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Robust Stabilization of Plants with both Parameter Perturbation and Unstructured Uncertainty

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Abstract– In this paper a robust stabilization problem is discussed for plant with both time-varying parameter perturbations and unstructured uncertainty. It is shown that, a robust L_2-stabilizing controller can be obtained by solving an H_∞ standard problem with a scaling parameter. Using an H_∞ design method, a robust L_2-stabilizing controller is drived. Finally, a numerical example is given.

Notation

R	the field of real numbers
R^n	the real n-dimensional space
A^T	transpose of matrix A
A^*	complex-conjugate transpose of matrix A
$\lambda_{max}(A)$	the maximum eigenvalue of A
$\bar{\sigma}(A)$	the maxmum singular value of A
$\|\|u\|\|_2$	L_2 norm defined by $\|\|u\|\|_2 := \int_0^\infty u^T u dt$
$\|\|F(s)\|\|_\infty$	H_∞ norm defined by $\|\|F\|\|_\infty := \sup_{\omega \in R} \bar{\sigma}\{F(j\omega)\}$
RH_∞	the space of all proper real-rational function matrices which have no poles in $Res \geq 0$
$BH_\infty(\varepsilon)$	the set of $F(s)$ in RH_∞ which satisfy $\|\|F(s)\|\|_\infty < \varepsilon$ for a given $\varepsilon > 0$
Ω	bounded set of time-varying matrices $\Omega := \{\Sigma(t) \mid \Sigma^T(t)\Sigma \leq 1,\ \Sigma(t) \in R^{n \times n}\}$
Ω_0	bounded set of constant matrices $\Sigma \in \Omega$.
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	$:= D + C(sI - A)^{-1}B$

1 Introduction

In the past years robust stabilization problems have received a great deal of attention. It is well known that the most effective method to solve the problem is H_∞ design approach. For unstructured uncertainty described in frequency domain, robust stabilization problem is equivalent to an H_∞ standard design problem [1], which can be solved using solutions of two Riccati equations [2],[14]. For time-varying parameter perturbation described in the state space such as $E\Delta(t)F$, an effective method dealing with robust stability is quadratic stabilization [4]~[5], and it is recently shown in [6] that the quadratic stabilization problem can be reduced to an H_∞ standard design problem.

In this paper we discuss the robust stabilization problem for plants with both time-varying parameter perturbation and unstructured uncertainty. The plant is represented by

$$\tilde{P}(s,t) = P(s, \Sigma(t))(I + \Delta(s)) \tag{1.1}$$

where $\Delta(s)$ denotes unstructured uncertainty which is assumed to be real rational, analytic in the closed right half plane of s and bounded by a prespecified $\varepsilon > 0$, e.t. $\Delta(s) \in BH_\infty(s)$. A state space realization of $P(s, \Sigma(t))$ is given by

$$\begin{aligned} \dot{x} &= (A + \Delta A(t))x + (B + \Delta B(t))v & (1.2) \\ y &= Cx + Dv & (1.3) \end{aligned}$$

$$[\Delta A(t)\ \ \Delta B(t)] = E\Sigma(t)[F_1\ \ F_2] \tag{1.4}$$

where A, B, C, D, E and F_i $(i = 1, 2)$ are known matrices and $\Sigma(t) \in \Omega$ denotes the unknown time-varying parameter perturbations.

The following question naturally raises. Is it possible to find a robust stabilizing controller for the plant $\tilde{P}(s,t)$? If so, how to obtain a desired controller? In this paper it is shown that a robust L_2-stabilizing controller can be obtained by solving an H_∞ standard design problem with a scaling parameter. This paper is organized as follows. In Section 2, some definitions and preliminary results about quadratic stability and L_2-stability are given. In Section 3, algebraic Riccati inequality with coefficient perturbations is invastigated which is a basic technique used in this paper. Note that the results given in Section 3 is an extention of the results given by [5]. Using the result and the ARE-based techniques[7]~[12], a robust L_2-stabilizing controller is designed in Section 4. Finally, a numerical example is given in Section 5.

