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Filtering of Spatially Invariant Image Sequences with One Desired Process

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This paper reports several mathematical properties of the filter vector developed for processing linearly-additive spatially-invariant image sequences. In this filtering of an image sequence into a single filtered image, the information about the image components originally distributed over the entire sequence is compressed into the one new image in a way that the desired component is enhanced and the undesired (interfering) components and noise are suppressed.

Introduction

Spatially-invariant image sequence is a sequence of images of the same scene or object with no relative object-sensor motion. The basic characteristic of a spatially-invariant image sequence is that all image features are in the same spatial position in each image of the sequence. The sequence is obtained by the variation of some property of the object or imaging system (other than motion), such that the image intensity (gray scale value) of the features (but not the location) changes from image to image.

An image sequence processing algorithm, based on a feature reduction technique used in statistical pattern recognition known as simultaneous diagonalization, was introduced by Miller[1, 2] and extended by Abd-Allah [3]. The new technique, called the simultaneous-diagonalization (SD) transform, has been applied to the filtering of spatially-invariant image sequences to generate a single new image in which a desired process or feature is enhanced while other undesired processes or features are suppressed. The SD transform was developed for analyzing medical image sequences with multiple physiological processes.

Spatially-Invariant Image Sequences

In a spatially-invariant image sequence, since the formation processes are spatially-invariant, the process contribution $\theta_m(i,j,k)$ is decomposable into the product of two factors:

$$\theta_m(i,j,k) = s_m(k)\varphi_m(i,j), \tag{1}$$

where $\varphi_m(i,j)$ is the spatial distribution (gray-scale map) of the mth process and $s_m(k)$ is the signature of the mth process over the sequence.

Many spatially invariant image sequences have the properties that the various image components contribute linearly and additively (perhaps after a point transformation, as the logarithm for x-ray images) to each image of the sequence. For such processes, the gray-scale value representation g(i,j,k) of the pixel at location (i,j) in the kth image of a linearly additive spatially-invariant image sequence corrupted by noise is

$$g(i,j,k) = \sum_{m=1}^{p} s_m(k)\varphi_m(i,j) + n(i,j,k). \qquad (2)$$

Linear Filtering of Image Sequences

The linear filtering of an image sequence to obtain a single new image is described by

$$r(i,j) = \sum_{k=1}^{n} x(k)g(i,j,k) \qquad (3)$$

in which x(k) is the kth element of the filter weighting function and r(i,j) is the scalar gray-scale value at (i,j) of the new image that results from filtering.

Applied to the linearly additive spatially-invariant image sequence (2), the filtered image is

$$r(i,j) = \sum_m \varphi_m(i,j)\sum_k x(k)s_m(k) + \sum_k x(k)n(i,j,k), \qquad (4)$$

or in vector notation,

$$r(i,j) = \sum_m \varphi_m(i,j)\langle x,s_m\rangle + \langle x,n(i,j)\rangle, \qquad (5)$$

$\langle\ \rangle$ denotes the inner or dot product of two vectors.

One desirable goal of the filter vector x is to collect or compress the information of a specific image formation process which is distributed over the image sequence into one composite image while also suppressing other image formation processes and noise.

In this form, it is clear that $\langle x, s_m\rangle$ determines the amount of the mth component present in the filtered image, while $\varphi_m(i,j)$ describes its distribution over the image. It is this dot product of the filter vector x with the image sequence component signature s_m that guides the selection of x to enhance a desired component (with signature s_m denoted d) and suppress v undesired components (with signatures s_m denoted u_1 to u_v). It is assumed, and is required for the following results, that $p = 1 + v$ is less than or equal to the number n of images in the sequence and that the set of signature vectors $\{ s_m \}$ are linearly independent. Indeed, if x were selected as a vector orthogonal to the set of undesired processes $\{ u_q \}$, they would be completely eliminated from the filtered image (5).

