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Mixed H_2/H_∞ Robust Control With Diagonal Structured Uncertainty

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Abstract

Mixed H_2/H_∞ robust control synthesis is considered for finite dimensional linear time-invariant systems under the presence of diagonal structured uncertainties. Such uncertainties arise for instance when there is real perturbation in the nominal model of the state space system or when modeling multiple (unstructured) uncertainty at different locations in the feedback loop. This synthesis problem is reduced to convex optimization problem over a bounded subset of matrices as well as diagonal matrix having certain structure. For computational purpose, this convex optimization problem is further reduced into Generalized Eigenvalue Minimization Problem where a powerful algorithm based on interior point method has been recently developed.

1 Introduction

Mixed H_2/H_∞ control theory offers a way of combining disturbance attenuation system which is guaranteed by H_∞-norm of a certain closed-loop transfer function, and quadratic performance which is measured by H_2-norm of another transfer function[2,3,5,7]. In this control problem we seek control-laws that minimize a quadratic performance index subject to an H_∞ constraint. Due to inherent conservativeness of H_∞-norm measure for robust stability[11,19], however, mixed H_2/H_∞ design may not be of practical use when the designer know some of the structure of uncertainty. This form of uncertainty arises for example when there is real perturbation in the nominal model of the state space system or when modeling multiple (unstructured) uncertainty at different locations in the feedback loop[11,13,16]. As far as the authors concern, there exists only a few works on mixed H_2/H_∞ control theory that were also devoted to deal with structured uncertainties. The first is the work of Yeh *et al.*[9] which extend the standard mixed H_2/H_∞ to include real parameter variations based on surrogate system concept. The second is the work of Madiwale[6] which discussed the problem of computing optimal values of a real block diagonal scaling matrix to reduce conservatism in the standard mixed H_2/H_∞ control theory, and may be considered as time domain approach to μ synthesis. The results of Madiwale however, requires iterations for updating scaling matrix where in each iteration a coupled Riccati equations need to be solved. At present, there is no efficient method for solving coupled Riccati equations other than homotopic continuation method. The approach of Yeh *et al.* also requires solving coupled Riccati equations.

In this paper, similar problem as that of Madiwale is discussed, but different approach is employed. We show that mixed H_2/H_∞ state feedback problem with diagonal structured uncertainties can be reduced to the convex optimization problem over a bounded subset of symmetric matrices as well as diagonal matrices with certain structure, avoiding solving coupled Riccati equations.

By employing solution to a generalized Riccati equation, an upperbound to quadratic performance is defined in order to enforce H_∞-norm constraint for all admissible diagonal structured perturbations. By a change of variable technique, we reduce the above problem into convex optimization problem over bounded subset of symmetric matrices and block diagonal matrix having certain structure. Due to convexities established in this paper, we can adopt any optimization method with global optimality properties.

Now, let us outline the content of this paper. In Section 2, we formulate mixed H_2/H_∞ robust control problem with diagonal structured uncertainty, and present some preliminaries. In Section 3, we will show by a change of variable technique, that this problem can be reduced to a convex optimization problem involving a Riccati equation and a diagonal structured matrix as the problem constraints. For computational purpose, in Section 4 we will further reduce the resulting convex optimi zation problem in to Generalized Minimization Problem where an Interior Point method has been developed recently to find its solution. Finally, in Section 5 we draw some conclusion of the present paper.

2 Problem Formulation and Preliminaries

2.1 Problem Formulation

In this subsection we formulate mixed H_2/H_∞ control problem with diagonal structured uncertainties. Consider finite-dimensional time-invariant linear systems described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1u_1(t) + B_2u_2(t) + B_3u_3(t) \\ z_1(t) &= C_1x(t) + D_{12}u_2(t) \\ z_2(t) &= C_2x(t) + D_{22}u_2(t) \\ z_3(t) &= C_3x(t) + D_{32}u_2(t) \\ y(t) &= x(t) \\ u_3(t) &= \Delta z_3(t), \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbf{R}^{n_x}$ is the state; $u_1(t) \in \mathbf{R}^{n_{u_1}}$ is the disturbance; $u_2(t)$ is the control; $u_3(t) \in \mathbf{R}^{n_{u_3}}$ is the fictitious input; $z_1(t) \in \mathbf{R}^{n_{z_1}}$ is the performance variable associated with H_∞ constraints; $z_2(t) \in \mathbf{R}^{n_{z_2}}$ is the performance variable associated with H_2 criterion; $y(t) \in \mathbf{R}^{n_y}$ is the measurement; and $z_3(t) \in \mathbf{R}^{n_{z_3}}$ is the fictitious output. In the description (2.1), $n_{u_3} = n_{z_3}$. Let (A, B_2) be stabilizable. Assume that the perturbation matrix Δ consists of repeated scalar blocks and full blocks. Thus, Δ is a subset of the block structure $\boldsymbol{\Delta}$(a prescribed set of block diagonal matrices) defined by:

$$\boldsymbol{\Delta} \;:=\; \{diag[d_1I_{r_1}, ..., d_sI_{r_s}, \Delta_{n_1}, ..., \Delta_{n_f}] :$$

$$d_j \in \mathbf{C}, \ \ \Delta j \in \mathbf{C}^{n_j \times n_j}\}, \tag{2.2}$$

where $\bar{\sigma}$ denotes maximum singular value. The two integers s and f in the above expression represent scalar and full blocks, respectively. The $i'th$ scalar blocks is $r_i \times r_i$, while the $j'th$ full blocks is $n_j \times n_j$. In other words, Δ takes the form of complex-valued, block diagonal perturbations, comprised of $d_i I_{ri}$ repeated scalar blocks, and Δ_{n_j} full blocks. The fullblocks can arise from multivariable neglected dynamics, while the repeated scalar block may represent affine parametric uncertainty in the nominal description of the system (2.1). We will need the bounded subsets of Δ, and we introduce the following notation

$$\mathbf{B\Delta} = \{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) \leq 1\}. \tag{2.3}$$