2 Definition and preliminary results

Consider a dynamical output feedback controller $K(s)$ for plant (1.1)

$$\begin{aligned} \dot{\xi} &= A_c\xi + B_c y & (2.1) \\ u &= C_c\xi & (2.2) \end{aligned}$$

The closed loop system of plant (1.1) with the controller is given in Fig.1. A state space realization of $G(s)$ is given by

$$\begin{aligned} \dot{x} &= Ax + Ew_1 + Bw_2 + Bu & (2.3) \\ z_1 &= F_1 x + F_2 w_2 + F_2 u & (2.4) \\ z_2 &= u & (2.5) \\ y &= Cx + Dw_2 + Du & (2.6) \end{aligned}$$

Definition 1. *For any $\Delta(s) \in BH_\infty(\varepsilon)$, the system given in Fig.1 is said to be quadratically stable, if there exist a* positive definite matrix $P > 0$ and constant $\alpha > 0$ such that for any $\Sigma(t) \in \Omega$

$$L(x, \Delta, \Sigma) := 2\zeta^T P \tilde{A}(\Delta, \Sigma(t))\zeta \leq -\alpha||\zeta|| \qquad (2.7)$$

holds for all $t \in R$ and $\zeta \in R^{n+n_c+n_\delta}$, where n, n_c and n_δ are dimensions of x, ξ and Δ respectively and $\tilde{A}$ is the system matrix of the closed loop system given in equation (2.14).

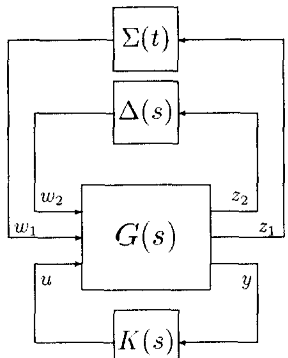


Figure 1: The closed loop system

For examination of L_2-stability of the closed loop system an auxiliary input signal e is introduced as shown in Fig.2.

Definition 2. *The closed loop system given in Fig.2 is said to be robust L_2-stable, if there exist an $\beta_1 > 0$ and $\beta_2 > 0$ such that*

$$||z_2||_2 \leq \beta_1||e||_2 + \beta_2, \ ^\forall e \in L_2 \qquad (2.8)$$

holds for all $\Delta(s) \in BH_\infty(\varepsilon)$ and $\Sigma(t) \in \Omega$.

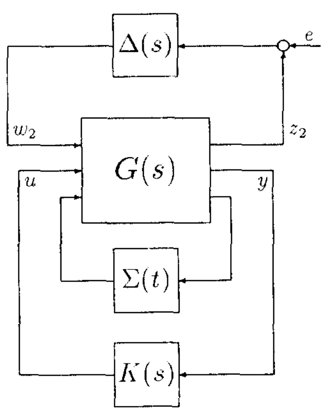


Figure 2: The equivalent system

It is not difficult to show that the system is robust L_2-stable if and only if the operator $\{I - T_{z_2w_2}(s)\Delta(s)\}^{-1}$ is L_2-stable for all $\Delta \in BH_\infty(\varepsilon)$ and $\Sigma(t) \in \Omega$, where $T_{z_2w_2}$ denotes the operator from w_2 to z_2. So that, we have the following Lemma.

Lemma 1. *The closed loop system of (1.1) with the controller (2.1), (2.2) is robust L_2-stable if and only if the operator $T_{z_1w_1}$ satisfies $||T_{z2w2}|| < \varepsilon^{-1}$ for all $\Sigma(t) \in \Omega$, where $||\cdot||$ denote the L_2 induced norm of the operator*

$$||T_{z_2w_2}|| := \sup_{w \neq 0} \frac{||z_2||_2}{||w_2||_2} \qquad (2.9)$$

▮

Lemma 2. *The closed loop system of (1.1) with the controller (2.1), (2.2) is quadratically stable for any $\Delta(s) \in BH_\infty(\varepsilon)$ if and only if the transfer function from w_1 to z_1 satisfies*

$$||T_{z_1w_1}(s, \Delta)||_\infty < 1, \ ^\forall \Delta \in BH_\infty(\varepsilon) \qquad (2.10)$$

Proof. Let a state space realization of $\Delta(s)$ be given by

$$\dot{\eta} = A_\delta\eta + B_\delta z_2 \qquad (2.11)$$

$$w_2 = C_\delta\eta + D_\delta z_2 \qquad (2.12)$$

Then, the closed loop system can be described as

$$\dot{x}_s = \tilde{A}(\Delta, \Sigma)x_s \qquad (2.13)$$

where $x_s^T := [x^T \ \xi^T \ \eta^T]$ and

$$\begin{aligned} \tilde{A}(\Delta, \Sigma) &= A_s + E_s\Sigma(t)F_s \qquad (2.14) \\ A_s &= \begin{bmatrix} A & B(I + D_\delta)C_c & BC_\delta \\ B_cC & A_c + B_cD(I + D_\delta)C_c & B_cDC_\delta \\ 0 & B_\delta C_c & A_\delta \end{bmatrix} \\ E_s &= \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix} \\ F_s &= [F_1 \ \ F_2(C_c + D_\delta C_c) \ \ F_2C_\delta] \end{aligned}$$

