COMPARISON OF SOME ORDER-REDUCTION METHODS FOR LINEAR SYSTEMS

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Abstract: Some well known order reduction methods are briefly described and a new order reduction technique is introduced. A comparison of the various classes of order reduction approaches are indicated. Furthermore,the question is raised how order reduction should be executed with respect to controller design. Finally,by means of an example,results of the discussed approaches are compared.

Introduction

The linearized original system may be given,as result of theoretical analysis, by a state space representation or by a transfer function of high order. On the other hand,as result either of experiments with the real plant or from a simulation study,the original system may be also represented by recorded data of its response to any input function or/and by discrete values of its frequency response. Since both parametric and non-parametric representations are used either in the time domain or in the frequency domain many well known methods for order reduction are based either on time domain or frequency domain representations.
It is common knowledge that two different systems may have frequency responses which correspond well over a certain range of frequencies but,in the time domain, they may have diverging input-output behaviour and vice versa. Therefore,a powerful tool for order reduction has to consider simultaneously both time domain and frequency domain behaviour. However,this can not be done without being prepared to compromise. We certainly know the relation between time domain and frequency domain. There is no order reduction method which enables to simultaneously eliminate all errers both in the time domain and frequency domain. Therefore,such complete error elimination can not be used as a design criterion. When using a method which is based exclusively on frequency domain representations,subjective optimal compromises are already inevitable within the relevant interval of frequencies. In the course of this,no time domain design criterion can be considered. Corresponding to this,when applying a pure time domain method,it is again necessary to compromise within various time intervals but no frequencydomain design criterion can be taken into account.
We already stated that both input-output performance and frequency domain behaviour have to be considered simultaneously. By doing so the number of available design criterions increases considerably. If it is inevitable to compromise it is then obvious that any compromise is the better the more criterions are considered. Thus, the result of this reflections is an approximating approach based on minimizing a cost function vector. The components of this vector must be design criterions defined in both domains.
In this paper we shall be discussing in detail the basic ideas behind the four

methods. These will also be illustrated by means of an example. Finally, a comparison between the methods will be made, for a single-input single-output system.

Statement of the problem

Consider a linear time-invariant system described by the equations

$$\dot{\underline{x}}_o(t) = \underline{A}_o\underline{x}_o(t) + \underline{b}_o u(t) \qquad (1)$$
$$\underline{y}_o(t) = [\underline{c}^T|\underline{0}]\underline{x}_o = \underline{c}^T\underline{x}_r(t)$$

where $\underline{x}_o \in R^q$, $y_o \in R^m$, $u \in R^p$. It will be assumed that this system is both controllable and observable, i.e., it is a minimum realization.

The objective of order reduction is to obtain the low-order model

$$\dot{\underline{x}}_m(t) = \underline{A}_m \underline{x}_m(t) + \underline{b}_m u(t)$$
$$\underline{y}_m(t) = \underline{c}^T \underline{x}_m(t) \qquad (2)$$

where $x_m \in R^r$ and $r \ll q$, such that $y_m \in R^m$ is a close approximation to $y_o(t)$ for all inputs $u(t)$.

Modal techniques: They are based on transforming (1) to the diagonal form

$$\dot{\underline{z}} = \underline{V}^{-1}\underline{A}_o\underline{V}\underline{z} + \underline{V}^{-1}\underline{b}_o u = \underline{A}^*\underline{z} + \underline{b}^* u \qquad (3)$$
$$\underline{y}_o = \underline{c}^T\underline{V}\underline{z} = \underline{c}^{*T}\underline{z}, \qquad (4)$$

by means of

$$\underline{x}_o = \underline{V}\underline{z} = \begin{bmatrix}\underline{x}_r \\ \underline{x}_s\end{bmatrix} = \begin{bmatrix}\underline{V}_{11} & \underline{V}_{12} \\ \underline{V}_{21} & \underline{V}_{22}\end{bmatrix}\begin{bmatrix}\underline{z}_r \\ \underline{z}_s\end{bmatrix} \qquad (5)$$

where the modal matrix $\underline{V}$ consists of the eigenvectors of $\underline{A}_o$. There are several modal approaches discussed in literature. A most frequently used and meanwhile well known approach was proposed by /1/.

