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A Modal Analysis for a Hung Euler-Bernoulli Beam with a Lumped Mass

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Abstract

In this paper, a modal analysis is applied for a hung Euler-Bernoulli beam with a lumped mass. We first derive the equations of motion using the Hamillton's principle. Then regarding the tension of beam as constant, we characterize the eigenfrequencies and the feature of eigenfunctions. The approximation employed here is corresponding that the lumped mass is sufficiently large than that of beam. Finally we compare the eigenfrequencies derived here with those obtained based on the Southwell's method.

1 Introduction

For control of the flexible space structures, the key step in the synthesis is to clarify he modes and the feature of corresponding mode shapes are identified.

In the literature, there are many studies for the typical Euler-Bernoulli beam which discuss the mode and the corresponding mode shapes. Though almost all of these studies are restricted to the case where the tension due to the gravity is negligible or the case where it has no tip mass. In a few studies which account the influence of gravity are concerned with so-called Southwell's approximation method. However, it is not obvious that the Southwell's method is applicable for the hung beam with a lumped mass.

In this paper, we consider a hung Euler-Bernoulli beam with a lumped mass. The beam we focus on can be regarded as a simplified model of the flexible bodies such as an antenna in spacecraft systems. And further, the beam has a feature that the motion is essentially affected by the tension.

In the following, we first derive equations of motion using the Hamillton's principle. Then, regarding the tension of beam as constant, we determine the eigenfrequencies and the feature of the eigenfunctions. The approximation employed here is corresponding that the lumped mass is sufficiently large than that of beam. Finally we compare the eigenfrequencies derive here with those estimated based on the two types of Southwell's approximation method.

2 Modal Analysis

2.1 Model of system

Consider a hung Euler-Bernoulli beam with a lumped mass (Figure 1).

The parameters of this beam are defined as follows: E is Young's modulus, I is moment of inertia, ρ is density, A is cross section, g is acceleration due to gravity, m is mass ,L is length and t is time. $\theta(t)$ is rotation of the beam. x_r[m] denotes the length from supported point and $y_r(x_r,t)$[m] is the transverse deformation at x_r.

In following, we will refer the non-dimensional values:

$$x := \frac{x_r}{L}, \quad y := \frac{y_r}{L} \tag{1}$$

as the length from supported point and the transverse deformation respectively.

2.2 Equations of motion

Hamillton's principle is used to derive equations of motion

$$\int_{t1}^{t2} (\partial T - \partial U)dt \equiv 0, \tag{2}$$

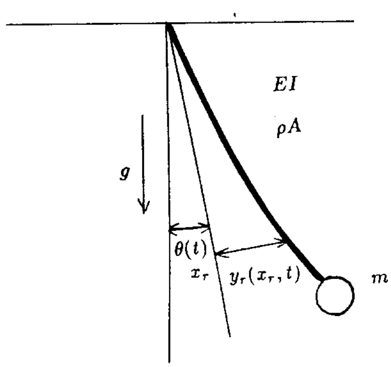


Figure 1: A hung beam model

$$
\begin{aligned}
T := & \\
& \frac{1}{2}\int_0^1 \left\{x^2\dot\theta^2(t) + 2x\dot y(x,t)\dot\theta(t)\right\} dx \\
& +\frac{1}{2}\int_0^1 \left(\dot y^2(x,t) + y^2(x,t)\dot\theta^2(t)\right) dx \\
& +\frac{m}{2\rho AL}\left\{(\dot\theta^2 y^2(1,t) + \dot\theta^2 + 2\dot\theta\dot y(1,t) + \dot y^2(1,t)\right\}
\end{aligned}
$$

$$
\begin{aligned}
U := & \\
& \frac{1}{2}\int_0^1 \left(\frac{\partial^2 y(x,t)}{\partial x^2}\right)^2 dx \\
& +\frac{mgL^2}{2EI}\int_0^1 \left(\frac{\partial y(x,t)}{\partial x}\right)^2 dx \\
& +\frac{\rho AgL^3}{2EI}\int_0^1 (1-x)\left(\frac{\partial y(x,t)}{\partial x}dx\right)^2 \\
& +\frac{\rho AgL^3}{EI}\int_0^1 \left(\frac{x\theta^2}{2} + y(x,t)\theta\right) dx \\
& +\frac{mgL^2}{EI}\left(\frac{\theta^2}{2} + y(1,t)\theta\right)
\end{aligned}
$$

where T and U are kinetic and potential energies of this system. Since the moment of inertia is sufficiently smaller than the other parameters, we ignore energies of shearing, rotary inertia and torsional.