By Theorem 2.7 in [6], the closed loop system is quadratically stable if and only if $\tilde{A}(\Delta, 0)$ is a stable matrix and

$$||F_s(sI - A_s)^{-1}E_s||_\infty < 1 \qquad (2.15)$$

It is not difficult to show that this is equivlent to that $T_{z_1w_1}$ is stable and $||T_{z_1w_1}||_\infty < 1$. ▮

Based on the above results, the following facts are clear. First, the design problem of quadratically stabilizing controller is equivalent to find a stabilizing controller $K(s)$ such that $||T_{z_1w_1}||_\infty < 1$ for all $\Delta \in BH_\infty(\varepsilon)$ when the path from z_1 to w_1 in Fig. 1 is broken. This problem can be solved by using the μ-synthesis method[3]. Second, if one can find such a controller $K(s)$ that $||T_{z_2w_2}|| < 1, ^\forall \Sigma \in \Omega$ after breaking the path from z_2 to w_2, then the closed loop system with the controller will be robust L_2-stable for all $\Delta(s) \in BH_\infty(\varepsilon)$ and $\Sigma(t) \in \Omega$. In this paper we will focus our attention on this robust stabilizing controller. We will extend the ARE-based technique to obtain a robust L_2-stabilizing controller for given plant with both parameter perturbation and unstructured uncertainty.

3 Riccati inequality with perturbations

Consider the algebraic Riccati inequality (ARI)

$$A^TX + XA + \frac{1}{\gamma^2}XBB^TX + C^TC < 0 \qquad (3.1)$$

We will discuss the ARI with coefficient perturbations such as $[A \ \ B] - [A + \Delta A \ \ B + \Delta B]$, $[\Delta A \ \ \Delta B] = E\Sigma[F_a \ \ F_b]$. The techniques used in this section are based on the literatures [5]. Note that the case of $\Delta B = 0$ has been investigated in [4], [5], [7] and [9].

The problem discussed in this section is to find under what conditions a positive definite matrix X satifying (3.1) satisfies (3.2).

$$(A+\Delta A)^T X + X(A+\Delta A) + C^T C + \frac{1}{\gamma^2} X(B+\Delta B)(B+\Delta B)^T X < 0 \quad (3.2)$$

$$[\Delta A \ \ \Delta B] = E\Sigma[F_a \ \ F_b], \ \ \Sigma \in \Omega_0 \quad (3.3)$$

Substitution (3.3) into (3.2) gives

$$\begin{aligned} & A^T X + XA + \frac{1}{\gamma^2} XBB^T X + C^T C \\ & + XE\Sigma(F_a + \frac{1}{\gamma^2}F_b B^T X) + (F_a + \frac{1}{\gamma^2} F_b B^T X)^T \Sigma^T E^T X \\ & < -\frac{1}{\gamma^2} XE\Sigma F_b F_b^T \Sigma^T E^T X \end{aligned} \quad (3.4)$$

Using Lemma 3.1 and Lemma 3.2 in [5], it is not difficult to show that the left hand side in the above Riccat inequality is less than $-\beta I$, $(\beta > 0)$ for all $\Sigma \in \Omega_0$ if and only if there exists a $\lambda > 0$ such that

$$\begin{aligned} & A^T X + XA + \frac{1}{\gamma^2} XBB^T X + C^T C \\ & + \lambda^2 XEE^T X + \frac{1}{\lambda^2}(F_a + \frac{1}{\gamma^2} F_b B^T X)^T (F_a + \frac{1}{\gamma^2} F_b B^T X) \\ & < -\beta I \end{aligned} \quad (3.5)$$

So that, a sufficient condition where $X > 0$ satisfies (3.2) $^\forall \Sigma \in \Omega_0$ is that there exists a $\lambda > 0$ such that

$$\begin{aligned} & A^T X + XA + \frac{1}{\gamma^2} XBB^T X + C^T C \\ & + \lambda^2 XEE^T X + \frac{1}{\lambda^2}(F_a + \frac{1}{\gamma^2} F_b B^T X)^T (F_a + \frac{1}{\gamma^2} F_b B^T X) \\ & < -\frac{1}{\gamma^2} \bar{\sigma}(F_b) XEE^T X \end{aligned} \quad (3.6)$$

This becomes also a necessary condition which is shown in the following theorem.

