최적 FIR 필터를 사용한 강인제어기

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Robust Controller with Optimal FIR Filter

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Abstract

In this paper, an output feedback controller is proposed for continuous time-invariant linear systems. The proposed controller, LQ-FIR consists of an LQ control gain and an optimal FIR filter. The LQ-FIR controller is derived, and the stability of the closed loop system is proved. The bounds of parameter variations guaranteeing the closed loop stability are obtained, when the LQ-FIR controller is applied to the system with model uncertainties.

1 Introduction

It is well known that the LQ controller is robust with respect to the multiplicative perturbation because of its good gain and phase margins[1,2]. Unlike the LQ controller, the LQG controller is known to be not so robust[4]. However, the LQG controller can be made to have as good a robustness as that of the LQ controller by using the LQG/LTR(Linear Quadratic Gaussian with Loop Transfer Recovery) method, with which the loop transfer matrix of the LQG controller becomes the same as that of the LQ controller[3,5,9]. Meanwhile, the robustness with respect to parameter variations in the state space models is quite different from the robustness with respect to the multiplicative perturbation. It is known that in some cases, the LQ controller may be unstable even with small parameter variations and the stability of the LQG controller may be very sensitive to small parameter variations even when the LQG/LTR method is used[6,7,8].

The optimal FIR filter, one of limited memory filters, which uses finite measurements on $[t-T, t]$ to estimate the state at time t, was derived by Kwons. It is composed of an FIR structure which guarantees BIBO stability and is known to be robust to modeling errors[11,12]. Moreover it is recently proved that FIR filter is a deadbeat observer in noise free systems[13,14]. According to these properties of the optimal FIR filters, it can be expected that the performance and the robustness of the controller using an FIR filter as a state observer are more improved rather than using the Kalman filter,i.e LQG. Deadbeat property can make the response more rapid and the robustness of the optimal FIR filter can improve the robustness of the controller using it.

In this paper, we propose a new controller named LQ-FIR consisting of an LQ control gain and an FIR filter, and analyze its properties. The proposed LQ-FIR controller can be compared with the LQG controller in their structure, because two controllers use the same control gain and different state observer. Specially, the measurement interval as a design parameter causes various effects on the characteristics of the LQ-FIR controller.

This paper is organized as follows. In Section 2, the LQ-FIR controller is derived and the closed loop stability is proved. In Section 3, the bounds of allowable parameter variations of the LQ-FIR controller are obtained. And the results are compared with those of the LQG controller. Finally we conclude this paper in Section 4.

2 LQ-FIR Controller

Let us consider the continuous-time state-space model:

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t) + Gw(t) \\ y(t) &= Cx(t) + v(t) \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is a state, $u(t) \in R^m$ is an input, and $y(t) \in R^l$ is an output. A,B, and C are time invariant matrices with appropriate dimensions, and w and v are zero-mean white Gaussian noises with covariances Q, R respectively, and it is assumed that w and v are uncorrelated with each other.

The proposed LQ-FIR controller(LQ-FIRC) is obtained as the following procedure. First, the control gain F is obtained from solving an LQ problem as follows :

$$F = R_c^{-1}B^TP. \tag{2}$$

where P staisfies the ARE and $Q_c \geq 0$, $R_c > 0$ are weighting matrices. Second, the state estimator is an optimal FIR filter given by[13]

$$\hat{x}(t \mid T) = \int_{t-T}^{t} H(t-\tau)y(\tau)d\tau + \int_{t-T}^{t} H_u(t-\tau)u(\tau)d\tau \tag{3}$$

where

$$H(t-s) = L^{-1}(0)\Theta(t-s)C^TR^{-1} \tag{4}$$

$$H_u(t-s) = L^{-1}(0)\Theta(t-s)L(t-s)B, \tag{5}$$

$$t-T \leq s \leq t$$

$$\begin{aligned} \frac{\partial\Theta(\sigma)}{\partial\sigma} &= -\Theta(\sigma)[A^T + L(\sigma)GQG^T] \\ &= [A^T + L(\sigma)GQG^T]\Theta(\sigma), \quad 0 \leq \sigma \leq T \\ \Theta(0) &= I_n \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{\partial L(\sigma)}{\partial\sigma} &= L(\sigma)A + A^TL(\sigma) - C^TR^{-1}C \\ &\quad + L(\sigma)GQG^TL(\sigma) \\ L(T) &= 0. \end{aligned} \tag{7}$$

Finally, the control input of the LQ-FIRC is given by

$$u(t) = -F\hat{x}(t \mid T). \tag{8}$$

For all design parameters $T > 0, Q \geq 0$ and $R > 0$, it reduces to a deadbeat observer ,i.e. $\hat{x}(t \mid T) = x(t)$ for $t \geq T$, in case the system has no noise input. With this deadbeat property of the optimal FIR filter, the following theorem shows the closed loop stability conditions.

