Unified Approach to Continuous and Discrete Nehari Problems

Koon-Seok Lee and Oh-Kyu Kwon[0]
Department of Electronical Engineering,
Inha University, 402-751, Korea

Abstract

A unified approach to continuous and discrete-time Nehari problems, based on recently developed results by the authors for the one-block and Hankel-norm model reduction problems, is proposed. First, we derive discrete-time solutions in delta domain where numerical error is small and then we show that the derived form becomes same as the continuous form when the sampling interval approaches to zero.

1. Introduction

Recent days, the role and use of the Nehari problem has become very important, especially in the area of control theory. In particular, it plays a main role in solving the one-block and Hankel-norm model reduction problems.

The works on this subject are found in [1], [2], [3], [5] and [9] for continuous-time systems and in [1], [7] and [8] for discrete-time systems. In [1] and [7], Nehari problem of balanced state-space model for system is considered. On the other hand, the direct methods without balancing were given in [2], [3] and [9]. In [8], a generalized state-space representation for causal as well as noncausal system is used in the derivations. However, they have not referred any relationship between continuous and discrete-time cases and hence we have a difficulty that we have to solve them with each other method. Thus, the derivation of unified solutions to continuous and discrete-time cases gives more insights into the problem and is therefore of interest.

In this paper, we derive the unified solutions for Nehari problem in delta domain. The benifits and connections between continuous and discrete-time systems by using delta form are sufficiently discussed in [4]. The delta form approach has the numerical properties superior to those of usual shift form. Also, owing to the similar structure of the delta operator with differential operator, it can generally use the continuos-time insights in discrete problem and it directly represents continuous form as sampling interval approaches zero. These indicate that the delta form approach to the Nehari problem may offer a powerful tool to solve the discrete-time H_∞ control design problem for continuous-time plants. The derivation procedure in this paper is based on the immediate consequences of the all-pass properties.

This paper is structured as follows: After a simple discussion of delta transformation and some definitions are given, then a precise statement of the problem is formulated in Section 2. The unified solutions to Nehari problem are given in Section 3 and a numerical example is then shown in Section 4 to illustrate the usefulness of the method proposed here. Concluding remarks are given in Section 5.

2. Problem Formulation in Delta Domain

A. Definitions and Nomenclatures

Consider the delta operator

$$\delta := \frac{q-1}{\Delta} \tag{2.1}$$

where q denote the forward shift operator and Δ is the sampling interval. Then we can represent the discrete-time model with delta operator as follows:

$$\delta x(k) = Ax(k) + Bu(k) \tag{2.2}$$
$$y(k) = Cx(k)$$

where
$$A = \Omega A_c \tag{2.3}$$
$$B = \Omega B_c \tag{2.4}$$
$$C = C_c \tag{2.5}$$
$$\Omega = \frac{1}{\Delta}\int_0^\Delta \exp(A_c\tau)\, d\tau \tag{2.6}$$

and A_c, B_c and D_c are the matrices in the continuos-time model. Note that δ-transformation for (2.2) does not produce the new unstable zeros but not Z-transformation. An interesting observation from (2.3) - (2.6) is that they reveal the close connection between discrete form and the underlying continuous form since $\Omega \to I$ as $\Delta \to 0$. And when we desire to find an algorithm for a copmputer in δ-domain so that the digital system approximates the transfer fuction matrices (TFM) G(s), we may use the following bilinear transformation in δ-domain:

$$G(\gamma) = G(s)|\ s = \gamma/(1+\Delta\gamma/2) \tag{2.7}$$

or inversely we can approximate $G(\gamma)$ to G(s) as follows:

$$G(s) = G(\gamma)|\ \gamma = s/(1-\Delta\gamma/2) \tag{2.8}$$

Notice that $|\gamma|^2\Delta/2 + Re(\gamma) < 0$ iff $Re(s) < 0$, and $|\gamma|^2\Delta/2 + Re(\gamma) = 0$ iff $Re(s) = 0$. And then we can see that the stability boundary for the case of delta operator is the contour $\gamma = (\exp(j\omega\Delta) - 1)/\Delta$ in frequency domain.