All techniques define dominant eigenvalues of the original system and assume only these as eigenvalues of the model. The various approached differ from each other essentially by the definition of dominance. /1/ defined his measure of dominance for the eigenvalues as

$$D_k = |b_k^* c_k^* / \lambda_k| \qquad (6)$$

with $b_k^* c_k^*$ the respective components of $\underline{b}^*$ and $\underline{c}^*$. Thus, not only the eigenvalues influence on stability but also on controllability and observability is considered. After computing the measures (6), the state equation (3) is splitted into

$$\dot{\underline{z}}_r = \underline{A}_1^*\underline{z}_r + \underline{b}_1^* u, \underline{A}_1^* = \mathrm{Diag}\{\lambda_1, \ldots\} \qquad (7)$$
$$\dot{\underline{z}}_s = \underline{A}_2^*\underline{z}_s + \underline{b}_2^* u, \underline{A}_2^* = \mathrm{Diag}\{\lambda_{r+1}, \ldots\} \qquad (8)$$

(7) is the dominant part. For reason of simplification it is assumed here that the number of dominant eigenvalues corresponds with the number of essential state variables. If (8) would be neglected completely, steady state agreement of $x_r(t)$ would be violated. Therefore, one receives from (5)

$$\dot{\underline{z}}_r = \underline{V}_{11}\underline{z}_r + \underline{V}_{12}\tilde{\underline{z}}_s = (\underline{V}_{11} + \underline{V}_{12}\underline{E})\underline{z}_r \qquad (9)$$
$$= \underline{M}\underline{z}_r$$

with the approximation $\tilde{\underline{z}}_s = \underline{E}\underline{z}_r$. The matrix $\underline{E}$ is calculated such that $|\tilde{z}_s(t) - z_s(t)|$ is minimized and steady state agreement of the reduced-order model is achieved. Insertion of

$$\underline{z}_r = \underline{M}^{-1}\tilde{\underline{x}}_r = \underline{M}^{-1}\underline{x}_m \qquad (10)$$

from (9) into (7) yields the reduced-order model

$$\dot{\underline{x}}_m(t) = \underline{A}_m\underline{x}_m(t) + \underline{b}_m u(t) \qquad (11)$$

with

$$\underline{A}_m = \underline{M}\,\underline{A}_1^*\underline{M}^{-1}, \quad \underline{b}_m = \underline{M}\,\underline{b}_1^* \qquad (12)$$

When having a critical look at modal techques, also at /1/ method, some serious questions may arise. Is the adoption of unchanged dominant eigenvalues really ideal? It can be shown that results areimproved greatly by slightly shifting both the remaining dominant eigenvalues and the zeros.

Balanced realization: In some way related to this class of methods is the idea of using the minimal realization theory and to eliminate weak subsystems which contribute little to the impulse response of the system. The original model (1) can be transformed to a balanced representation for which

$$\underline{P} = \int_0^\infty e^{\underline{A}t}\,\underline{b}\,\underline{b}^T\,e^{\underline{A}^T t}\,dt, \quad \underline{A}\,\underline{P} + \underline{P}\,\underline{A}^T + \underline{b}\,\underline{b}^T = \underline{0}$$

$$\underline{Q} = \int_0^\infty e^{\underline{A}^T t}\,\underline{c}\,\underline{c}^T\,e^{\underline{A}t}\,dt, \quad \underline{A}\,\underline{Q} + \underline{Q}\,\underline{A}^T + \underline{c}\,\underline{c}^T = \underline{0}$$

$$\underline{P} = \underline{Q} = \mathrm{Diag}\{\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_q\}$$

are valid. If $\sigma_r >> \sigma_{r+1}$, the components smaller than σ_r are neglected. Thus, from the balanced and partitioned original model

