_d(k), k \geq 0$ then the closed-loop system $P \circ \bar{Q}$ is identical to $P \circ Q$. Since the compensator $\bar{Q}$ also solves the local exact tracking problem, there exists an $\epsilon > 0$ such that $y^{P \circ \bar{Q}}(k, 0, 0, Y_d(k)) = y_d(k), k \geq 0$, whenever $P \circ \bar{Q}$ is initialized at 0, and $\|v(k)\| < \epsilon, k \geq 0$. This implies that $y^{P\circ\bar{Q}}(k) = v(k-N) + v(k-2N) + v(k-3N) + \cdots, \forall k \geq 0$, with $v(l) = 0$ for $l < 0$. This closed-loop system is identical to the $\xi$-subsystem of the compensator $\bar{Q}$, which is a linear system. In other words, the closed-loop system $P \circ \bar{Q}$ exhibits the local linear input-output behavior from the origin. □

## 5 A Regularity Condition for Tracking

In this Section, it is shown that if a plant is invertible and accessible from zero and the local exact tracking problem from the origin is solvable while keeping the closed-loop system invertible, then the plant necessarily has a well-defined relative degree at zero. The invertibility is just a technical assumption due to the status of the currently available results on controllability [11]. We have seen that a compensator solving the local exact tracking problem from the origin can be easily modified to yield a closed-loop system having a local linear input-output behavior from the origin. In the next step, we show that whenever the plant possesses the accessibility property, it is always possible to construct a third compensator yielding a linear input-output behavior in an open neighborhood of the origin. This result will imply that the plant necessarily has a well-defined relative degree at zero. To go further, we need some concepts on controllability of discrete-time nonlinear systems.

With the concepts on accessibility, we can show the following result on the linear input-output behavior.

**Lemma 5.1** *Consider a system of the form (2.1) with the origin as an equilibrium point. Suppose that the system is accessible from the origin with an arbitrarily small open ball as a control set $U$. Then the system has a local linear input-output behavior in an open neighborhood of the origin if and only if it has a local linear input-output behavior from the origin.*

**Proof:** We have only to prove sufficiency. Let $O$ be a simply connected nonempty open subset of $A^+(x)$. Then for each point $p \in O$, there exist $\bar{k}$ and a control $u^p$ such that $x(\bar{k}-1, 0, u^p) = p$ and $\max_{0 \leq k \leq \bar{k}-1} \|u^p(k)\| < \epsilon_1$, for arbitrarily small $\epsilon_1 > 0$. By the definition 4.1, for $(T+\bar{k})$, there exist $\epsilon > 0$ and $(A, B, C, D)$ such that

$$y(k, 0, u) = \sum_{j=0}^{k-1} CA^{k-1-j}Bu(j) + Du(k), \quad 0 \leq k \leq T + \bar{k},$$

for all $u$ such that $\|u(k)\| < \epsilon$. Define $\tilde{u}(k)$ by

$$\tilde{u}(k) = \begin{cases} u^p(k+\bar{k}), & -\bar{k} \leq k \leq -1, \\ u(k), & 0 \leq k \leq T, \end{cases}$$

where $\|u(k)\| < \epsilon, \quad 0 \leq k \leq T$. Since the system is time invariant,

$$\begin{aligned}
& y(k, 0, \tilde{u}) \\
= & \sum_{j=-\bar{k}}^{k-1} CA^{k-1-j}B\tilde{u}(j) + D\tilde{u}(k), \\
= & \sum_{j=-\bar{k}}^{-1} CA^{k-1-j}Bu^p(j+\bar{k}) + \sum_{j=0}^{k-1} CA^{k-1-j}Bu(j) + Du(k),
\end{aligned}$$

Define $w_0(k,p) := \sum_{j=-\bar{k}}^{-1} CA^{k-1-j}Bu^p(j+\bar{k})$. Then by the uniqueness of the solution of the system (2.1)

$$y(k, p, u) = y(k, 0, \tilde{u}) = w_0(k, p) + \sum_{j=0}^{k-1} CA^{k-1-j}Bu(j) + Du(k),$$

for all $p \in O$ and for all $u(k) \in B_\epsilon(0)$, $0 \leq k \leq T$. In particular, it is true on an open neighborhood of $p = 0$. □

**Corollary 5.2** *If a system is accessible from the origin with an arbitrarily small open ball as a control set and has a local linear input-output behavior from the origin, and the corresponding matrices $(A, B, C, D)$ have a (finite) relative degree, then the system has a well-defined relative degree at the origin.*

We recall some definitions from differential geometry. Other basic definitions are referred to the texts [10] or [15].