Let discrete system $G(\gamma) = D + C(\gamma I - A)^{-1}B$ and the approximating continuous system $G_1(s) = D_1 + C_1(sI - A_1)^{-1}B_1$, then a straight calculation by transformation (2.8) using matrix inversion lemma gives

$$A_1 = \hat{A}_1 A \tag{2.9}$$

$$B_1 = \hat{A}_1 B \tag{2.10}$$

$$C_1 = C\hat{A}_1 \tag{2.11}$$

$$D_1 = D - \frac{\Delta}{2} C\hat{A}_1 B \tag{2.12}$$

where $\hat{A}_1 = (I + \frac{\Delta}{2} A)^{-1}$, provided A and $(I + \frac{\Delta}{2} A)$ are invertible.

If $G(\gamma)$ is stable but not necessarily minimal with a state-space realization in (2.2), then the controllability and observability gramians, P and Q respectively, are defined as the solutions to the following unified Lyapunov equations:

$$AP + PA^T + BB^T + \Delta APA^T = 0 \tag{2.13}$$

$$A^TQ + QA + C^TC + \Delta A^TQA = 0. \tag{2.14}$$

The Hankel singular values of $G(\gamma)$ are defined as

$$\{ \sigma_i := \lambda_i(PQ),\ 1 \le i \le n \}$$

and the Hankel norm denoted $\|\cdot\|_H$ is the largest of these. Substituting (2.9), (2.10) and (2.11) into corresponding continuous Lyapunov equation to (2.13) and (2.14), then we see that the controllability and observability gramians of δ-model are the same ones with corresponding continuous-time.

State-space system is denoted

$$G(\gamma) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.15}$$

where $G(\gamma) = C(\gamma I - A)^{-1}B + D$. Then the G's conjugate system is defined as

$$G(\gamma)^* = \left[\begin{array}{c|c} -A^T\hat{A} & \hat{A}C^T \\ \hline -B^T\hat{A} & D^T - \Delta B^T\hat{A}^TC^T \end{array}\right] \tag{2.16}$$

where $\hat{A} = (I + \Delta A)^{-1}$, provided A and $(I + \Delta A)$ are invertible. This formulation is easily derived by direct calculation with matrix inversion lemma.

For a given

$$P(\gamma) = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] \tag{2.17}$$

we define the lower linear fractional tranformation by

$$F_L(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{12}. \tag{2.18}$$

The H_∞-norm of a TFM $G(\gamma)$ is denoted

$$\| G(\gamma) \|_\infty := \sup_\omega \sigma_{max}[(\exp(j\omega\Delta) - 1)/\Delta] \tag{2.19}$$

RL_∞ denotes the space of proper, real rational function with no poles on $\gamma = (\exp(j\omega\Delta) - 1)/\Delta$ with bounded norm denoted $\|\cdot\|_\infty$. RH_∞ denotes the subspace of RL_∞ with no poles outside the open stability boundary contour and RH_∞^- denotes the space of orthogonal complementary of RH_∞ in RL_∞ with no poles on the stability boundary contour.

The 'inertia' of a square matrix A, written In_dA or InA, is defined as the triple of integers

$$In_dA = \{ \pi_d(A),\ \nu_d(A),\ \delta_d(A) \}, \tag{2.20}$$

$$InA = \{ \pi(A),\ \nu(A),\ \delta(A) \}, \tag{2.21}$$

where $\pi_d(A)$, $\nu_d(A)$ and $\delta_d(A)$ denote the number of eigenvalues of A lying outside, inside and on the stability boundary contour, respectively, and $\pi(A)$, $\nu(A)$ and $\delta(A)$ denote the number of eigenvalues of A lying in RHP, LHP and on the imaginary axis, respectively.