THEOREM 1 *If the system (1) is completely controllable and completely observable, then the LQ-FIRC stabilizes the system (1).*

Proof: The closed loop system without noise is given as follows:

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ &= Ax(t) - BF\hat{x}(t \mid T) \\ &= (A - BF)x(t) + BFe(t) \qquad (9)\end{aligned}$$

where $e(t)$ is a state estimation error defined as $e(t) := x(t) - \hat{x}(t \mid T)$. In the above equations, complete observability condition guarantees that $e(t) = 0$ for $t \geq T$, since the optimal FIR filter is a deadbeat observer in noise free systems. It means that $BFe(t)$ is finite value and the stability is judged by only $A - BF$. By the properties of LQ, $x(t)$ is asymptotically stable under complete controllability condition. Consequently, the closed loop system is asymptotically stable.

<Q.E.D.>

In the derived LQ-FIRC, the control gain F and the filter gains $H(\cdot), H_u(\cdot)$ are time invariant whenever the system is time invariant. So, they can be determined by off-line computation. Structurally, the LQ-FIRC can be compared with the LQGC(LQG controller) composed of an LQ control gain and the Kalman filter. Since two controllers use the same control gain, the differences between two controllers are due to the characteristics of the optimal FIR filter and the Kalman filter. It is remarkable that the LQ-FIRC has the measurement interval T as a design parameter besides Q, R, Q_c and R_c.

3 Stability Robustness Bound

We consider the plant with parameter uncertainties as follows :

$$\begin{aligned}\frac{dx(t)}{dt} &= (A + \Delta A)x(t) + Bu(t) \\ y(t) &= (C + \Delta C)x(t) \qquad (10)\end{aligned}$$

where ΔA and ΔC are time invariant matrices. In order to obtain the guaranteed bounds of parameter variations in case the LQ-FIRC for (1) is applied to the system (10), the following lemma is required.

LEMMA 1 *When the LQ-FIRC for (1) is applied to the system (10), the state estimation error is given by*

$$\begin{aligned}e(t) &:= x(t) - \hat{x}(t) \\ &= \int_{t-T}^{t} L^{-1}(0)\Theta(t-\tau)[L(t-\tau)\Delta Ax(\tau) \\ &\quad -C^T R^{-1}\Delta Cx(\tau)]d\tau. \qquad (11)\end{aligned}$$

Proof: From (3)-(7) and (10), $\hat{x}(t)$ is represented as follows :

$$\begin{aligned}&\hat{x}(t) \\ &= \int_{t-T}^{t} L^{-1}(0)\Theta(t-\tau)[C^T R^{-1}(C + \Delta C)x(\tau) \\ &\quad +L(t-\tau)Bu(\tau)]d\tau \\ &= \int_{t-T}^{t} L^{-1}(0)\Theta(t-\tau)\{C^T R^{-1}Cx(\tau) \\ &\quad +L(t-\tau)[\frac{dx(\tau)}{d\tau} - (A + \Delta A)x(\tau)]\}d\tau \\ &\quad +\int_{t-T}^{t} L^{-1}(0)\Theta(t-\tau)C^T R^{-1}\Delta Cx(\tau)d\tau \\ &= \int_{t-T}^{t} L^{-1}(0)\Theta(t-\tau)\{[C^T R^{-1}C - L(t-\tau)A]x(\tau) \\ &\quad +L(t-\tau)\frac{dx(\tau)}{d\tau}\}d\tau + \int_{t-T}^{t} L^{-1}(0)\Theta(t-\tau) \\ &\quad \cdot[C^T R^{-1}\Delta Cx(\tau) - L(t-\tau)\Delta Ax(\tau)]d\tau. \qquad (12)\end{aligned}$$