Theorem 1. *Assume that $F_b F_b^T = I$ and Σ is square matrix. Let a positive definite matrix $X > 0$ satisfy Riccati inequality (3.1). Then, the X satisfies (3.2) for all $\Sigma \in \Omega_0$ if and only if there exists an $\lambda > 0$ such that*

$$A_\lambda^T X + XA_\lambda + \frac{1}{\gamma^2} XB_\lambda B_\lambda^T X + C_\lambda^T C_\lambda < 0 \quad (3.7)$$

where

$$A_\lambda = A + \frac{1}{\gamma^2 \lambda^2} BF_b^T F_a \quad (3.8)$$

$$B_\lambda = [BR \ \ \rho E] \quad (3.9)$$

$$C_\lambda = \begin{bmatrix} C \\ \frac{1}{\lambda} F_a \end{bmatrix} \quad (3.10)$$

$$R^2 = I + \frac{1}{\gamma^2 \lambda^2} F_b^T F_b \quad (3.11)$$

$$\rho^2 = 1 + \lambda^2 \gamma^2 \quad (3.12)$$

Proof. Since (3.7) is equivalent to (3.6), the sufficiency is clear from the above discussion. For proof of the necessity assume that (3.2) holds for any $\Sigma \in \Omega_0$. Then, for any vector $\xi \neq 0$, we have

$$\begin{aligned} & \xi^T Y \xi + 2\xi^T XE\Sigma(F_a + \frac{1}{\gamma^2} F_b B^T X)\xi \\ < & -\frac{1}{\gamma^2} \xi^T XE\Sigma\Sigma^T E^T X\xi, \ ^\forall \Sigma \in \Omega_0 \end{aligned} \quad (3.13)$$

where

$$Y = A^T X + XA + \frac{1}{\gamma^2} XBB^T X + C^T C$$

From Lemma 3.1 in [5], there exist a $\Sigma(\xi) \in \Omega$ such that

$$\begin{aligned} & \max_{\Sigma \in \Omega} | \xi^T XE\Sigma(F_a + \frac{1}{\gamma^2} F_b B^T X)\xi | \\ = & \ \xi^T XE\Sigma(\xi)(F_a + \frac{1}{\gamma^2} F_b B^T X)\xi \end{aligned} \quad (3.14)$$

and

$$\Sigma(\xi)\Sigma^T(\xi) = I$$

Thus, from (3.13)

$$\begin{aligned} & \xi^T Y \xi + 2\xi^T XE\Sigma(F_a + \frac{1}{\gamma^2} F_b B^T X)\xi \\ & \leq \xi^T AR(H, X)\xi + 2\xi^T XE\Sigma(\xi)(F_a + \frac{1}{\gamma^2} F_b B^T X)\xi \\ & < -\frac{1}{\gamma^2} \xi^T XEE^T X\xi \end{aligned} \quad (3.15)$$

Hence, we have

$$\begin{aligned} & \xi^T \{A^T X + XA + \frac{1}{\gamma^2} XBB^T X + C^T C + \frac{1}{\gamma^2} XEE^T X\}\xi \\ < & -2\xi^T XE\Sigma(F_a + \gamma^{-2} F_b B^T X)\xi, \ \forall \Sigma \in \Omega \end{aligned} \quad (3.16)$$

Using the technique similar to that used in the proof of Theorem 3.3 in [5], it follows that there exist an $\epsilon > 0$ such that

$$\begin{aligned} & \epsilon^2 XEE^T X + (F_a + \frac{1}{\gamma^2} F_b B^T X)^T (F_a + \frac{1}{\gamma^2} F_b B^T X) \\ + & \ \epsilon\{A^T X + XA + \frac{1}{\gamma^2} XBB^T X + C^T C + \frac{1}{\gamma^2} XEE^T X\} \\ < & \ 0 \end{aligned} \quad (3.17)$$

Therefore, (3.7) follows from (3.17) with $\lambda = \sqrt{\epsilon}$. ∎

4 Design of robust stabilizing controller

Now, we design a robust L_2-stabilizing controller for the plant (1.1). From Lemma 1 it is clear that a desired controller can be obtained by finding such a controller $K(s)$ given by (2.1) and (2.2) that the operator $T_{z_2 w_2} : L_2 - L_2$ satisfies $||T_{z_2 w_2}|| < \varepsilon^{-1}$ for all $\Sigma(t) \in \Omega$. Next Lemma provides a feasible way to solve this problem using the ARE-based techniques.