B. Problem Formulation

Nehari problem[5] (or so-called Hankel-norm approximation problem with zero order) is posed as follows: Given G in RL_∞ with $\|G^*\|_H < \alpha$, $G(\gamma) \in RH_\infty^-$, find all X's in RH_∞ such that

$$\| G + X \|_\infty \le \alpha. \tag{2.22}$$

We may assume that G is proper and analytic inside the stability boundary contour, i.e. $G \in RH_\infty^-$. Otherwise, factor G uniquely as

$$G = G_1 + G_2$$

$$G_1^*,\ G_2 \in RH_\infty,\ G_1 \text{ proper}.$$

So to solve the Nehari problem for G, solve it for G_1, i.e. find all X_1's in RH_∞ such that $\| G_1 + X_1 \|_\infty < \alpha$ and then set $X = X_1 + G_2$. Thus, without loss of generality, we can assume that $G(\gamma) \in RH_\infty^-$.

3. Main results

In section 2, some definitions and basic results for solving problem are given. In this section, we derive all solutions to Nehari problem in delta domain using all-pass properties and show that the solutions become to continuous form as sampling interval approaches to zero.

Theorem 3.1 Given $G(\gamma) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, $G \in RH_\infty^{-(n \times m)}$, such that

$$\|G^*\|_H < \alpha \tag{3.1}$$

a parameterization of unified all solutions to Nehari problem is given by the lower linear fractional transformation

$$X(\gamma) = F_L(Q(\gamma), K(\gamma)),\ X \in RH_\infty \tag{3.2}$$

where $K \in RH_\infty$, $\|K\|_\infty < \alpha^{-1}$

and $Q(\gamma) = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right]$

$$= \left[\begin{array}{c|cc} (-A^T - B_1B^T)\hat{A}^T & B_1 & R^{T-1}\hat{A}^TC^TD_{12} \\ \hline -CP - D_{11}B^T\hat{A}^T & D_{11} - D & D_{12} \\ -D_{21}B^T\hat{A}^T & D_{21} & 0 \end{array}\right] \tag{3.3}$$

where $R = PQ - \alpha^2 I$,

$B_1 = R^{T-1}(\hat{A}^TC^TD_{11} + QB)$,

and D_{11}, D_{12} and D_{21} are obtained as fllows:

$$D_{11} = -\Delta\alpha^2(I + \Delta C\hat{A}R^{-1}P\hat{A}^TC^T)^{-1}C\hat{A}R^{-1}B \tag{3.4}$$

$$D_{12}^T(I + \Delta C\hat{A}R^{-1}P\hat{A}^TC^T)D_{12} = \alpha^2 I \tag{3.5}$$

$$D_{21}^TD_{21} = -\Delta\alpha^2B^TR^{T-1}\hat{A}^TC^TD_{11} - \alpha^2(\Delta B^TQR^{-1}B - I). \tag{3.6}$$

The following lemma gives sufficient conditions for a TFM to be square all-pass.

Lemma 3.1 Given $G(\gamma) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ such that

$$AP + PA^T + BB^T + \Delta APA^T = 0, \; P = P^T \quad (3.7)$$
$$A^TQ + QA + C^TC + \Delta A^TQA = 0, \; Q = Q^T \quad (3.8)$$
$$DB^T + CP + \Delta CPA^T = 0 \quad (3.9)$$
$$D^TC + B^TQ + \Delta B^TQA = 0 \quad (3.10)$$
$$DD^T + \Delta CPC^T = I \quad (3.11)$$
$$D^TD + \Delta B^TQB = I \quad (3.12)$$
$$PQ = I \quad (3.13)$$

then $G^*G = GG^* = I$. (3.14)

Proof : The proof is shown at Appendix B

□ □ □

It will be necessary to determine the exact number of stable poles for certain matrices satisfying Lyapunov equations in proving Theorem 3.1. The connection between a solution to the Lyapunov equation and the poles of system are now stated.

Lemma 3.2 Given the nxn and nxm matrices A and B, there exists a symmtric matrix P satisfying

$$AP + PA^T + BB^T + \Delta APA^T = 0, \quad (3.15)$$

then

(i) there exists a unique solution to (3.15) if and only if $\lambda_i(A) + \lambda_j(A)^* + \Delta\lambda_i(A)\lambda_j(A)^* \neq 0 \;\forall i,j$

(ii) if $\delta(P) = 0$, then $\pi_d(A) \le \nu(P)$, $\nu_d(A) \le \pi(P)$

(iii) if $\delta_d(A) = 0$, then $\pi(P) \le \nu_d(A)$, $\nu(P) \le \pi_d(A)$

(iv) if (A,B) is controllable, then $\pi_d(A) = \nu(P)$, $\nu_d(A) = \pi(P)$ and $\delta_d(A) = \delta(P)$.