**Definition 5.3 ([10])** *A distribution $\Delta$ on a manifold $M$ is a choice of subspace $\Delta(x)$ of $T_xM$, tangent space of $M$ at a point $x$, for each $x \in M$. $\Delta$ is called smooth (analytic, respectively) if for each $x \in M$, there is a neighborhood $O$ of $x$ and there is a set of smooth (analytic, respectively) vector fields $\{X_1, \cdots, X_d\}$ on $O$ which span $\Delta(x)$ at each point $x$ of $O$. A vector field $X$ on $M$ is said to belong to the distribution $\Delta$ (denoted by $X \in \Delta$ ) if $X(x) \in \Delta(x)$ for each $x \in M$. A smooth (analytic, respectively) distribution $\Delta$ is called* involutive *if $[X, Y] \in \Delta$ for all $C^\infty$ (analytic, respectively) vector fields $X, Y \in \Delta$ where $[,]$ is the Lie bracket.*

**Definition 5.4 ([15])** *A submanifold $N \subset M$ is called* invariant *for the system*

$$\dot{x} = f(x) \tag{5.5}$$

*if $f(x) \in T_x N$, for all $x \in N$.*

If $N$ is connected then this immediately implies that the solution of (5.5) for $x(0) \in N$ remains in $N$ for all $t \geq 0$. Recall also that an analytic manifold is a topological manifold endowed with an analytic differentiable structure.

Now we have the following result regarding relative degrees of a compensator and the closed-loop system on a certain manifold.

**Lemma 5.5** *Consider a SISO analytic plant of the following form:*

$$P: \begin{array}{rcl} x(k+1) & = & f(x(k), u(k)), \;\; x \in \boldsymbol{R}^n, u \in \boldsymbol{R}, \\ y(k) & = & h(x(k), u(k)), \;\; y \in \boldsymbol{R}, \end{array} \tag{5.6}$$

*and a compensator $Q^a$*

$$Q^a: \begin{array}{rcl} z(k+1) & = & b(x(k), z(k), v(k)), \;\; z \in \boldsymbol{R}^\nu, v \in \boldsymbol{R}, \\ u(k) & = & \beta(x(k), z(k), v(k)). \end{array} \tag{5.7}$$

*Let $N_0$ be an analytic submanifold of $\boldsymbol{R}^n \times \boldsymbol{R}^\nu$ containing the origin and suppose that $N_0$ is an invariant manifold for the closed-loop system $P \circ Q^a$. Then, if $P \circ Q^a$ restricted to $N_0$, denoted $P \circ Q^a|_{N_0}$, has a well-defined relative degree at the origin with respect to $v$ and $y$, it also has a well-defined relative degree at the origin with respect to $v$ and $u$.*

**Proof:** Let $n_1$ be the relative degree from $v$ to $y$ of $P \circ Q^a|_{N_0}$, $n_2$ the (possibly not well-defined) relative degree from $v$ to $u$ of $P \circ Q^a|_{N_0}$. Define $\hat{h} \circ \hat{f}^0(x, u) = h(x, u)$ and $\hat{h} \circ \hat{f}^{i+1}(x, u) = \hat{h} \circ \hat{f}^i(f(x, u), 0), i \geq 0$. Let $n_3$ be the smallest integer $k \geq 1$ such that

$$\frac{\partial}{\partial u}(\hat{h} \circ \hat{f}^k)_{(x_0, u_0)} \neq 0,$$

for some $(x_0, u_0)$. As in the proof of the Lemma 2.4, the chain rule yields $n_1 = n_2 + n_3$. The rest of the proof is the same as in Lemma 2.4. □

Before we proceed further, we need to put a technical restriction about the compensator $Q^a$ due to the requirements of the currently available results on accessibility as shown in Appendix C. That is, we only allow compensators $Q^a$ of the form (5.7), with the components of $a$ and $\alpha$ analytic, which always keeps the closed-loop system $P \circ Q^a$ invertible. This requires that the linearization of the closed-loop should not have a pole at the origin.

It is obvious that once such a compensator $Q^a$ satisfying the above assumption is found the invertibility of the closed-loop system $P \circ Q^a$ is preserved under a coordinate change for the closed-loop system.

Suppose further that the compensator $Q^a$ solves the local exact tracking problem from the origin. Then we can prove the desired results of this Section as follows. Let $\tilde{f}$ denote the transition map of the closed-loop system $P \circ Q^a$. Under the invertibility assumption, consider the distribution $L^-(x)$ (Appendix C), associated with $\tilde{f}$, of the closed-loop system $P \circ Q^a$, i.e.,

$$L^- = \text{Lie}\{Ad_0^k X_u^- | k \leq 0, u \in U\}.$$

Then the distribution $L^-(x)$ is analytic, and also involutive since it is a Lie algebra. Thus $L^-(x)$ has the maximal integral manifold property [10, Corollary 2.1.9]. Let $\mathcal{M}^-$ denote the maximal integral manifold of $L^-$ containing the origin. Then it holds that dim $L^-(x)$ = dim $\mathcal{M}^-$ for all $x \in \mathcal{M}^-$. This property implies that the closed-loop system $P \circ Q^a$ restricted to $\mathcal{M}^-$, denoted $P \circ Q^a|_{\mathcal{M}^-}$, is *accessible*. Hence let us call $L^-(x)$ the "strong accessibility distribution," following [3].