Then for uncertainty block $\Delta \in \mathbf{\Delta}$ to be the subset of $\mathbf{B\Delta}$, we will require in (2.2) that $|d_j| \leq 1$ and $\bar{\sigma}(\Delta_j) \leq 1$. In this paper, we will assume that we were given the plant description (2.1) (see [11,13,19] for the discussion on how the affine parametric uncertainty and/or unmodeled dynamic can be represented in the form of equation (2.1)).

The mixed H_2/H_∞ robust control problem considered in this paper is formulated as follows: for the plant given by (2.1), determine state feedback controller described by

$$u_2(t) = Kx(t) \tag{2.4}$$

such that the following design criteria are satisfied,

1. If $u_1(t)$ is an L_2 deterministic signal, the closed-loop transfer function from $u_1(t)$ to $z_1(t)$ satisfies

$$\sup_{\Delta \in \mathbf{B\Delta}} \|T_{z_1 u_1}(s)\|_\infty < 1; \tag{2.5}$$

2. If $u_1(t)$ is a white noise signal with unit strength, H_2 performance criterion defined by

$$J := \lim_{T\to\infty} \mathcal{E}\{\frac{1}{T}\int_0^T [z_2(t)' z_2(t)]dt\} \tag{2.6}$$

is minimized, where $\mathcal{E}$ denotes the expectation;

3. The closed-loop system is asymptotically stable for all perturbations $\Delta \in \mathbf{B\Delta}$.

Suppose $T_{z_3 u_3}(s)$ denotes the transfer function with compensator loop closed and uncertainty loop $u_3 \to y_3$ open. Then by the small gain theorem, the stability condition (3) is implied by

$$\|M T_{z_3 u_3} M^{-1}\|_\infty < 1, \tag{2.7}$$

where M is a scaling matrix that is commute with Δ,

$$\begin{aligned} M \ := \ & \{diag[M_1, ..., M_s, m_1 I_{n_1}, ..., m_f I_{n_f}] : \ M_i \in \mathbf{C}^{r_i \times r_i}, \\ & M_i = M_i^* > 0, m_j \in \mathbf{R}, m_j > 0\}. \end{aligned} \tag{2.8}$$

With the aid of scaling parameter M, however, the condition in (2.7) gives less conservative test for robust stability than the familiar infinity norm measure derived from the small gain theorem.

The closed-loop system can be written as

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}_1 u_1 + \tilde{B}_3 u_3 \\ z_1 = \tilde{C}_1 \tilde{x}, \ \ & z_2 = \tilde{C}_2 \tilde{x}, \ \ z_3 = \tilde{C}_3 \tilde{x} \\ u_3 &= \Delta z_3, \end{aligned} \tag{2.9}$$

where

$$\tilde{A} = A + B_2 K, \ \ \tilde{B}_1 = B_1, \ \ \tilde{B}_3 = B_3$$
$$\tilde{C}_1 = C_1 + D_{12}K, \ \ \tilde{C}_2 = C_2 + D_{22}K, \ \ \tilde{C}_3 = C_3 + D_{32}K.$$

Suppose that the nominal closed-loop system (2.9) is internally stable. Then under the absence of diagonal structured uncertainty, H_2 performance in (2.5) can be expressed as

$$J = \|T_{z_2 u_1}\|_2^2 = tr\ [\tilde{C}_2' \tilde{C}_2 P], \tag{2.10}$$

where P is positive definite solution to Lyapunov equation

$$\tilde{A}P + P\tilde{A}' + \tilde{B}_1 \tilde{B}_1' = 0. \tag{2.11}$$

2.2 Preliminaries

In this subsection, we collect some results which are useful in developing convex programming for the above synthesis problem.

The following well-known lemma characterizes H_∞-norm bound of a transfer function matrix in terms of a Riccati equation.

Lemma 2.1 *Let transfer function matrix $G(s) := C(sI-A)^{-1}B$ be given, and suppose that $\|G(s)\|_\infty < \gamma$. Then there exists $X = X' \geq 0$ satisfying*

$$AX + XA' + \gamma^{-2} XC'CX + BB' = 0 \tag{2.12}$$

such that $[A + \gamma^{-2} BB'P]$ is asymptotically stable.