Proof : The proof follows from Theorem 3.3 of Glover(1984) by using a bilinear transformation in δ-domain.

□ □ □

Lemma 3.3 A_Q defined by Theorem 3.1 does not have any eigenvalues on the stability boudary contour, that is,

$$\delta_d(A_Q) = 0. \quad (3.16)$$

Proof : See Appendix C.

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Next lemma plays a key role to approximate the anticausal TFM $G(\gamma)$ by causal TFM $F(\gamma)$.

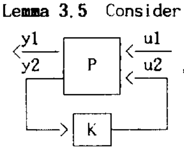
Lemma 3.4 Let $G(\gamma) \in RH_\infty^-$, then

$$\| G(\gamma)^* \|_H = \inf_{F \in RH_\infty} \| G + F \|_\infty. \quad (3.17)$$

The proof follows directly from Theorem 6.1 of Glover (1984).

The next lemma considers lower linear fractional transformations with all-pass matrices and is based on the work of Doyle([11], Lemma 15).

Lemma 3.5 Consider the following feedback system:



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad P_{12}, P_{21}, P_{22} \in RH_\infty$$

Suppose that $\alpha^{-1}P$ is all-pass, $P_{21}^{-1} \in RH_\infty$, $\|P_{22}\|_\infty < \alpha$ and K is a proper rational matrix. Then the following are equivalent.

(i) $F_L(P,K) \in RH_\infty$ and $\|T_{y_1u_1}\|_\infty \le \alpha$

(ii) $K \in RH_\infty$ and $\|K\|_\infty \le \alpha^{-1}$

The proof is derived similarily as that of [11].

Proof of Theorem 3.1 : We can now prove Theorem 3.1 from Lemma 3.1 - 3.5 (See Appendix A).

Now we can directly show that Theorem 3.1 represents a continuous form as sampling interval approaches to zero from (2.2) - (2.6) and following:

$$\hat{A} = I, \; D_{11} = 0 \text{ and } D_{12} = D_{21} = \alpha I \text{ as } \Delta \rightarrow 0. \quad (3.18)$$

4. Example

To illustrate the solution developed in the paper, the following example is taken. Let $G^{\sim}(s)$ be given by following state-space realization:

$$G^{\sim}(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right], \quad G^{\sim}(s) \in RH_\infty^-$$

where

$$A_c = \begin{bmatrix} 5.12 & 20.18 & 9.85 \\ -3. & -4. & -2. \\ 6.12 & 17.18 & 9.85 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 1. & 2. & 0. \\ 6.12 & 22.18 & 9.85 \end{bmatrix}, \quad D_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The delta model $G(\gamma)$ with $\Delta = 0.01$(sec) corresponding to $G^{\sim}(s)$ is given as follows:

$$G(\gamma) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \quad G(\gamma) \in RH_\infty^-$$

where

$$A = \begin{bmatrix} 5.24 & 21.16 & 10.35 \\ -3.08 & -4.41 & -2.20 \\ 6.28 & 18.24 & 9.39 \end{bmatrix} \quad B = \begin{bmatrix} 1.08 \\ -2.59e-2 \\ 1.08 \end{bmatrix}$$

$$C = \begin{bmatrix} 1. & 2. & 0. \\ 6.11 & 22.1 & 9.85 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The result by developed solution in Theorom 3.1 with $\alpha = 1$ is shown in Fig. 4.1.