Note that a state space realization of $T_{z_2 w_2}$ is given by

$$\dot{x}_c = (\bar{A} + \bar{E}\Sigma(t)\bar{F}_a)x_c + (\bar{B} + \bar{E}\Sigma(t)\bar{F}_b)w_2 \quad (4.1)$$

$$z_2 = \bar{C}x_c \quad (4.2)$$

where

$$\bar{A} = \begin{bmatrix} A & BC_c \\ B_c C & A_c + B_c DC_c \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ B_c D \end{bmatrix}$$

$$\bar{C} = [0 \ \ C_c],$$

$$\bar{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \bar{F}_a = [F_1 \ \ F_2 C_c], \bar{F}_b = F_2$$

Lemma 3. *Let $\varepsilon > 0$ be given. If there exists a positive definite matrix $P > 0$ such that*

$$(\bar{A}+\bar{E}\Sigma(t)\bar{F}_a)^T P+P(\bar{A}+\bar{E}\Sigma(t)\bar{F}_a)+\bar{C}^T\bar{C} + \varepsilon^2 P(\bar{B}+\bar{E}\Sigma(t)\bar{F}_b)(\bar{B}+\bar{E}\Sigma(t)\bar{F}_b)^T P<0 \quad (4.3)$$

for all $\Sigma(t) \in \Omega$, then the operator $T_{z_2 w_2}$ satisfies $||T_{z_2 w_2}|| < \varepsilon^{-1}$ for all $\Sigma(t) \in \Omega$.

Proof. Stability of the operator follows immediately from the standard Lyapunov theory with (4.3). To show the validity of the norm condition, differentiate $x_c^T P x_c$ along the solution of state equation (4.1) and (4.2),

$$\frac{d}{dt} x_c^T P x_c = 2x_c^T(\bar{A} + \bar{E}\Sigma(t)\bar{F}_a)^T P x_c + w_2^T(\bar{B}+\bar{E}\Sigma(t)\bar{F}_b)^T P x_c + x_c P(\bar{B}+\bar{E}\Sigma(t)\bar{F}_b) w_2 \quad (4.4)$$

Use of (4.3) and some algebraic manipulation give

$$\frac{d}{dt} x_x^T P x_c \;<\; -x_c^T \bar{C}^T \bar{C} x_c + \varepsilon^{-2} w_2^T w_2 - \varepsilon^2 \phi^T \phi \quad (4.5)$$

$$\phi \;=\; (\bar{B} + \bar{E}\Sigma(t)\bar{F}_b)^T P x_c - \varepsilon^{-2} w_2 \quad (4.6)$$

Integration of (4.4) from $t = 0$ to $t = \infty$ with $x_c(0) = x_c(\infty) = 0$ gives

$$\| z_2 \|_2^2 < \varepsilon^{-2} \| w_2 \|_2^2 - \varepsilon^2 \| \phi \|_2^2 \quad (4.7)$$

which implies that $||T_{z_2 w_2}|| < \varepsilon^{-1}$. ∎

Clearly, the Riccati inequality (4.3) holds for all time-varying perturbation if and only if the positive definite matrix $P > 0$ satisfies Riccati inequality with time invariant perturbation of the form

$$(\bar{A} + \bar{E}\Sigma\bar{F}_a)^T P + P(\bar{A} + \bar{E}\Sigma\bar{F}_a) + \bar{C}\bar{C} + \varepsilon^2 P(\bar{B}+\bar{E}\Sigma\bar{F}_b)(\bar{B}+\bar{E}\Sigma\bar{F}_b)^T P < 0, \;\; \Sigma \in \Omega_0 \quad (4.8)$$

Hence, combining Lemma 3 and Theorem 1, we immediately obtain the following result.

Theorem 2. *Given $\varepsilon > 0$ and assume that $F_2 F_2^T = I$. If there exists a constant $\lambda > 0$ such that Riccati inequality*

$$\bar{A}_\lambda^T P + P\bar{A}_\lambda + \varepsilon^2 P\bar{B}_\lambda \bar{B}_\lambda^T P + \bar{C}_\lambda^T \bar{C}_\lambda < 0 \quad (4.9)$$

has a positive definite solution $P > 0$, then, the operator $T_{z_2 w_2}$ satisfies

$$||T_{z_2 w_2}|| < \frac{1}{\varepsilon^2} \quad (4.10)$$

for all $\Sigma(t) \in \Omega$, where

$$\bar{A}_\lambda = \bar{A} + \frac{\varepsilon^2}{\lambda^2}\bar{B}\bar{F}_b^T \bar{F}_a \quad (4.11)$$

$$\bar{B}_\lambda = [\bar{B}R \;\; \rho\bar{E}] \quad (4.12)$$

$$\bar{C}_\lambda = \begin{bmatrix} \bar{C} \\ \frac{1}{\lambda}\bar{F}_a \end{bmatrix} \quad (4.13)$$

$$R^2 = I + \frac{\varepsilon^2}{\lambda^2} F_2^T F_2 \quad (4.14)$$

$$\rho^2 = 1 + \frac{\lambda^2}{\varepsilon^2} \quad (4.15)$$

∎

Thus, we can obtain a desired controller by finding a controller $\{A_c, B_c, C_c\}$ which ensures the existence of $\lambda > 0$ such that (4.9) has positive definite solution. Our next result shows that this can be found by solving an H_∞ standard problem.