Under the assumptions that the transverse deflection $y(x,t)$ and its derivatives $\frac{dy(x,t)}{dx}$ are regarded as small values, we obtain the equations of motion:

$$
\begin{aligned}
& \int_0^1 x\ddot y(x,t)dx + c\ddot y(1,t) \\
& \quad + d\int_0^1 y(x,t)dx + cdy(1,t) = 0
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
& \frac{\partial^4 y(x,t)}{\partial x^4} + \ddot y(x,t) - \frac{\partial}{\partial x}\left(S(x)\frac{\partial y(x,t)}{\partial x}\right) \\
& \quad - cd\frac{\partial^2 y(x,t)}{\partial x^2} = 0
\end{aligned} \tag{4}
$$

$$
y(0,t) = \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=0} = 0 \tag{5}
$$

$$
\left.\frac{\partial^3 y(x,t)}{\partial x^3}\right|_{x=1} = c\ddot y(1,t), \qquad \left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=1} = 0 \tag{6}
$$

$$
c = \frac{m}{\rho AL}, \qquad d = \frac{\rho AgL^3}{EI}, \qquad S(x) = d(1-x), \tag{7}
$$

where x, y are the non-dimensional values defined in equation (1) and $S(x)$ denotes the tension force.

In the sequel, we will assume that the tension force $S(x)$ can be regarded as constant along the beam:

$$
S(x) = d.
$$

The approximation employed here is corresponding that the lumped mass is sufficiently large than that of beam.

2.3 Modal analysis of the Beam with a Lumped Mass

For the analysis of the free vibration, a harmonic motion is assumed in the following form:

$$
y(x,t) = \phi(x)q(t), \tag{8}
$$

$$
\phi(x) = e^{i\delta x}\phi_0, q(t) = e^{i\omega t}q_0
$$

where ω is the natural frequency and $\phi(x)$ is the corresponding eigenfunction. Substituting equation (8) into equation (4), (5) and (6), we first obtain the following equations:

$$
\ddot q(t) + \omega^2 q(t) = 0, \tag{9}
$$

$$
\frac{d^4\phi(x)}{dx^4} - d(1+c)\frac{d^2\phi(x)}{dx^2} = \omega^2\phi(x). \tag{10}
$$

$$\phi(0) = \left.\frac{d\phi(x)}{dx}\right|_{x=0} = 0, \tag{11}$$

$$\left.\frac{d^3\phi(x)}{dx^3}\right|_{x=1} = -c\omega^2\phi(1), \quad \left.\frac{d^2\phi(x)}{dx^2}\right|_{x=1} = 0. \tag{12}$$

Then using the relation derived from (8) and (10):

$$\delta^4 + d(1+c)\delta^2 - \omega^2 = 0, \tag{13}$$

we obtain the following equations.

$$\begin{aligned}\phi(x) = \\ & A\{(\delta^2\cos\delta + (\frac{\omega}{\delta})^2\cosh\frac{\omega}{\delta})(\sinh\frac{\omega}{\delta}x - \frac{\omega}{\delta^2}\sin\delta x) \\ & -(\omega\sin\delta + (\frac{\omega}{\delta})^2\sinh\frac{\omega}{\delta})(\cosh\frac{\omega}{\delta}x - \cos\delta x)\}\end{aligned} \tag{14}$$

$$\begin{aligned}& 2\frac{\omega^3}{\delta}\cos\delta\cosh\frac{\omega}{\delta} + (\omega^2\delta - \frac{\omega^4}{\delta^3})\sin\delta\sinh\frac{\omega}{\delta} \\ & +\omega\delta^3 + c(\frac{\omega^4}{\delta^2} + \omega^2\delta^2)\cos\delta\sinh\frac{\omega}{\delta} \\ & +(\frac{\omega}{\delta})^5 - c(\omega^3 + \frac{\omega^5}{\delta^4})\sin\delta\cosh\frac{\omega}{\delta} \\ & = 0.\end{aligned} \tag{15}$$

where A is a constant factor. The equations (13), (15) determine the value of the natural frequency ω and the equation (14) provides the feature of eigenfunctions.