Next, we present some known results from μ analysis[11,19]. For this purpose, consider interconnection depicted in Figure 1. $G(s)$ can be partitioned so that input-output map from u_1 to z_1 can be expressed as the following linear fractional transformation(LFT)

$$z_A = F_l(G, \Delta) u_A,$$

where

$$F_l(G,\Delta) := G_{11} + G_{12}\Delta(I - G_{22}\Delta)^{-1} G_{21}$$
$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}.$$

The structured singular value $\mu(G(j\omega))$ of a complex matrix $G(j\omega)$ with respect to the block structure $\mathbf{\Delta}$ is based on Multivariable Nyquist theorem and is defined as

$$\mu((G(j\omega)) = \left\{ \begin{array}{c} \{\min_{\Delta \in \mathbf{\Delta}} [\bar{\sigma}(\Delta) : \ det(I + G(j\omega)\Delta) = 0]\}^{-1} \\ or\ \ 0, \ det(I + G(j\omega)\Delta) \neq 0 \end{array} \right\}.$$

This says that μ is the inverse of the smallest magnitude of a destabilizing perturbation of G. Robust stability and performance is demonstrated in terms of μ in the following theorem, the proof of which can be found in [11,19].

Theorem 2.1

1. **Robust Stability**

$F_l(G,\Delta)$ is stable for all $\Delta \in \mathbf{B\Delta}$ iff $\sup_\omega \mu(G_{22}(j\omega)) \leq 1$, where μ is computed with respect to the structure $\mathbf{\Delta}$.

2. **Robust Performance**

$F_l(G,\Delta)$ is stable and $\|F_l(G,\Delta)\|_\infty \leq 1$ for all $\Delta \in \mathbf{B\Delta}$ iff $\sup_\omega \mu(G(j\omega)) \leq 1$ where here μ is computed with respect to augmented structure $\tilde{\mathbf{\Delta}} = \{diag\{\Delta, \Delta_P\}\}, \Delta \in \mathbf{\Delta}$.

The above theorem demonstrates that stability and (H_∞) performance robustness is guaranteed by a necessary and sufficient stability condition involving μ. See Figure 3.

To this end, let us restrict the matrix M_i in (2.8) to be real symmetric matrices[20]. The following proposition gives a condition that guarantee the satisfaction of closed-loop stability and H_∞ performance bound for all structured perturbations.

Proposition 2.1 *Consider the closed-loop system described in (2.9), with Δ given in (2.2). Suppose that the following condition is satisfied,*

$$\|\hat{M}\hat{C}(sI - \tilde{A})^{-1}\hat{B}\hat{M}^{-1}\|_\infty < 1 \quad (2.13)$$

where

$$\hat{M} := \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \; \hat{C} := \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_3 \end{bmatrix}, \; \hat{B} := \begin{bmatrix} \tilde{B}_1 & \tilde{B}_3 \end{bmatrix}.$$

Then,

1. *the closed-loop system is asymptotically stable for all perturbations $\Delta \in \mathbf{B\Delta}$;*
2. *the worst-case disturbance attenuation satisfies*

$$\sup_{\Delta\in\mathbf{B\Delta}} \|T_{z_1u_1}\|_\infty < 1;$$

3. *there exists $\tilde{P} \geq 0$ such that*

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \tilde{B}_1\tilde{B}_1' + \tilde{P}\tilde{C}_3'W\tilde{C}_3\tilde{P} + \tilde{B}_3W^{-1}\tilde{B}_3' = 0 \quad (2.14)$$

and such that $[\tilde{A} + (\tilde{B}_1\tilde{B}_1' + \tilde{B}_3(M'M)^{-1}\tilde{B}_3')\tilde{P}]$ is asymptotically stable, where $W := M'M$.

Proof
The condition

$$\|\hat{M}\hat{C}(sI - \tilde{A})^{-1}\hat{B}\hat{M}^{-1}\|_\infty < 1$$

implies that

$$\sup_\omega \bar{\sigma}(\hat{M}\hat{G}(j\omega)\hat{M}^{-1}) < 1,$$

where $\hat{G} := \hat{C}(sI - \tilde{A})^{-1}\hat{B}$. It is well known that the last condition is an upperbound to μ[11], and thus

$$\sup_\omega \mu(\hat{G}(j\omega)) < 1,$$

where μ is computed with respect to the structure $\mathbf{B\Delta} = \{diag\{\Delta, \Delta_P\}\}$. See Figure 3, with $G \leftarrow \hat{G}, u_A \leftarrow u_1, u_B \leftarrow u_3, z_A \leftarrow z_1$, and $z_B \leftarrow z_3$ Now, in view of Theorem 2.1, we conclude that $F_l(\hat{G}, \Delta)$ is stable and $\|F_l(\hat{G},\Delta)\|_\infty < 1$ for all $\Delta \in \mathbf{B\Delta}$. Thus, the closed-loop system depicted in Figure 1 is asymptotically stable for all $\Delta \in \mathbf{B\Delta}$ and the worst case disturbance attenuation satisfies (2.5).