$$\begin{bmatrix}\dot{\underline{x}}_r \\ \dot{\underline{x}}_2\end{bmatrix} = \begin{bmatrix}\underline{A}_r & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22}\end{bmatrix}\begin{bmatrix}\underline{x}_r \\ \underline{x}_2\end{bmatrix} + \begin{bmatrix}\underline{b}_r \\ \underline{b}_2\end{bmatrix} u$$
$$y = [\underline{c}_r^T | \underline{c}_2^T]\begin{bmatrix}\underline{x}_r \\ \underline{x}_2\end{bmatrix} \qquad (16)$$

the reduced-order model with the order r is received as /2/

$$\dot{\underline{x}}_r(t) = \underline{A}_r \underline{x}_r(t) + \underline{b}_r u(t)$$
$$y = \underline{c}_r^T \underline{x}_r \qquad (17)$$

Error minimization technique: This was established by /3/. In the view of the present contribution, it seems to be the most powerful approach to order reduction in the frequency domain for the time being. It belongs to the class of methods which are oriented on minimization of cost functions in the frequency domain. The original model $G_o(s)=Z(s)/N(s)$ should be approximated by the reduced-order model $G_m(s)= P(s)/Q(s)$ such that both frequency responses and step responses are as similar as possible. This requires correspondence of the frequency responses $G_o(j\omega)$ and $G_m(j\omega)$ within a large range of frequencies. This correspondence is attained by minimizing the cost function

$$\Phi_f = \sum_{i=1}^{k} q_i \left| N(j\omega_i)P(j\omega_i) - Z(j\omega_i)Q(j\omega_i) \right|^2 \qquad (18)$$

ω_i; i=1,2,...k are discrete values of frequencies within the range of correspondence. These values have to be laid down with respect to certain rules. The weigting factors are given in their most general formulation

$$q_i = \frac{1}{|N(j\omega_i)|^2 \ \omega_i^{2\delta}} \left| \frac{Z(j\omega_i)}{N(j\omega_i)} \right|^{\beta} \qquad (19)$$

The method thus offers the transparent control parameter $\omega_i, q_i=Q_i(\omega_i,\delta,\beta)$ where the number of k can be rather low. /3/ proposed an algorithm to find the optimal control parameters.

New order reduction technique: A high-order transfer function $G_{so}(s)$ should be approximated by a transfer function $G_{sm}(s,r_s)$ with an order as low as possible.

The pole-zero difference of the reduced order model $G_{sm}(s,r_s)$ should, whenever possible, agree with the original model. Since the low-order transfer function is not calculated by an analytic method but designed by means of an approximation approach, order reduction will be an iterative procedure in most cases. However, in majority, the designed results are better than results calculated by any analytic order reduction technique. Since the original model is represented by measurement results the structure of the reduced-order model has to be estimated first. Some a priori knowledge, e.g., from theoretical consideration is helpful. If the model was controlled when executing the measurements, the transfer function of the controller belongs to the fixed part of the reduced -order model. As above mentioned, known transfer functions of subsystems belongs to the fixed part of the reduced-order model; After having estimated the model structure and having specified the requirements by a set of cost function, optimization of the parameters r_s is carried out. If the result of the approximation is not satisfactory the model structureor/and the cost function must be altered. The systematic bottom-up approach is based on factorization of transfer functions. After factorization, the original transfer function is splitted into reduced-order components. Zero or poles in the origin belong to the fixed part ; non-minimimum phase components are treated as separate subsystems. This decomposition into subsystems must not be carried out

arbitrarily but has to follow certain rules which are programmed in the package. Thus, for the purpose of order reduction,a rule based tool was developed. The first rule is that the order of each subsystems must not be higher than four. Each partial transfer function is reduced to its lowest possible order not violating the pole_zero difference. Thus,subsystems with one zero and two poles or systems with two zeros abd three poles are reduced to first order systems. Subsystems with one zero and three poles or with two zeros and four poles are reduced to second order systems with two poles The parameters of the reduced subsystems are the initial parameters for the first step of the optimization procedure and are calculated from simple formulas. The reduction of subsystems is based on the principle of time_moments matching.