In case we focus on the gravity-free system, the natural frequency and the corresponding eigenfunction are given in the following form:

$$\begin{aligned}\phi_s(x) = \\ & A_s\{(\cos\delta_s + \cosh\delta_s)(\sin\delta_s x - \sinh\delta_s x) \\ & -(\sin\delta_s + \sinh\delta_s)(\cosh\delta_s x - \cos\delta_s x)\},\end{aligned} \tag{16}$$

$$\begin{aligned}& 1 + \cos\delta_s\cosh\delta_s \\ & +c\delta_s(\cos\delta_s\sinh\delta_s - \sin\delta_s\cosh\delta_s) = 0,\end{aligned} \tag{17}$$

$$\omega^2 = \delta_s^4, \tag{18}$$

where A_s denotes a constant factor.

Comparing the equations (13) and (15) with (17) and (18) which determine natural frequencies, we can see that it arises a new term $(\omega^2\delta - \frac{\omega^4}{\delta^3})$ $\sin\delta\sinh\frac{\omega}{\delta}$ in the case we focus on the influence of gravity.

Table 1: Parameters of the beam

Length	2.0m
Width	0.252m
Thickness	1mm
Linear density	8.8 $\times 10^3 kg/m^3$
Elastic modulus	1.11 $\times 10^{11} N/m^2$
Moment of inertia	1.25 $\times 10^{-11} m^4$
Cross section	2.54 $\times 10^{-4} m^2$
A lumped mass	44.4kg

Table 2: Natural frequencies
(a) the gravity-free case
(b) under the influence of gravity

mode No.	(a) $[Hz]$	(b) $[Hz]$
1	0.0862	1.3924
2	2.4700	14.3102
3	7.9719	29.1769
4	16.6157	45.1166

3 Numerical Example

Based on the parameters stated in Table 1, we calculate the lower four eigenfrequencies(Table 2.) and the corresponding mode shapes(Figure 2.).

Comparing the both cases; the case we focus on the influence of gravity (13)-(15) and the gravity-free case (16)-(18), we can see that the eigenfrequencies under the influence of gravity is about three to seventeen times greater than the eigenfrequencies of gravity-free case.

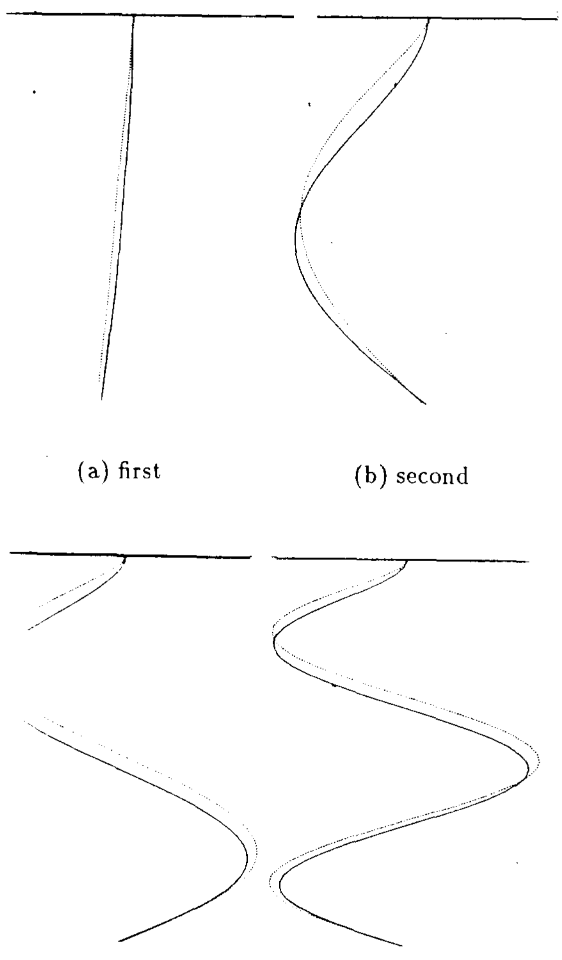
Figure 2 shows that the mode shapes under the influence of gravity are quite different from those of gravity-free case. There facts indicate that the mode shapes are affected very strongly by the tension, especially in the lower mode shapes.

4 Compared with the Southwell's method

In this chapter, we apply the Southwell's approximation method for the hung beam, then compare the Southwell's method with the previous result stated in chapter 2 in the case of the hung beam.

In applying the Southwell's method for this beam, the two types of the approach are employed.