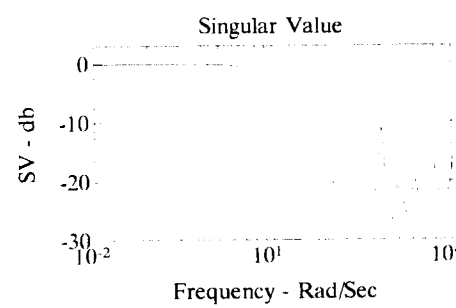


Figure 4.1. Singular values plots of (G + X) with $\alpha = 1$,
——— : $\Delta = 0.01$ sec
······ : $\Delta = 0$ sec.

5. Conclusion

We have proposed a method to obtain all solutions of the Nehari problem in delta domain, based on developed results by the authors for the one-block and Hankel-norm model reduction problems. Also, we have shown that the solutions are the unified form for the continuous and discrete-time Nehari problems, which eliminates the difficulty that we must solve the problem with each other method in continuous and discrete-time case.

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Appendix A

Proof of Theorem 3.1 :

First construct augmented system $G_a(\gamma) \in RH_\infty^{-(n+m)\times(n+m)}$ and $Q(\gamma) \in RH_\infty^{(n+m)\times(n+m)}$ such that $\alpha^{-1}[G_a(\gamma) + Q(\gamma)]$ is square all-pass, where

$$G_a(\gamma) = \begin{bmatrix} G(\gamma) & 0 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} A_a & B_a \\ \hline C_a & D_a \end{array}\right]. \quad (A.1)$$

Let $Q(\gamma)$ have state-space realization $\left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right]$

$$\text{then } G_a(\gamma) + Q(\gamma) = \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right] \quad (A.2)$$

where $A_e = \begin{bmatrix} A_a & 0 \\ 0 & A_Q \end{bmatrix}$, $B_e = \begin{bmatrix} B_a \\ B_Q \end{bmatrix}$, $C_e = [C_a \ C_Q]$, $D_e = D_a + D_Q$.

$\alpha^{-1}[G_a(\gamma) + Q(\gamma)]$ is square all-pass by lemma 2.1 if there exist symmetric matrices P_e and Q_e satisfying

$$A_eP_e + P_eA_e^T + B_eB_e^T + \Delta A_eP_eA_e^T = 0 \quad (A.3)$$
$$A_e^TQ_e + Q_eA_e + C_e^TC_e + \Delta A_e^TQ_eA_e = 0 \quad (A.4)$$
$$D_eB_e^T + C_eP_e + \Delta C_eP_eA_e^T = 0 \quad (A.5)$$
$$D_e^TC_e + B_e^TQ_e + \Delta B_e^TQ_eA_e = 0 \quad (A.6)$$
$$D_eD_e^T + \Delta C_eP_eC_e^T = \alpha^2 I \quad (A.7)$$
$$D_e^TD_e + \Delta B_e^TQ_eB_e = \alpha^2 I \quad (A.8)$$
$$P_eQ_e = \alpha^2 I. \quad (A.9)$$

Now let the two solutions to Lyapunov equations of $G(\gamma)$ or $G_a(\gamma)$ be P and Q, and hence satisfy (2.13) and (2.14), respectively.

Assuming dimension(A) = dimension(A_e), one solution to (A.9) is then,

$$P_e = \begin{bmatrix} P & I \\ I & QR^{-1} \end{bmatrix}, \quad Q_e = \begin{bmatrix} Q & -R^T \\ -R & RP \end{bmatrix}. \quad (A.10)$$

Now given any D_e satisfying (A.7) and (A.8), B_Q is obtained from (1,1) block of (A.6), C_Q from the (1,1) block of (A.5), and A_Q from (1,2) block Of (A.3) as,

$$B_Q = R^{T-1}(\hat{A}^TC_a^TD_e + QB_a) \quad (A.11)$$
$$C_Q = -C_aP - D_eB_a^T\hat{A}^T \quad (A.12)$$
$$A_Q = (-A^T - B_QB_a^T)\hat{A}^T. \quad (A.13)$$

For (A.10)-(A.13) the remaining block of (A.3)-(A.6) can be verified by long manupulation and hence omitted here.