Filter Design Criterion

To provide a quantitative criterion for the selection of the filter vector, (5) suggests the energy ratio

$$r_E(x) = \frac{E_d}{E_u + E_n} \qquad (6)$$

where the desired component energy is

$$E_d = \langle x,d\rangle^2 = x^t d d^t x \qquad (7)$$

the undesired component energy is

$$E_u = \sum_{q=1}^{v} \langle x, u_q\rangle^2 = x^t[\sum_{q=1}^{v} u_q u_q^t]x \qquad (8)$$

and the noise energy for white noise with covariance matrix ωI is

$$E_n = x^t[\omega I]x = \omega x^t x. \qquad (9)$$

Denoting $U = \left[u_1\, u_2 \ldots u_v \right]$, the filter design criterion (6) to be maximized with respect to x can be written

$$r_E(x) = \frac{x^t d d^t x}{x^t(UU^t + \omega I)x} = \frac{x^t A x}{x^t B x} \qquad (10)$$

which is a generalization of Rayleigh's quotient[4].

Setting the gradient with respect to x of (10) equal to zero to maximize $r_E(x)$, (10) can be transformed into the generalized eigenvector problem

$$Ax = r_E(x)Bx. \qquad (11)$$

Or, since B is positive definite for $\omega > 0$, (11) can be expressed as

$$B^{-1}Ax = r_E(x)x \qquad (12)$$

in which the energy ratio $r_E(x)$ is an eigenvalue and the filter vector x is an eigenvector of $B^{-1}A$. Since the objective here is to obtain the filter x_{max} which will maximize the energy of the desired image formation process relative to the undesired image formation process and noise, x_{max} is chosen to be the eigenvector associated with the largest eigenvalue $r_E(x_{max})$. The n x n matrix B is nonsingular (rank n) and positive definite for $\omega > 0$; the n x n matrix A is singular (rank 1) and positive semidefinite, with positive eigenvalue d^td and corresponding eigenvector d. The n x n matrix $B^{-1}A$ is singular (rank 1), with only one nonzero (positive) eigenvalue. That nonzero eigenvalue $r_E(x_{max})$ and the corresponding eigenvector $x = x_{max}$ satisfying

$$(UU^t + \omega I)^{-1}dd^tx = r_E(x)x \qquad (13)$$

are the desired filter vector $x = x_{max}$ and maximum filtered energy ratio $r_E(x_{max})$.

Filter Derivation

Although (13) can be readily solved with standard algorithms that calculate eigenvalues and eigenvectors, important properties of this filter can be observed by obtaining an explicit solution for the filter vector x. The following is a completely general derivation of the filter.

In (13), the maximum eigenvalue $r_E(x_{max})$ can be obtained directly, since the matrix $(UU^t + \omega I)^{-1}dd^t$ is of rank 1. Being of rank 1, $(UU^t + \omega I)^{-1}dd^t$ has only one non-zero eigenvalue and the trace of $(UU^t + \omega I)^{-1}dd^t$ equals the maximum eigenvalue (the only non-zero value). So, the maximum eigenvalue is

$$r_E(x_{max}) = \mathrm{Trace}\left((UU^t + \omega I)^{-1}dd^t \right)$$

$$= \mathrm{Trace}(d^t(UU^t + \omega I)^{-1}d)$$

$$= d^t(UU^t + \omega I)^{-1}d, \qquad (14)$$

since the latter expression is a scalar. Substituting this expression for $r_E(x_{max})$ into (13), we get

$$(UU^t + \omega I)^{-1}dd^tx = xd^t(UU^t + \omega I)^{-1}d, \qquad (15)$$

which is satisfied by the filter vector solution

$$x_{max} = \chi(UU^t + \omega I)^{-1}d \qquad (16)$$

for arbitrary non-zero scalar χ. That is, $r_E(x)$ is maximized by the eigenvector x_{max} in (16) with eigenvalue $r_E(x_{max})$,

$$r_E(x_{max}) = d^t(UU^t + \omega I)^{-1}d. \qquad (17)$$

Limiting Cases of Filter Formulas

The explicit expression in (16) provides the basis for some interesting special cases, involving the relative orientation of d and U and of large noise and small noise energy compared with the energy of the interfering image formation processes.