Definition 3. *Given plant $G(s)$ as shown in Fig.3. A controller $K(s)$ is said to be a solution of H_∞ standard problem with plant $G(s)$, if $K(s)$ is stabilizing controller for $G(s)$ and H_∞ norm of transfer matrix from w to z is less than a given $\gamma > 0$.*

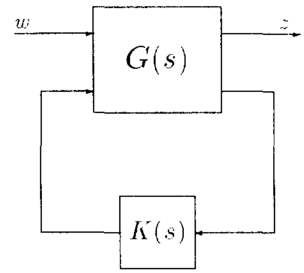


Figure 3: H_∞ standard problem

Theorem 3. *Let $\gamma = \varepsilon^{-1}$, and $\lambda > 0$. There exists a strictly proper controller (2.1),(2.2) such that the Riccati inequality (4.9) has positive definite solution P if and only if the H_∞ standard problem with the plant $G_\lambda(s)$ given as follows has a solution.*

$$G_\lambda(s) = \left[\begin{array}{c|cc} A + \frac{\varepsilon^2}{\lambda^2} B F_2^T F_1 & [BR \;\; \rho E] & BR^2 \\ \hline \begin{bmatrix} 0 \\ \frac{1}{\lambda} F_1 \end{bmatrix} & 0 & \begin{bmatrix} I \\ \frac{1}{\lambda} F_2 \end{bmatrix} \\ C + \frac{\varepsilon^2}{\lambda^2} D F_2^T F_1 & [DR \;\; 0] & DR^2 \end{array}\right] \quad (4.16)$$

Proof. Note that a state space realization of the system given in Fig.3 with plant (4.16) is described by

$$\dot{x}_s = \bar{A}_\lambda x_s + \bar{B}_\lambda w \quad (4.17)$$

$$z = \bar{C}_\lambda x_s \quad (4.18)$$

where $\bar{A}_\lambda$, $\bar{B}_\lambda$ and $\bar{C}_\lambda$ are given by (4.11) $\sim$ (4.13). Thus, the Theorem follows immediately from Lemma 2.2 in [13]. ∎

From Theorem 2 and Theorem 3 we can obtain a robust L_2-stabilizing controller by solving the H_∞ standard problem, if it is solvable for a given $\lambda > 0$. We now give a result obtained by existing H_∞ method under the following assumptions.

$A1$. (A, B) is stabilizable.

$A2$. The nominal plant $\tilde{P}$ with $\Sigma(t) = 0$ and $\Delta(s) = 0$ is minimum phase.

$A3$. $F_2 F_2^T = I$.

Theorem 4. *If there exists a $\lambda > 0$ such that the followings hold,*

(1). *Riccati equation*

$$(\hat{A}-\hat{B}_2\hat{D}_{12}^T\hat{C}_1)^T X + X(\hat{A}-\hat{B}_2\hat{D}_{12}^T\hat{C}_1) + X(\varepsilon^2\hat{B}_1\hat{B}_1^T - \hat{B}_2\hat{B}_2^T)X + \hat{C}_1^T(I-\hat{D}_{12}\hat{D}_{12}^T)\hat{C}_1 = 0 \quad (4.19)$$

has a semi-positive definite solution $X \geq 0$.

(2). *Riccati equation*

$$(\hat{A}-\hat{B}_1\hat{D}_{21}^T\hat{C}_2)Y+Y(\hat{A}-\hat{B}_1\hat{D}_{21}^T\hat{C}_2) + Y(\varepsilon^2\hat{C}_1^T\hat{C}_1-\hat{C}_2^T\hat{C}_2)Y+\hat{B}_1(I-\hat{D}_{21}^T D_{21})\hat{B}_1^T=0 \quad (4.20)$$

has a semi-positive definite solution $Y \geq 0$.

(3). $\lambda_{max}(XY) < \frac{1}{\varepsilon^2}$.

Then, a robust L_2-stabilizing controller is given by

$$\begin{aligned} K(s) &= \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] && (4.21)\\ C_c &= -M_\lambda^{-1}(\hat{D}_{12}^T\hat{C}_1+\hat{B}_2^T X)(I-\varepsilon^2 YX)^{-1} && (4.22)\\ B_c &= (Y\hat{C}_2^T+\hat{B}_1\hat{D}_{21}^T)N_\lambda^{-1} && (4.23)\\ A_c &= \hat{A}-B_c N_\lambda\hat{C}_2+\varepsilon^2 Y\hat{C}_1^T\hat{C}_1+ \\ &\quad (\hat{B}_2+\varepsilon^2 Y\hat{C}_1^T\hat{D}_{12})M_\lambda C_c - B_c D R^2 C_c && (4.24)\end{aligned}$$