Considering the inertia of QR^{-1} ,

$$\begin{aligned} In(QR^{-1}) &= In(RQ^{-1}) = In(-(Q_1P_1 - \alpha^2 I)P_1^{-1}) \\ &= In(-Q_1 + \alpha^2 P_1^{-1}) = In(-P_1^{1/2}Q_1P_1^{1/2} + \alpha^2 I) \\ &= In(diag(-\sigma_i^2 + \alpha^2 I)). \end{aligned} \quad (A.14)$$

where P_1, Q_1 are solutions to Lypunov equations of $G^*(\gamma)$ and equal to -Q and -P, respectively, and $\sigma_i^2 = \lambda_i(P_1Q_1)$, then by Lemma 3.4 QR^{-1} has all eigenvalues in RHP. Thus A_Q has all eigenvalues inside stability boundary contour by Lemma 3.2 and Lemma 3.3.

Now find De satisfying (A.6) or (A.7). Substituting (A.10)-(A.13) into (A.8),

$$D_e^T(I + \Delta\hat{A}^TR^{-1}P\hat{A}^TC^T)D_e + \Delta\alpha^2 D_e^TC\hat{A}R^{-1}B + \Delta\alpha^2 B^TR^{T-1}\hat{A}^TC^TD_e + \alpha^2(\Delta B^TQR^{-1}B - I) = 0. \quad (A.15)$$

and partitioning D_e with proper dimensions as

$$D_e = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} \quad (A.16)$$

then D_{11}, D_{12} and D_{21} satisfy (3.4) - (3.6).

To complete the proof of Theorem 3.1 with Lemma 3.5 we show finally that $Q_{21}^{-1} \in RH_\infty$ and $\|Q_{22}\|_\infty < \alpha$.

First, from (3.3), 'A' term of Q_{21}^{-1} is

$$(-A^T - R^{T-1}(\hat{A}^TC^TD_{11} + QB)B^T)\hat{A}^T + R^{T-1}(\hat{A}^TC^TD_{11} + QB)B^T\hat{A}^T = -A^T\hat{A}^T. \quad (A.17)$$

Since $|\lambda_i(A)|^2\Delta/2 + Re(\lambda_i(A)) > 0$,

$$Q_{21}^{-1} \in RH_\infty. \quad (A.18)$$

Next, since $Q_{12}^*Q_{12} + Q_{22}^*Q_{22} = \alpha^2 I$,

$$\|Q_{22}\|_\infty < \alpha \quad \text{if} \quad Q_{12}^*Q_{12} \neq 0 \text{ for all } \omega,$$

that is, $\det(Q_{12}((\exp(j\omega\Delta)-1)/\Delta)) \neq 0$ for all ω.

From (3.3), $\det(Q_{12})$ is

$$\begin{aligned} &\det(D_{12})\cdot\det\{\gamma I + [A^T + R^{T-1}(\hat{A}^TC^TD_{11}B^T + QBB^T)\hat{A}^T] \\ &\qquad - R^{T-1}\hat{A}^TC^T(CP + D_{11}B^T)\hat{A}^T\} \div \det(\gamma I - A_Q) \\ &= \det(D_{12})\cdot\det(\gamma I + A^T\hat{A}^T + R^{T-1}QBB^T\hat{A}^T - R^{T-1}\hat{A}^TC^TCP) \\ &\qquad \div \det(\gamma I - A_Q). \end{aligned} \quad (A.19)$$

where

$$\begin{aligned} &\det(\gamma I + A^T\hat{A}^T + R^{T-1}QBB^T\hat{A}^T - R^{T-1}\hat{A}^TC^TCP) \\ &= \det\{\gamma I + R^{T-1}[-(QAP + \Delta QAPA^T)\hat{A}^T - \alpha^2 A^T\hat{A}^T - \hat{A}^TC^TCP]\} \\ &= \det[\gamma I + R^T(-QA^TP - \alpha^2 A^T\hat{A}^T - \hat{A}^TC^TCP)] \end{aligned}$$

$= \det[\gamma I + R^{T-1}\hat{A}^T(A^TQP - \alpha^2 A^T)]$

$= \det(\gamma I + R^{T-1}\hat{A}^T A^T R^T).$ (A.20)

Since $\lambda_i(-\hat{A}^T A^T)$ are not on stability boundary contour

$\det(Q_{12}) \neq 0$ if $\det(D_{12}) \neq 0.$ (A.21)

This completes the proof.