1. Orthogonal d and U

If the desired process d is orthogonal to each of the undesired processes,

$$U^t d = 0, \tag{18}$$

then the filter vector (16) reduces to

$$x = \alpha d. \tag{19}$$

In this case, the filtered energy ratio (17) becomes

$$r_E(x_{max}) = \frac{1}{\omega} d^t d. \tag{20}$$

This is an ideal case. By (18) and (19),

$$U^t x = U^t(\alpha d) = \alpha(U^t d) = 0. \tag{21}$$

That is, the filter vector x is also orthogonal to the undesired processes U. From (5), it is clear that this means that the υ undesired processes u_q in U will be completely eliminated by x in the filtered image. The resulting filtered energy ratio (17) is the ratio of only the desired process (signal) energy $d^t d$ and the noise energy ω.

2. Large Noise Case

When the image sequence is contaminated by very large noise, the interfering processes have little effect on the filtering scheme. Setting the undesired process matrix U = 0 (compared to the size of ω), the resulting filter emphasizes the desired process d and deemphasizes only noise. In this case, the filter vector in (16) reduces to

$$x = \alpha d, \tag{22}$$

and the energy ratio in (17) becomes

$$r_E(x_{max}) = \frac{1}{\omega} d^t d. \tag{23}$$

This special case corresponds to the well-known matched filter, since the filter vector (16) is a scaled version of the desired signature vector d. As may be recalled, the matched filter is the optimal solution for filtering a known signal corrupted by additive noise based on the ratio of the signal-to-noise (S/N) energy, which has the form (22) for the case of white noise.

3. Small Noise Case

If the noise energy is small, then the filter can focus its effect on the undesired processes. Through a direct consequence of a well-known matrix identity, the filter expression in (16) is changed into

$$x_{max} = \alpha\left[I - U(U^tU + \omega I)^{-1}U^t\right]. \tag{24}$$

As $\omega \to 0$, the filter vector (24) approaches

$$x_f = \lim_{\omega \to 0} x_{max} = \alpha\left[I - U(U^tU)^{-1}U^t\right]d, \tag{25}$$

and the criterion (17) approaches

$$\lim_{\omega \to 0} r_E(x_f) = \infty. \tag{26}$$

4. Noise-Free Case

In this section, the properties of the noise-free filter x_f in (25) and the origin of the infinite energy ratio in (26) are explored. When there is no noise present, the criterion function is now

$$r_E(x) = \frac{x^t d d^t x}{x^t U U^t x}, \tag{27}$$

in which the denominator matrix $B = UU^t$ is singular.

Calculating the inner product of

$$U^t x_f = \alpha(U^t - U^t)d = 0. \tag{28}$$

It is this property that makes the denominator of (27) zero and the ratio infinite.

We observe that x_f is orthogonal to every undesired signature vector u_q, which tells us that the filtering by this vector eliminates all undesired processes in the filtered image. This is a very desirable property. That is, this filter vector suppresses all of the interfering components of the image sequence, while the energy associated with the desired process is kept at the maximum possible value.