Comparison of different methods of order redoction for a system

A comparison of four different methods of order reduction will be made for the case of a system. The considered are (a) modal (b) balanced realization,(c) error minimization,(d) new order reduction. We shall consider the example of a 20.order model. The eigenvalues of the system are located at

zeros	poles
0.000	- 0.063 ± j 32.575
0.000	- 1.096 ± j 14.363
- 0.742 ± j 30.642	- 1.502
- 1.027 ± j 3.280	- 1.969 ± j 45.547
- 2.607 ± j 11.983	- 3.712 ± j 5.606
- 2.851 ± j 30.895	- 7.466 ± j 18.383
- 7.461 ± j 48.693	-18.714 ± j 43.103
-20.584 ± j 29.988	-25.636 ± j 6.060
-22.450	-34.596 ± j 24.022
-32.265 ± j 18.499	-48.611
-38.273	-66.310 ± j 40.810
-66.724 ± j 85.126	

Table 1. 10.reducad-order model by (a).

zeros	poles
$3,8.10^{-6}$	- 0.063 ± j 32.575
- 0.082 ± j 30.973	- 1.096 ± j 14.363
- 3.102 ± j 7.019	- 1.969 ± j 45.547
- 3.732 ± j 33.196	- 18.714 ± j 43.103
- 6.566 ± j 48.920	- 34.596 ± j 24.022
- 52.571 ± j 67.590	- 48.611

Table 2. 10.reduced_order model by (b)

zeros	poles
0.000, 1.740	- 0.064 ± j 32.576
- 0.233 ± j 30.454	- 0.810 ± j 14.421
- 1.804 ± j 14.435	- 1.824 ± j 45.520
- 2.525 ± j 32.479	- 2.793 ± j 15.945
- 7.034 ± j 46.206	- 10.073 ± j 36.175

Table 3. 1o. reduced_order model by (c)

zeros	poles
0.000, 0.000	- 0.062 ± j 32.576
- 0.593 ± j 29.678	- 1.083 ± j 14.325
- 1.364 ± j 31.954	- 1.921 ± j 45.530
- 2.235 ± j 12.650	- 5.396 ± j 13.560
- 8.399 ± j 47.207	- 12.086 ± j 29.991

Table 4. 10. reduced_order model by (d)

zeros	poles
0.000, 0.000	- 0.346 ± j 32.561
- 0.185 ± j 30.502	- 0.716 ± j 0.697
- 1.172 ± j 1.251	- 1.355 ± j 15.046
- 6.012 ± j 34.280	- 2.174 ± j 45.168
- 10.631 ± j 47.795	- 16.517 ± j 42.001

The response y(t) for the original model, as well as for each reduced_order model, is shown in Fig. 1.

Concluding remarks

We have considered a number of different methods for obtaining low_order models for high_order model. From the point of view of practical applications,the new order method and the balanced realization appear to be the most promising.

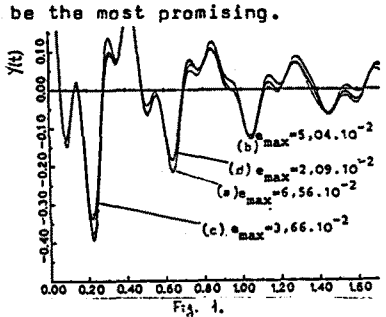

Fig. 1.

1Litz, L. (1979). Reduktion der Ordnung linearer Zustandsraummodelle mittels modularer Verfahren. Hochschulverl. Stuttgart.

2Moore, B.C. (1981). Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction. IEEE Trans. on Autom. Control, AC-26, No.1, Pp.17-32.

3Kiendl, H., Post, K. (1988). Invariante Ordnungsreduktion mittels transparenter Parametrierung. Automatisierungstechnik, 36, S.92-101.