where

$$\begin{aligned} \hat{A} &= A+\frac{\varepsilon^2}{\lambda^2}BF_2^TF_1 \\ \hat{B}_1 &= [BR\ \ \rho E], \quad \hat{B}_2 = BR^2M_\lambda^{-1} \\ \hat{C}_1 &= \begin{bmatrix} 0 \\ \frac{1}{\lambda}F_1 \end{bmatrix}, \quad \hat{D}_{12} = \begin{bmatrix} I \\ \frac{1}{\lambda}F_2 \end{bmatrix} M_\lambda^{-1} \\ \hat{C}_2 &= N_\lambda^{-1}(C+\frac{\varepsilon^2}{\lambda^2}DF_2^TF_1), \quad \hat{D}_{21} = N_\lambda^{-1}[DR\ \ 0] \\ M_\lambda^2 &= I+\frac{1}{\lambda^2}F_2^TF_2 \\ N_\lambda^2 &= DR^2D^T \end{aligned}$$

Proof. By some algebraic manipulations, it is not difficult to show that the $K(s)$ is a solution of the H_∞ standard problem with the plant $G_\lambda(s)$ if and only if the controller $\hat{K}(s)$ with state space realization given by

$$\begin{aligned} \hat{K}(s) &= \left[\begin{array}{c|c} A_c+B_cDR^2C_c & B_cN_\lambda \\ \hline M_\lambda C_c & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} \hat{A}_c & \hat{B}_c \\ \hline \hat{C}_c & 0 \end{array}\right] && (4.25)\end{aligned}$$

is a solution of H_∞ standard problem with the plant given by

$$\hat{G}_\lambda(s) = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & 0 & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right] \quad (4.26)$$

Note that $\hat{D}_{12}^T\hat{D}_{12} = I$ and $\hat{D}_{21}\hat{D}_{21}^T = I$. Since (A, B) is stabilizable, $(\hat{A}, \hat{B}_2)$ is stabilizable for any $\lambda > 0$. From the assumption $A2$, it is easy to show that $(\hat{C}_2, \hat{A})$ is detectable for any $\lambda > 0$. Hence, from the Theorem A.1 in the appendix, the solvability of the H_∞ standard problem is equivalent to the condition (1) ~ (3), and a solution $\hat{K}(s)$ is given by

$$\begin{aligned} \hat{C}_c &= -(\hat{D}_{12}^T\hat{C}_1+\hat{B}_2^TX)(I-\varepsilon^2YX)^{-1} && (4.27)\\ \hat{B}_c &= (Y\hat{C}_2^T+\hat{B}_1\hat{D}_{21}^T) && (4.28)\\ \hat{A}_c &= \hat{A}-B_c\hat{C}_2+\varepsilon^2Y\hat{C}_1^T\hat{C}_1 \\ &\quad +(\hat{B}_2+\varepsilon^2Y\hat{C}_1^T\hat{D}_{12})C_c && (4.29)\end{aligned}$$

Using the relation (4.25), a desired controller $K(s)$ can be obtained as (4.22)~(4.24). ∎

5 Numerical example

Consider plant (1.1) with unstructured uncertainty $||\Delta(s)||_\infty < 0.3$ and $P(s, \Sigma(t))$ described by the state equation

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1+0.4f(t) \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0.2f(t) \\ 1 \end{bmatrix} v && (5.1)\\ y &= [1\ \ 5]x + v && (5.2)\end{aligned}$$

Let

$$E = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, \Sigma(t) = f(t), \quad F_1 = [0\ \ 2], \quad F_2 = 1$$

Then, (5.1) and (5.2) can be represented in the form of (1.2)~(1.4) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1\ \ 5], \quad D = 1$$

It is easy to verify that the plant satisfies the assumptions $A1 \sim A3$. Let $\varepsilon = 0.3$ and $\lambda = 0.001 > 0$, the Riccati equations in Theorem 4 have positive definite solutions

$$\begin{aligned} X &= \begin{bmatrix} 72.6230 & 4.3579 \\ 4.3579 & 33.4779 \end{bmatrix} && (5.3)\\ Y &= \begin{bmatrix} 0.0224 & -0.0182 \\ -0.0182 & 0.0252 \end{bmatrix} && (5.4)\end{aligned}$$

and $\lambda_{max}(YX) = 2.0316 < \varepsilon^{-2} = 11.1111$. Thus, we obtain a robust L_2-stabilizing controller $K(s)$ from (4.22)~(4.24) as follows

$$K(s) = -\frac{5.6741s+0.1149}{s^2+1.7336s+2.6025} \quad (5.5)$$

This $K(s)$ ensures L_2-stability of the closed loop system for all $\Sigma(t) \in \Omega$ and $||\Delta(s)||_\infty < 0.3$. Fig.4(a)~ (c) show the initial state responces of the closed loop system, the state of the plant (a), the state of the controller (b) and the state of $||\Delta(s)$ (c) under the following conditions

$$\begin{aligned} f(t) &= \sin 30t && (5.6)\\ \Delta(s) &= \frac{0.2}{s^2+0.3s+1} && (5.7)\end{aligned}$$