Now, by Lemma 2.1, the H_∞-norm bound (2.13) implies the existence of positive semidefinite matrix $\hat{P} = \hat{P}' \geq 0$ satisfying

$$\tilde{A}\hat{P} + \hat{P}\tilde{A}' + \hat{P}\hat{C}'\hat{M}'\hat{M}\hat{C}\hat{P} + \hat{B}(\hat{M}'\hat{M})^{-1}\hat{B}' = 0$$

such that $[\hat{A} + \hat{B}(\hat{M}'\hat{M})^{-1}\hat{B}'\hat{P}]$ is asymptotically stable. This in turn implies the existence of positive semidefinite matrix $\tilde{P} = \tilde{P}' \geq 0$ satisfying

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \tilde{B}_1\tilde{B}_1' + \tilde{P}\tilde{C}_3'W\tilde{C}_3\tilde{P} + \tilde{B}_3W^{-1}\tilde{B}_3' = 0$$

such that $[\tilde{A} + (\tilde{B}_1\tilde{B}_1' + \tilde{B}_3(M'M)^{-1}\tilde{B}_3')\tilde{P}]$ is asymptotically stable.

□

For convenience in stating some of the results of this paper, let us define

$$R(M,\tilde{P}) := \tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \tilde{B}_1\tilde{B}_1' + \tilde{P}\tilde{C}_3'W\tilde{C}_3\tilde{P} + \tilde{B}_3W^{-1}\tilde{B}_3'. \quad (2.15)$$

Suppose that the condition in Proposition 2.1 is satisfied. Then, the following conditions can be easily verified[3,7],

$$0 \leq P \leq \tilde{P} \leq \hat{P}, \quad (2.16)$$
$$J \leq \tilde{J} := tr[\tilde{C}_2'\tilde{C}_2\hat{P}], \quad (2.17)$$

where $\hat{P}$ denotes any real symmetric solution to the Riccati inequality $R(M,\hat{P}) < 0$, with R defined by (2.15). Note that $\tilde{J}$, which is given in terms of solution to Riccati equation $R(M,\tilde{P}) = 0$, is an upperbound to the quadratic cost J. Instead of minimizing the quadratic cost it self, we will minimize this upperbound in our optimization problem defined later.

The following results can be established in the same way as that of [3, Lemma 2.1], the different being that in the present paper Riccati equation involved in the definition of the upper bound involves a scaling matrix, as well as an additional quadratic term.

Lemma 2.2 *Consider the closed-loop system described in (2.9) and let T_{zu} denote the transfer function matrix from (u_1,u_3) to (z_1,z_2,z_3). Suppose that $\|\hat{M}\hat{C}(sI - \tilde{A})^{-1}\hat{B}\hat{M}^{-1}\|_\infty < 1$. Then*

$$J(T_{zu}) = inf\{tr(\tilde{C}_2'\tilde{C}_2\tilde{P}) : \tilde{P} = \tilde{P}' > 0 \;\; such \;\; that \;\; R(M,\tilde{P}) < 0\}. \quad (2.18)$$

Now, let the transfer function of the plant (2.1) be denoted by $\mathcal{P}$. The transfer function of the overall closed-loop system will be denoted by

$$T_{zu} := \begin{bmatrix} T_{z_1u_1} \\ T_{z_2u_1} \\ T_{z_3u_3} \end{bmatrix}.$$

We call a controller K *admissible* if K internally stabilizes the plant $\mathcal{P}$ *for all structured perturbations* $\Delta \in \mathbf{B\Delta}$. Introduce the following sets :

$$\begin{aligned} \mathcal{A}(\mathcal{P}) &:= \{K : K \text{ is admissible}\} \\ \mathcal{A}_\infty(\mathcal{P}) &:= \{K \in \mathcal{A}(\mathcal{P}) : \; \|\hat{M}\hat{C}(sI - \tilde{A})^{-1}\hat{B}\hat{M}^{-1}\|_\infty < 1, \\ &\quad \hat{M} = \{diag\{I, M\}\}, \;\; M \in \mathcal{M}\}. \end{aligned} \quad (2.19)$$

In the above expression, $\mathcal{M}$ is the set of matrices commuting with all elements of Q, where Q is the subset of $\mathbf{C}^{n_{u_3}\times n_{u_3}}$ describing the uncertainty structure, and every element Δ of Q has the form (2.2). Thus, every element M of $\mathcal{M}$ has the form (2.8).

In view of Proposition 2.1 and Lemma 2.2, we consider the following synthesis problem which may be considered as an extension of "suboptimal H_2/H_∞ controller synthesis" introduced by Khargonekar and Rotea[3] to the finite dimensional plant under the presence of input diagonal structured uncertainty.

Synthesis Problem: "Compute the mixed performance measure

$$\theta_m(\mathcal{P}) := inf\{J(T_{zu}) : K \in \mathcal{A}_\infty(\mathcal{P})\}, \quad (2.20)$$

and, given any $\theta > \theta_m$, find a controller $K \in \mathcal{A}_\infty(\mathcal{P})$ such that $J(T_{zu}) < \theta$".

3 Convex Optimization Approach

In this section we will develop a convex optimization approach for solving the controller synthesis problem introduced above. Motivated by the result of [3], where it is proved that all memoryless state feedback mixed H_2/H_∞ controllers cannot be improved upon by the use of dynamic "full information" controllers, we are interested in the computation of constant state feedback matrices for the minimization of $J(\mathcal{P},K)$. The set of such controllers will be denoted by

$$\mathcal{A}_{\infty,m}(\mathcal{P}) := \{K \in \mathcal{A}_\infty(\mathcal{P}) : K \in \mathbf{R}^{n_{u_2}\times n_y}\}. \quad (3.1)$$

It will be shown that the optimal performance $\theta_m(\mathcal{P})$ defined in (2.20) is the value of (finite dimensional) convex optimization problem. Further, given any $\theta > \theta_m$, one can find K such that $J(\mathcal{P}, K) < \alpha$ by solving a convex programming problem.