On the other hand, since it is assumed that the compensator $Q^a$ solves the local exact tracking problem from the origin, by Remark 4.4, the closed-loop system $P \circ Q^a$ exhibits a local linear input-output behavior from the origin. In addition, $P \circ Q^a|_{\mathcal{M}^-}$ also has the same local linear input-output behavior from the origin. Therefore, by Lemma 5.1, $P \circ Q^a|_{\mathcal{M}^-}$ has a well-defined relative degree at zero. Note that since an invertible system restricted to a subset of the state space is still invertible, $P \circ Q^a|_{\mathcal{M}^-}$ is invertible.

Following the usual way of realizing a dynamic state feedback as a precompensator, let $C = P \circ Q^a|_{\mathcal{M}^-}$, with input $v$ and output $u$. Then $P \circ C$ has the same input-output behavior from the origin as $P \circ Q^a|_{\mathcal{M}^-}$ from $v$ to $y$. Furthermore, $\mathcal{M}^-$ is an analytic submanifold of $\boldsymbol{R}^n \times \boldsymbol{R}^\nu$ containing the origin, and invariant under the dynamics of $P \circ Q^a$ since $(d\tilde{f}_v)X \in L^-$ for any vector field $X \in L^-$. Therefore, the compensator $C$ has a well-defined relative degree at zero as shown in Lemma 5.5.

Now, using the regularity of $C$, one can deduce a third compensator $\tilde{C}$ of the form

$$\begin{array}{rcl} \xi(k+1) & = & c(x(k), \xi(k), v(k)), \\ u(k) & = & d(x(k), \xi(k), v(k)), \end{array} \tag{5.8}$$

such that $P \circ \tilde{C}$ is accessible from zero and has a local linear input-output behavior from the origin by the following lemma.

**Lemma 5.6** *Consider a SISO analytic plant of the form (5.6) and suppose that the plant $P$ is accessible from the origin. Let $C$ be an analytic compensator such that $C$ has a well-defined relative degree at the origin and $P \circ C$ is invertible. Then there exists a compensator $\tilde{C}$ of the form (5.8) such that $P \circ \tilde{C}$ is accessible from the origin and has the same linear input-output behavior from the origin as $P \circ C$.*

Putting all the above results together yields the main result.

**Theorem 5.7** *Consider an analytic SISO plant $P$ of the form (5.6). Suppose that $P$ is accessible from the origin and is invertible. Then, if the local exact tracking problem from the origin is solvable with the invertibility of the closed-loop system, $P$ has a well-defined relative degree at the origin. In particular, this is the case if the local asymptotic tracking problem is solvable.*

The proof is similar to that of Theorem 4.6 in [6]; it is omitted.

Consequently, the following also holds.

**Corollary 5.8** *Consider a SISO analytic, invertible plant $P$ of the form (5.6). Let $\mathcal{M}^-$ be the maximal integral manifold of the strong accessibility distribution of $P$ containing the origin. Then, if the local exact tracking problem from the origin is solvable with the invertibility of the closed-loop system, the plant $P$ restricted to $\mathcal{M}^-$ has a well-defined relative degree at the origin.*

## 6 Conclusions

In this study, it was shown that a well-defined relative degree at the origin is a necessary regularity condition for asymptotic tracking (in discrete-time systems), which includes most adaptive control. To show this, the asymptotic and exact tracking problems were defined, and it was proven that if a compensator solves the asymptotic tracking problem, then the same compensator solves the exact tracking problem. Then the concept of a linear input-output behavior of a discrete-time system was introduced in relation to a well-defined relative degree. It was next shown that if a compensator solves the local exact tracking problem from the origin, then the closed-loop system has a local linear input-output behavior from the origin, which provided the connection between tracking problem and a well-defined relative degree. Finally, with the help of accessibility concepts, it was proven that, for a single-input single-output system, if a plant is invertible and accessible from the origin and a compensator solves the local asymptotic tracking problem while keeping the closed-loop system invertible, then the plant necessarily has a well-defined relative degree at the origin. This result confirms that a well-defined relative degree is one of the basic requirements for tracking or model reference type control.

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