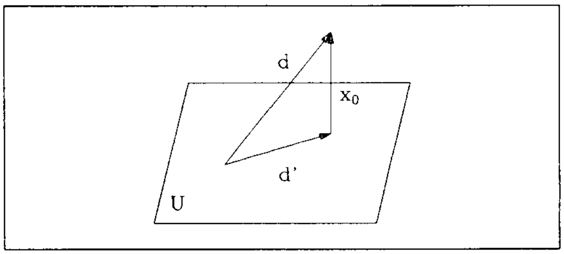

Figure A

The vector space interpretation of x_f is illustrated in Figure A. Let d′ be the projection of d into the plane spanned by the u_q's (denoted by U in the figure). Then $x_f = d - d'$ is one possible form of the vector orthogonal to all u_q's. Of all vectors orthogonal to the space of the u_q's, this x_f has the greatest correlation with d. The direction of x_f can be provided more directly by well-known relations in linear algebra. The matrix operator $U(U^tU)^{-1}U^t$ calculates the component of a vector in the space of U, so $d' = U(U^tU)^{-1}U^td$. The operator $I - U(U^tU)^{-1}U^t$ calculates the orthogonal component, so $d - d' = \left[I - U(U^tU)^{-1}U^t\right]d$. Hence, x_f is in the direction of the component of the desired process d orthogonal to the undesired processes U.

The vector x_f in (25) is the limiting form of the vector x_{max} from (16) but it is not the only vector orthogonal to U. There is a set of vectors orthogonal to U, for which the denominator is zero and the ratio is infinite.

Observe that the n x n matrix UU^t is of rank $\upsilon < n$, so that there are non-zero vectors x for which $UU^t x = 0$. Let the null space of UU^t be denoted by Ψ. Because $U_{n\times\upsilon}$ for $n < \upsilon$ is of full rank υ, the null space Ψ is

$$\Psi = \{x \mid UU^tx = 0\} = \{x \mid U^tx = 0\}. \quad (29)$$

If we let $H_0 = I - U(U^tU)^{-1}U^t$, then we can represent the null space Ψ with the eigenspace of the matrix H_0 with eigenvalue 1. That is,

$$\Psi = \{x \mid H_0x = x\}. \quad (30)$$

Since H_0 is the operator that gives the component of a vector orthogonal to U, Both (29) and (30) define the space of vectors x orthogonal to U. The following are several properties of the matrix H_0 :

(a) H_0 is symmetric. (31)

(b) H_0 is idempotent. $H_0H_0 = H_0$ (32)

(c) H_0 is normal. $H_0^tH_0 = H_0H_0^t$ (33)

(d) H_0 has eigenvalues only of 0 or 1. (34)

(e) $\mathrm{rank}(H_0) = \mathrm{Trace}(H_0)$

$$= \mathrm{Trace}\left[I_n - U(U^tU)^{-1}U^t\right]$$

$$= \mathrm{Trace}(I_n) - \mathrm{Trace}\left[U(U^tU)^{-1}U^t\right]$$

$$= \mathrm{Trace}(I_n) - \mathrm{Trace}(I_\upsilon) = n - \upsilon. \quad (35)$$

Here, I_n is the n x n identity matrix and I_υ is the $\upsilon \times \upsilon$ identity matrix. So, the dimension of the null space Ψ is

$$\dim(\Psi) = \mathrm{rank}(H_0) = n - \upsilon. \quad (36)$$

Also, the same class given by (29) and (30) can be found as the collection of vectors x of the form $x = H_0 y$ for some y, because H_0 is idempotent and for some y,

$$x = H_0 y \implies H_0 x = H_0 H_0 y = H_0 y = x. \quad (37)$$

Therefore, when noise is not present or negligible in the imaging process, the energy ratio will go to infinity for filter vectors x which construct the class

$$\Psi = \{x \mid x = H_0 y, \text{ for some } y\}. \quad (38)$$

It is easily seen from (38) that the vector x_f in (25) corresponds to a vector in Ψ. This x_f is the unique vector in Ψ with the greatest correlation with the desired signature vector d.

Conclusion

This paper considers the selection of a filter vector for linear filtering of a sequence of spatially invariant images of an object or scene to maximize the ratio of desired component energy to undesired component and noise energy in the filtered image. The filtered image is a weighted linear combination of the images of the sequence, and the filter vector is the set of weights. Special results are also given for the cases in which the filter is designed only to suppress undesired processes or only noise.

References

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