Let Ξ denote the set of $n_x \times n_x$ real symmetric matrices, and define

$$\Omega := \{(X, M, \tilde{P}) \in \mathbf{R}^{n_{u_2} \times n_x} \times \mathbf{R}^{n_{u_3} \times n_{u_3}} \times \Xi : \tilde{P} > 0, M \in \mathcal{M}\}. \quad (3.2)$$

Observe that Ω is an open strictly convex subset of $\mathbf{R}^{n_{u_2} \times n_x} \times \mathbf{R}^{n_{u_3} \times n_{u_3}} \times \Xi$. Given $(X, M, \tilde{P}) \in \Omega$, define

$$f(X, M, \tilde{P}) := tr[(C_2\tilde{P} + D_{22}X)\tilde{P}^{-1}((C_2\tilde{P} + D_{22}X)'] \quad (3.3)$$

and, for $(X, M, \tilde{P}) \in \mathbf{R}^{n_{u_2} \times n_x} \times \mathbf{R}^{n_{u_3} \times n_{u_3}} \times \Xi$, let

$$\begin{aligned}\hat{R}(X, M, \tilde{P}) &:= A\tilde{P} + \tilde{P}A' + B_2X + X'B_2' + B_1B_1' \\ &+(C_1\tilde{P} + D_{12}X)'(C_1\tilde{P} + D_{12}X)) + B_3(M'M)^{-1}B_3' \\ &+(MC_3\tilde{P} + D_{32}X)'(MC_3\tilde{P} + D_{32}X). \end{aligned} \quad (3.4)$$

Define also the set of real matrices:

$$\Phi(\mathcal{P}) := \{(X, M, \tilde{P}) \in \Omega : \hat{R}(X, M, \tilde{P}) < 0\} \quad (3.5)$$

and consider the optimization problem

$$\tau(\mathcal{P}) := inf\{f(X, M, \tilde{P}) : (X, M, \tilde{P}) \in \Phi(\mathcal{P})\}. \quad (3.6)$$

Theorem 3.1 *Consider the plant $\mathcal{P}$ defined in (2.1). Let T_{zu} denote its transfer matrix, and $\mathcal{A}_{\infty,m}(T_{zu})$ denote the set of controllers defined in (3.1). Let $\Phi(\mathcal{P})$ be given by (3.5). Let θ_m and $\tau(\mathcal{P})$ be as defined in (2.20) and (3.6), respectively. Then,*

$$\mathcal{A}_{\infty,m}(\mathcal{P}) \neq \emptyset \quad (3.7)$$

if, and only if,

$$\Phi(\mathcal{P}) \neq \emptyset \quad (3.8)$$

with $\emptyset$ denote empty set. In this case,

$$\theta_m(\mathcal{P}) = \tau(\mathcal{P}). \quad (3.9)$$

Furthermore, given any $\alpha > \theta_m(\mathcal{P})$, there exists $(X, M, \tilde{P}) \in \Phi(\mathcal{P})$ such that the state feedback gain $K := X\tilde{P}^{-1}$ satisfies

$$K \in \mathcal{A}_{\infty,m}(\mathcal{P}) \text{ and } J(T_{zu}, K) \leq f(X, M, \tilde{P}) < \alpha. \quad (3.10)$$

Proof

First, we will show that if $\Phi(\mathcal{P}) \neq \emptyset$ then $\mathcal{A}_{\infty,m}(\mathcal{P}) \neq \emptyset$, and $\theta_m(\mathcal{P}) \leq \tau(\mathcal{P})$. Suppose that $\epsilon > 0$ is given. From the definition of $\tau(\mathcal{P})$, it follows that there exists a $(X, M, \tilde{P}) \in \Omega$ such that

$$f(X, M, \tilde{P}) \leq \tau(\mathcal{P}) + \epsilon, \quad \hat{R}(X, M, \tilde{P}) < 0$$

where $f(X, M, \tilde{P})$ and $\hat{R}((X, M, \tilde{P})$ are defined in (3.3) and (3.4), respectively. We will construct state feedback matrix $K \in \mathcal{A}_{\infty,m}(\mathcal{P})$ such that $J(T_{zu}, K) \leq f(X, M, \tilde{P}) < \alpha$. Define the real matrix $K := X\tilde{P}^{-1}$, then closing the loop, we have the closed-loop system described in (2.9). Using standard algebraic manipulation, it can be easily verified that

$$\begin{aligned}\hat{R}(X, M, \tilde{P}) &= \tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \tilde{B}_1\tilde{B}_1' + \tilde{P}\tilde{C}_3'W\tilde{C}_3\tilde{P} \\ &+\tilde{B}_3W^{-1}\tilde{B}_3' =: R(M, \tilde{P}), \end{aligned} \quad (3.11)$$