Here, the first term of the right side of (12) equals to $x(t)$ [13]. Therefore the above equations give

$$\begin{aligned}\hat{x}(t) &= x(t) + \int_{t-T}^{t} L^{-1}(0)\Theta(t-\tau)[C^T R^{-1}\Delta Cx(\tau) \\ &\quad -L(t-\tau)\Delta Ax(\tau)]d\tau.\end{aligned}$$

<Q.E.D.>

According to Lemma 1, when ΔA and ΔC exist, the closed loop system including the LQ-FIRC can be expressed by

$$\begin{aligned}\frac{dx(t)}{dt} &= (A + \Delta A)x(t) + Bu(t) \\ &= (A + \Delta A)x(t) - BF\hat{x}(t) \\ &= (A + \Delta A)x(t) - BF[x(t) - e(t)] \\ &= (A - BF + \Delta A)x(t) + BF\int_{t-T}^{t} L^{-1}(0)\Theta(t-\tau) \\ &\quad \cdot[L(t-\tau)\Delta A - C^T R^{-1}\Delta C]x(\tau)d\tau. \qquad (13)\end{aligned}$$

In (13), the bounds of ΔA and ΔC which guarantee the closed loop stability are shown in the following theorem.

THEOREM 2 *Assume that the only one among ΔA and ΔC exists at a time. The closed loop system (13) is asymptotically stable if ΔA and ΔC satisfy the following inequalities, respectively,*

$$\bar{\sigma}(\Delta A) < \max_{\alpha} \frac{-\bar{\sigma}(P) + \sqrt{\bar{\sigma}^2(P) + (2\alpha - 1)\Pi}}{\alpha^2 \Pi} \quad (\Delta C = 0) \qquad (14)$$

$$\bar{\sigma}(\Delta C) < \frac{1}{T\bar{\sigma}(PBF)\max_{\tau\in[0,T]}[\bar{\sigma}^2(H(\tau))]} \quad (\Delta A = 0) \qquad (15)$$

where

$$\Pi := T^2\bar{\sigma}^2(PBF)\max_{\tau\in[0,T]}[\bar{\sigma}^2(K(\tau))]$$

and $\bar{\sigma}(\cdot)$ denotes the maximum singular value, α is a positive scalar, $H(\cdot)$ is defined in (3)-(7), P and $K(\cdot)$ satisfy the following equations

$$(A - BF)^T P + P(A - BF) = -2I_n \qquad (16)$$

$$K(t) = L^{-1}(0)\Theta(t)L(t). \qquad (17)$$

Proof: (i) (Derivation of the inequality (14).) Let $\Delta C = 0$, $\hat{A} := A - BF$, and

$$V(x(t)) = \alpha T x^T(t)Px(t) + \beta\int_0^T\int_{t-\tau}^{t} x^T(s)x(s)ds d\tau \qquad (18)$$

be a Lyapunov fuction, where $\alpha > 0$ and $\beta > 0$ will be chosen later to maximize the bound. Taking the derivative of (18) along (13) yields

$$\frac{dV(x(t))}{dt} = \int_0^T \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T M \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix} d\tau \qquad (19)$$

where

$$M = \begin{bmatrix} \alpha(\hat{A} + \Delta A)^T P + \alpha P(\hat{A} + \Delta A) + \beta I & \alpha TPBFK(\tau)\Delta A \\ \alpha T\Delta A^T K^T(\tau)F^T B^T P & -\beta I \end{bmatrix}. \qquad (20)$$

Combining the above equation with (16), the matrix M is negative definite[10] if

$$\begin{aligned}&-2\alpha I + \beta I + \alpha\Delta A^T P + \alpha P\Delta A \\ &+\frac{1}{\beta}T^2\alpha^2 PBFK(\tau)\Delta A\Delta A^T K^T(\tau)F^T B^T P < 0 \qquad (21)\end{aligned}$$

for all $0 < \tau \leq T$. This inequality holds if

$$f(\bar{\sigma}(\Delta A)) \equiv -2\alpha+\beta+2\alpha\bar{\sigma}(P)\bar{\sigma}(\Delta A)+\frac{1}{\beta}\alpha^2T^2$$

$$\cdot\bar{\sigma}^2(PBF)\max_{\tau\in(0,T]}[\bar{\sigma}^2(K(\tau))]\bar{\sigma}^2(\Delta A) < 0. \quad (22)$$

$\bar{\sigma}(\Delta A) < g(\alpha,\beta)$ where $g(\alpha,\beta)$ is the positive root of $f(\bar{\sigma}(\Delta A)) = 0$ guarantees that the above inequality holds. (14) is obtained by maximizing $g(\alpha,\beta)$ with $\beta = 1$.