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Apendix B

Proof of Lemma 3.1 : First show that $G(\gamma)^*G(\gamma) = I$. From (2.16), the state-space realization for G^*G is

$$G^*G = \left[\begin{array}{cc|c} -A^T + \Delta A^T\hat{A}^T A^T & C^TC - \Delta A^T\hat{A}^T C^TC & C^TD - \Delta A^T\hat{A}^T C^TD \\ 0 & A & B \\ \hline -B^T + \Delta B^T\hat{A}^T A^T & D^TC - \Delta B^T\hat{A}^T C^TC & D^TD - \Delta B^T\hat{A}^T C^TD \end{array}\right]. \quad (B.1)$$

Applying a state similarity transformation on (B.1) by $\begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix}$ with $Q = Q^T$ and setting (3.8) and (3.10), then

$G^*G = D^TD - \Delta B^TA^TC^TD$, and hence if

$D^TD - \Delta B^TA^TC^TD = I$ (B.2)

equivalently, from (3.10) and (B.2)

$D^TD + \Delta B^TQB = I,$ (B.3)

then $G^*G = I$.

Next we can show that $GG^* = I$ similarily as above by a state similarity transformation $\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix}$ with $P = P^T$.

Finally , we complete the proof to be square all-pass, that is, $G^*G = GG^* = I$, by showing $PQ = I$. Premultiplying (3.10) by D and then substituting (3.11), then we have that

$C(I - PQ) = 0.$ (B.4)

Thus $PQ = I$. This completes the proof.

□ □ □

Appendix C

Proof of Lemma 3.3 : We prove our claim by contradictions in two step using PBH test for controllibility.

i) A_Q does have not any uncontrollable mode on the stability boundary contour : Assume (A_Q, B_Q) has any uncontrollable mode on the stability boundary contour. Then there exist a λ and a vector x such that

$x^*A_Q = x^*(-A^T - B_1B^T)\hat{A}^T = \lambda x^*$. (C.1)

$x^*B_Q = x^*[B_1 \quad R^{T-1}\hat{A}^TC^TD_{12}] = [0 \quad 0].$ (C.2)

From (C.1) and (C.2) we conclude that

$-x^*A^T\hat{A}^T = \lambda x^*$ (C.3)

which is contradiction with the fact that $A^T\hat{A}^T$ does have not any eigenvalues on the stability boundary contour. Thus, we conclude that (A_Q, B_Q) does not have any uncontrollable mode on the stability boundary contour.

ii) A_Q does not have any controllable mode on the stability boundary contour : Assume (A_Q, B_Q) has any controllable mode on the stability boundary contour. Then there does not exist any λ and any vector x such that

$x^*A_Q = \lambda x^*,$ (C.4)

$x^*B_Q = [0 \quad 0].$ (C.5)

Take x satisfying (C.4) and (C.5), and multiply P-Lyapunov equation (C.6) by x^* from the left and x from the right.

$A_QQR^{-1} + QR^{-1}A_Q^T + B_QB_Q^T + \Delta A_QQR^{-1}A_Q^T = 0$ (C.6)

Then, we get

$x^*A_QQR^{-1}x + x^*QR^{-1}A_Q^Tx + x^*B_QB_Q^Tx + x^*\Delta A_QQR^{-1}A_Q^Tx$

$= \lambda x^*QR^{-1}x + \lambda^* x^*QR^{-1}x + x^*B_QB_Q^Tx + \Delta\lambda\lambda^* x^*QR^{-1}x$

$= (\lambda + \lambda^* + \Delta\lambda\lambda^*)x^*QR^{-1}x + x^*B_QB_Q^Tx$

$= 0.$ (C.7)

Since $(\lambda + \lambda^* + \Delta\lambda\lambda^*) = 0$, (C.7) implies that

$x^*B_Q = 0$ (C.8)

which contradicts the assumption and thus (A_Q, B_Q) does not have any controllable mode.

□ □ □