and that

$$f(X, M, \tilde{P}) = tr(\tilde{C}_2'\tilde{C}_2\tilde{P}). \quad (3.12)$$

Since $\tilde{P} > 0$ satisfies $\hat{R}(X, M, \tilde{P}) < 0$, it is standard fact that $\tilde{A}$, i.e. the nominal closed-loop system, is asymptotically stable, and that $\|\hat{M}\hat{C}(sI - \tilde{A})^{-1}\hat{B}\hat{M}^{-1}\|_\infty < 1$. Furthermore, by Proposition 2.1 the last condition implies that the perturbed closed-loop system is also asymptotically stable for all $\Delta \in \mathbf{B\Delta}$. Therefore, $K \in \mathcal{A}_{\infty,m}(\mathcal{P})$. In view of Lemma 2.2, we have $J(T_{zu}, K) \leq tr(\tilde{C}_2'\tilde{C}_2P)$. Hence,

$$\theta_m(\mathcal{P}) \leq J(T_{zu}, K) \leq f(X, M, \tilde{P}) \leq \tau(\mathcal{P}) + \epsilon.$$

Since ϵ is arbitrary positive real number, then $\theta_m(\mathcal{P}) \leq \tau(\mathcal{P})$.

Next we will show that if $\mathcal{A}_{\infty,m}(\mathcal{P}) \neq \emptyset$, then $\Phi(\mathcal{P}) \neq \emptyset$, and $\theta_m(\mathcal{P}) = \tau(\mathcal{P})$. Let ϵ be given. From the definition of $\theta_m(\mathcal{P})$ in (2.20), there exists $K \in \mathcal{A}_{\infty,m}(\mathcal{P})$ such that

$$J(T_{zu}, K) \leq \theta_m(\mathcal{P}) + \epsilon/2.$$

Using the controller K, the closed-loop system is given by (2.9). It follows from Lemma 2.2 that there exists $\tilde{P} = \tilde{P} > 0$ such that

$$\begin{aligned} tr(\tilde{C}_2'\tilde{C}_2\tilde{P}) &\leq J(T_{zu}, K) + \epsilon/2 \leq \theta_m(\mathcal{P}) + \epsilon, \\ R(M, \tilde{P}) &= \tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \tilde{B}_1\tilde{B}_1' \\ &+\tilde{P}\tilde{C}_3'W\tilde{C}_3\tilde{P} + \tilde{B}_3W^{-1}\tilde{B}_3' < 0. \end{aligned}$$

Define $X := K\tilde{P}$. Then,

$$(X, M, \tilde{P}) \in \Omega \text{ and } \hat{R}(X, M, \tilde{P}) = R(M, \tilde{P}) < 0$$

It follows that $(X, M, \tilde{P}) \in \Phi(\mathcal{P})$ and from (3.3), $f(X, M, \tilde{P}) = tr(\tilde{C}_2'\tilde{C}_2\tilde{P})$. Then, we have

$$f(X, M, \tilde{P}) \leq \theta_m(\mathcal{P}) + \epsilon.$$

Again, since ϵ is arbitrary positive number, we conclude that $\theta_m(\mathcal{P}) \geq \tau$.

The last part of this theorem follows immediately from the definitions and the construction for K.

□

From Theorem 3.1, it follows that the computation of $\tau(\mathcal{P})$ involves a search over the set $\Phi(\mathcal{P})$, where X, M, and $\tilde{P}$ serve as the decision variables. On the other hand $\theta_m(\mathcal{P})$ is computed by solving nonlinear programming problem with only the real matrix K as the decision variable. Furthermore, the set of feasible static feedback gains, $\mathcal{A}_{\infty,m}(\mathcal{P})$ does not necessarily convex, and therefore the original optimization problem for mixed H_2/H_∞ controller synthesis does not necessarily convex. We will show that the optimization problem defined in (3.6) is indeed a *convex problem.*

Theorem 3.2 *Let f and Φ be as defined in (3.3) and (3.5), respectively, and consider the optimization problem (3.6). Then, the set Φ is convex and the function $f : \Phi \to \mathbf{R}^+$ is convex and real analytic. Consequently, the optimization problem defined in (3.6) is convex.*

Proof

The fact that f is a real analytic function on the open set Ω follows immediately from (3.3). The convexity of f can be established in the same manner as that of Theorem 4.1 of [3].

The convexity of Φ is derived by showing that $\hat{R}(X, M, \tilde{P}) : \mathbf{R}^{n_{u_2} \times n_x} \times \mathcal{M} \times \Sigma \to \Sigma$ is a convex mapping(with respect to the cone of positive semidefinite matrices). Let us rewrite $\hat{R}$ as

$$\hat{R}(X, M, \tilde{P}) = \Phi_1(X, M, \tilde{P}) + \Phi_2(X, M, \tilde{P}), \quad (3.13)$$

where

$$\begin{aligned}\Phi_1(X, M, \tilde{P}) &:= A\tilde{P} + \tilde{P}A' + B_2X + X'B_2 + B_1B_1' \\ &\quad (C_1\tilde{P} + D_{12}X)'(C_1\tilde{P} + D_{12}X) \\ \Phi_2(X, M, \tilde{P}) &:= (C_3\tilde{P} + D_{32}X)'M'M(C_3\tilde{P} + D_{32}X) \\ &\quad +B_3(M'M)^{-1}B_3'. \end{aligned}$$

The convexity of Φ_1 has been established in [3]. Therefore, to prove the convexity of Φ it remains to show that Φ_2 is also convex in the domain $(X, M, \tilde{P})$.