Fig 4.(a)

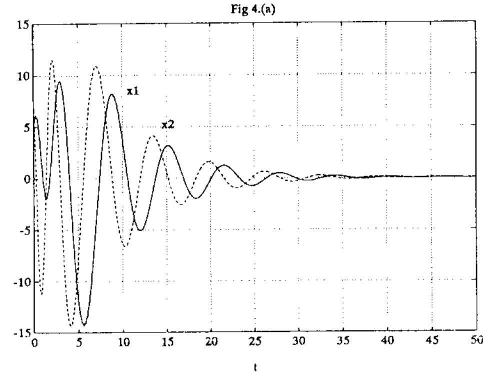


Fig 4.(b)

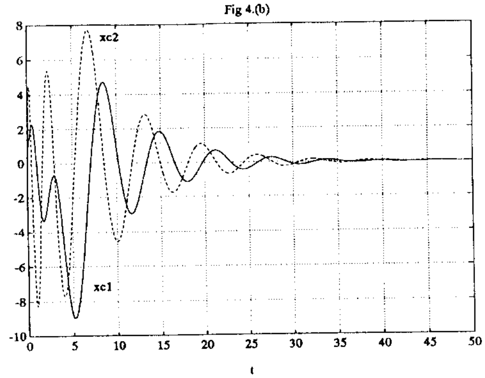
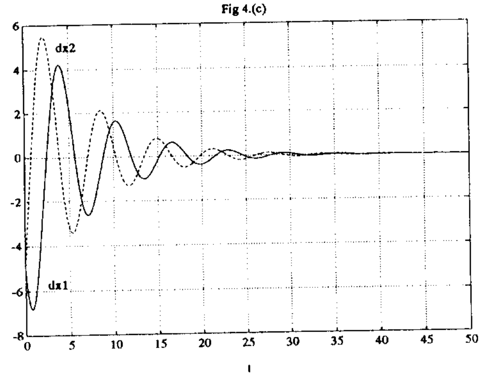


Fig 4.(c)



6 Conclusion

In this paper, robust stabilization problem for plant with both time-varying parameter perturbation and unstructured uncertainty is investigated. First, the result about algebraic Riccati inequality with perturbation given in [4] ~ [5] is extended. Based on this extention and the ARE-base method, a robust L_2-stabilizing controller for a given plant with the uncertainties is designed.

Appendix

Consider a plant described by

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned}$$

and the controller $u = K(s)y$ given by

$$\begin{aligned} \dot{\xi} &= A_k \xi + B_k y \\ u &= C_k \xi \end{aligned}$$

Assume that

1. (A, B) is stabilizable.
2. (A, C_2) is detectable.
3. $D_{12}^T D_{12} = I$ and $D_{21} D_{21}^T = I$.

Theorem A.1[14] *There exists a controller $K(s)$ such that the closed loop system of the plant with the controller is stable and $||T_{zw}||_\infty < \gamma$ if and only is the followings hold.*

(1). *Riccati equation*

$$\begin{aligned} &(A - B_2 D_{12}^T C_1)^T X + X(A - B_2 D_{12}^T C_1) \\ &+ X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X + C_1^T (I - D_{12} D_{12}^T) C_1 = 0 \end{aligned}$$

has a semi-positive definite solution $X \geq 0$.

(2). *Riccati equation*

$$\begin{aligned} &(A - B_1 D_{21}^T C_2) Y + Y(A - B_1 D_{21}^T C_2)^T \\ &+ Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y + B_1 (I - D_{21}^T D_{21}) B_1^T = 0 \end{aligned}$$

has a semi-positive definite solution $Y \geq 0$.

(3). $\lambda_{max}(XY) < \gamma^2$.

Then, a desired controller is given by

$$\begin{aligned} C_k &= -(D_{12}^T C_1 + B_2^T X)(I - \gamma^{-2} Y X)^{-1} \\ B_k &= Y C_2^T + B_1 D_{21}^T \\ A_k &= A - B_k C_2 + \gamma^{-2} Y C_1^T C_1 \\ &\quad + (B_2 + \gamma^{-2} Y C_1^T D_{12}) C_k \end{aligned}$$

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