(ii) (Derivation of the inequality (15).) Similar to (i).

<Q.E.D.>

For the LQGC, the guaranteed bounds were obtained by Kim[8] as follows:

$$\bar{\sigma}(\Delta A) < \frac{1}{\bar{\sigma}(P_c)+\bar{\sigma}(P_cBF)\bar{\sigma}(P_o)} \quad (\Delta B=0,\Delta C=0) \quad (23)$$

$$\bar{\sigma}(\Delta C) < \frac{1}{\bar{\sigma}(P_cBF)\bar{\sigma}(P_oK)} \quad (\Delta A=0,\Delta B=0) \quad (24)$$

$$(A-BF)^TP_c+P_c(A-BF) = -2I_n \quad (25)$$

$$(A-KC)^TP_o+P_o(A-KC) = -2I_n. \quad (26)$$

It is not easy to compare (14)(15) with (23)(24), since they are different in their forms. The following example show the comparison between the bounds of allowable parameter variations of the LQ-FIRC and those of the LQGC through some computation.

Example

We consider the system with

$$A=\begin{bmatrix}-2 & -5\\ 1 & 0\end{bmatrix}, B=\begin{bmatrix}1\\0\end{bmatrix}, C=\begin{bmatrix}1 & 1\end{bmatrix}. \quad (27)$$

Let $G = \begin{bmatrix}1 & 1\end{bmatrix}^T$. Table 1 and Table 2 show the bounds of $\bar{\sigma}(\Delta A)$ and $\bar{\sigma}(\Delta C)$ respectively along Q_c and Q with fixed T, R and R_c.

These results present that the guaranteed bounds of $\bar{\sigma}(\Delta A)$ and $\bar{\sigma}(\Delta C)$ of the LQ-FIRC are larger than those of the LQGC in [8]. Especially the cases of $Q = 10000$ in Table 1 and Table 2 mean that the LTR methods are used to the LQGC. Therefore it can be concluded that the LQ-FIRC is more robust to parameter variations than the LQGC and even the LQG/LTRC.

4 Conclusion

In this paper, we proposed an output feedback controller named LQ-FIRC using an optimal FIR filter as a state observer and made analyses of its robustness. The proposed LQ-FIRC is composed of an LQ control gain and an optimal FIR filter. It stabilizes the given system under the condition of complete controllability and complete observability. To check the robustness to parameter variations, the allowable bounds of parameter variations of the LQ-FIRC were obtained and compared with those of the LQGC and LQG/LTRC in an example.

To investigate the properties of the LQ-FIRC in details, more analytic studies are needed. It is required that its transfer function be found to analyze its many chracteristics in frequency domains. Additionally, since the optimal FIR filter can improve the response of the controller, it is expected that another controller using an optimal FIR filter can be designed and applied to tracking problems.

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Table 1. Bounds of $\bar{\sigma}(\Delta A)$ with $T = R = R_c = 1$

	$Q=$ 100		$Q=$ 1000		$Q=$ 10000	
	LQG	LQ-FIR	LQG	LQ-FIR	LQG	LQ-FIR
$Q_c = 100C^TC$	0.0263	0.0350	0.0280	0.0329	0.0286	0.0322
$Q_c = 10C^TC$	0.0825	0.1055	0.0869	0.1049	0.0887	0.1030
$Q_c = C^TC$	0.1993	0.2245	0.2042	0.2282	0.2060	0.2267

Table 2. Bounds of $\bar{\sigma}(\Delta C)$ with $T = R = R_c = 1$

	$Q=$ 100		$Q=$ 1000		$Q=$ 10000	
	LQG	LQ-FIR	LQG	LQ-FIR	LQG	LQ-FIR
$Q_c = 100C^TC$	0.0021	0.0030	6.8084e-4	9.2194e-4	2.1716e-4	2.9723e-4
$Q_c = 10C^TC$	0.0079	0.0113	0.0025	0.0034	8.0467e-4	0.0011
$Q_c = C^TC$	0.0406	0.0587	0.0130	0.0176	0.0041	0.0055