Let us define $\tilde{X} := (C_3\tilde{P} + D_{32}X)$. Then

$$\Phi_2(X, M, \tilde{P}) = \Phi_2(\tilde{X}, W) = \tilde{X}'W\tilde{X} + B_3W^{-1}B_3'.$$

In view of Proposition E.7.f on p. 459 of[1], the mappings from $(\tilde{X}, W) \to \tilde{X}'W\tilde{X}$, and $(\tilde{X}, W) \to B_3W^{-1}B_3'$, are convex. The map $(X, \tilde{P}) \to \tilde{X}$ is obviously convex due to linearity of $\tilde{X}$ in the variables X and $\tilde{P}$. Again by Proposition E.7.f of[1] the map $M \to W$ is convex. Therefore, $\Phi_2(X, M, \tilde{P})$ is a convex mapping, and the convexity of Φ follows from the convexity of Φ_1 and Φ_2.

Now, since the set Ω defined in (3.2) is convex, from the fact that the level sets of a convex mapping are convex, it follows that the set Φ defined in (3.5) is convex. Since the objective function $f(.)$ is convex on $\Omega \supset \Phi$, we conclude that the optimization problem defined in (3.6) is a convex problem.

□

Remark 3.1

Under certain condition that is counterpart of that of Lemma 4.6 in [3], we can show that the set Φ defined in (3.5) is bounded. This condition is useful in guaranteeing that a numerical algorithm can be effectively used to solve (3.6).

Remark 3.2

While the result of this paper only guarantees to provide an optimized nominal H_2 performance, in view of the results of [17] it may be extended to also provide a robust H_2 performance under the presence of structured uncertainty.

Let us consider again mixed H_2/H_∞ robust control synthesis for the state feedback plant $\mathcal{P}$. Suppose that $\alpha > 0$ is given. From Theorems 3.1 and 3.2, we know that there exists $K \in \mathcal{A}_{\infty,m}(\mathcal{P})$ such that $J(T_{zu}, K) < \alpha$ if only if there exists $(X, M, \tilde{P}) \in \Phi$ such that $f(X, M, \tilde{P}) < \alpha$. And in this case, the real matrix $K := X\tilde{P}^{-1}$ is a solution to the sub-optimal synthesis problem. The problem of finding $(X, M, \tilde{P}) \in \Phi$ such that $f(X, M, \tilde{P}) < \alpha$ is a convex *feasibility program* which is a (nonsmooth) convex optimization problem[12].

4 Reduction To Generalized Eigenvalue Minimization Problem

In this section, we will show that the optimization problem defined in (3.6) can be reduced to Generalized Eigenvalue Minimization Problem(GEMP) developed in [12]. This is the problem of minimizing the maximum generalized eigenvalue of a (symmetric, symmetric positive-definite) pair of matrices that depend affinely on a variable x that is subject to some constraints. In [12], a fast and attractive algorithm based on Interior Point Method has been applied to solve efficiently GEMP.

In the general case, GEMP with variables $x \in \mathbf{R}^m$ and $\lambda \in \mathbf{R}$ takes the form

$$\min_{\substack{\lambda G(x) - F(x) > 0 \\ G(x) > 0 \\ H(x) > 0}} \lambda \tag{4.1}$$

or equivalently,

$$\min_{\substack{G(x) > 0 \\ H(x) > 0}} \lambda_{max}(F(x), G(x)). \tag{4.2}$$

where λ_{max} denotes the generalized maximum eigenvalue. This is a function defined on a pair of matrices X, Y by $\lambda_{max}(X, Y) := max\{\lambda \in \mathbf{R} | det(\lambda Y - X) = 0\}$. In (4.1) and (4.2), F ,G and H are symmetric matrices that depend affinely on $x \in \mathbf{R}^m$:

$$F(x) := F_0 + \textstyle\sum_{i=1}^m x_i F_i, \quad G(x) := G_0 + \sum_{i=1}^m x_i G_i, \quad H(x) := H_0 + \sum_{i=1}^m x_i H_i, \tag{4.3}$$

where $F_i = F_i', G_i = G_i' \in \mathbf{R}^{r\times r}$, and $H_i = H_i' \in \mathbf{R}^{s\times s}$. Matrices $F(x)$ and $G(x)$ may be complex Hermitian.

Let us turn our attention to the optimization problem defined in (3.6). Let us rewrite the objective function (3.3) as:

$$\begin{aligned} f(X, M, \tilde{P}) \;=\; & tr(C_2\tilde{P}C_2' + C_2X'D_{22}' + D_2XC_2' \\ & + D_{22}X\tilde{P}^{-1}X'D_{22}'). \end{aligned} \tag{4.4}$$

The last term $\Theta(X, M, \tilde{P} := D_{22}X\tilde{P}^{-1}X'D_{22}'$ in the above equation can be equivalently expressed as

$$\Theta(X, M, \tilde{P}) = \min_{\begin{bmatrix} S & D_{22}X \\ X'D_{22}' & \tilde{P} \end{bmatrix} > 0} tr(S).$$

Let us introduce the change of variable $\tilde{W} = (M'M)^{-1}$. Let us further define

$$\begin{aligned} L_1(\lambda, X, \tilde{W}, \tilde{P}, S) &:= -tr(C_2\tilde{P}C_2' + C_2X'D_{22}' + D_{22}XC_2') \\ &\quad\; -tr(S) + \lambda \\ L_2(\lambda, X, \tilde{W}, \tilde{P}, S) &:= -\begin{bmatrix} L_{2a} & L_{2b} \\ L_{2c} & L_{2d} \end{bmatrix} \\ L_3(\lambda, X, \tilde{W}, \tilde{P}, S) &:= \begin{bmatrix} S & D_{22}X \\ X'D_{22}' & \tilde{P} \end{bmatrix} \\ L(\lambda, X, \tilde{W}, \tilde{P}, S) &:= diag(L_1, L_2, L_3), \end{aligned}$$

where

$$\begin{aligned} L_{2a} &= -(A\tilde{P} + \tilde{P}A' + B_2X + X'B_2' + B_1B_1' + B_3\tilde{W}B_3') \\ L_{2b} &= [(C_1\tilde{P} + D_{12}X)' \quad (C_3\tilde{P} + D_{32}X)'] \\ L_{2c} &= L_{2b}' \\ L_{2d} &= \begin{bmatrix} I & 0 \\ 0 & \tilde{W} \end{bmatrix}. \end{aligned}$$

Note carefully that $L_1(\lambda, X, \tilde{W}, \tilde{P}, S)$, $L_2(\lambda, X, \tilde{W}, \tilde{P}, S)$ and $L_3(\lambda, X, \tilde{W}, \tilde{P}, S)$ are affine matrix in the variables $(\lambda, X, \tilde{W}, \tilde{P})$.

Using the above constructions and employing the Schur complement formula, our optimization problem (3.6) can now be represented as

$$\min_{\substack{L(\lambda, X, \tilde{W}, \tilde{P}, S) > 0 \\ \tilde{W} \in \mathcal{M}}} \lambda. \tag{4.5}$$

Represented in the form of (4.1), symmetric affine matrices $F(x)$,$G(x)$ and $H(x)$ for the optimization problem (4.5) are given by

$$\begin{aligned} F(x) &:= diag([-tr(C_2\tilde{P}C_2' + C_2X'D_{22}' + D_{22}XC_2') - tr(S)], \\ &\qquad L_2, L_3) \\ G(x) &:= diag(1, 0, 0, 0) \\ H(x) &:= \tilde{W}. \end{aligned}$$

Vector x in (4.1) then contains the optimization variables which consist of the independent variables of $(\lambda, X, \tilde{W}, \tilde{P}, S)$. Since the constraint $L(\lambda, X, \tilde{W}, \tilde{P}, S) > 0$ has been expressed in terms of matrices that are affine in $(\lambda, X, \tilde{W}, \tilde{P}, S)$, the additional constraint $\tilde{W} \in \mathcal{M}$ in (4.5) can be handled easily using partition technique described on p. 57 in [11]. The GEMP can be effectively solved using Interior Point Method. Detailed algorithm as well as its convergence can be found in [12].

5 Conclusion

The problem of synthesizing mixed H_2/H_∞ robust controllers has been presented for finite dimensional linear time-invariant systems under the presence of structured uncertainty. This synthesis problem is well motivated since in addition to providing robust stability and robust H_∞ performance under the presence structured uncertainty, it also provide an optimized (nominal) quadratic performance. Thi suboptimal synthesis problem has been reduced to convex optimization problem over a bounded subset of symmetric matrices as well as diagonal matrix having certain structure, via the use of a Riccati equation and a change of variables. The resulting convex optimization problem has been reduced to the Generalized Eigenvalue Minimization Problem where a powerful algorithm based on interior point method(analytic center) has been developed to find its solution[12].

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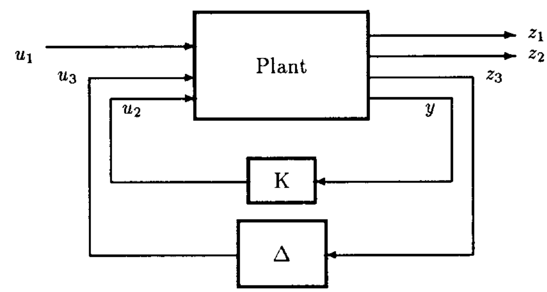

Figure 1: General framework for mixed H_2/H_∞ robust control design

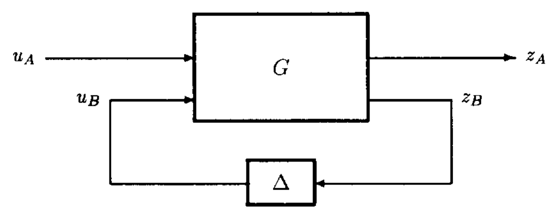

Figure 2: μ analysis framework

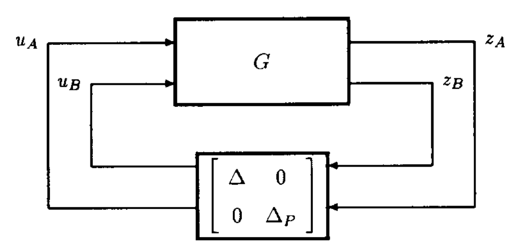

Figure 3: Block augmentation for robust performance