In the first approach, we approximate the eigenfrequencies by the equation:



(a) first (b) second (c) third (d) fourth

Figure 2: Lower four mode shapes. (— the gravity-free case ; ··· under the influence of gravity)

$$\omega^2 = \frac{\int_0^1 \left(\frac{d^2\phi_s(x)}{dx^2}\right)^2 dx}{\int_0^1 \phi_s^2(x)dx} + \frac{d(1+c)\int_0^1 \left(\frac{d\phi_t(x)}{dx}\right)^2 dx}{\int_0^1 \phi_t^2(x)dx} \tag{19}$$

where ϕ_s is defined in (16) and ϕ_t is defined as a solution to

$$d(1+c)\frac{d^2y(x,t)}{dx^2} = \ddot{y}(x,t). \tag{20}$$

In the second approach, we approximate the eigenfrequencies by the following equation.

$$\omega^2 = \frac{\int_0^1 \left(\frac{d^2\phi_s(x)}{dx^2}\right)^2 dx + d(1+c)\int_0^1 \left(\frac{d\phi_s(x)}{dx}\right)^2 dx}{\int_0^1 \phi_s^2(x)dx} \tag{21}$$

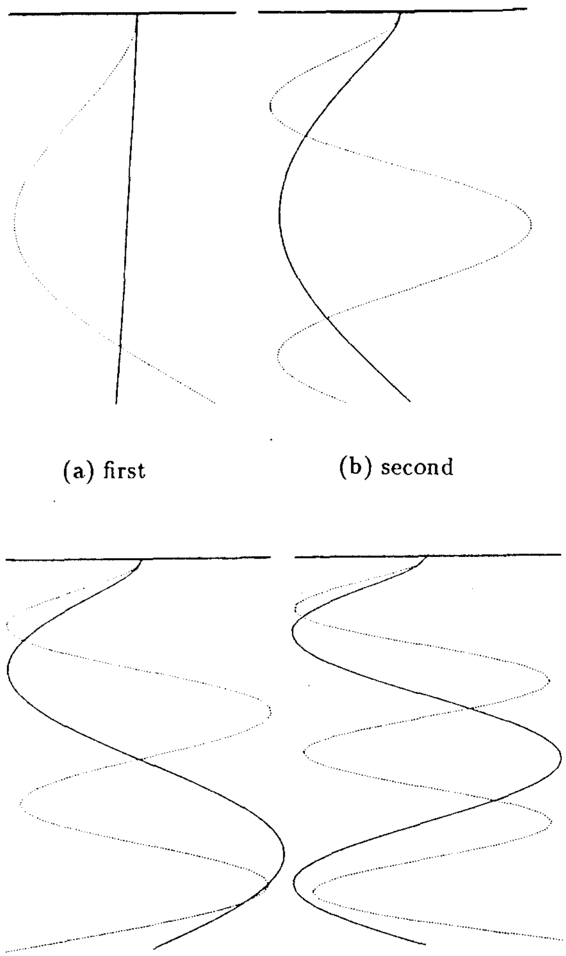
In Table 3 and 4, the Southwell's method is compared with the previous result stated in chapter 2. Table 3 shows approximated frequencies obtained by the first approach together with the exact eigenfrequencies stated in chapter 2. Table 4 shows the approximated eigenfrequencies obtained by the second approach with the exact ones.

From the Table 3, we can see that the approximated frequencies obtained by the first approach are very larger than the exact eigenfrequencies. While Table 4 shows that there is a small difference in low frequencies between the eigenfrequencies derived in chapter 2 with the approximated eigenfrequencies obtained by the second approach. Therefore, in case of the hung beam under the influence of gravity, we can say that the second approach is better than the first one in order to evaluate the eigenfrequencies. However, as for the mode shapes we can say the followings.

The mode shapes corresponding to Table 4 are illustrated in Figure 3. From Figure 3, we can see that the approximated mode shapes obtained by the second approach are quite different from the exact ones. Especially, the difference appears at the end of beam. Moreover, in the Southwell's approach, the first mode corresponding to the motion of flexible pendulum is vanished.

5 Conclusion

In this paper, we have analyzed a hung Euler-Bernoulli beam with a lumped mass. We first de-



(a) first (b) second

(c) third (d) fourth

Figure 3: Lower four mode shapes (—the exact mode shapes;- - -the approximated mode shapes)

Table 3: the validity of the Southwell's method(1)
(a):the exact frequencies
(b):the approximated eigenfrequencies obtained by the first approach

mode No.	(a) $[Hz]$	(b) $[Hz]$	the error [%]
1	1.3924	2.5770	70.1
2	14.3102	19.7798	38.2
3	29.1768	34.6664	18.8
4	45.1166	50.4349	11.7

Table 4: the validity of the Southwell's method(2)
(a):the exact frequencies
(b):the approximated eigenfrequencies obtained by the second approach

mode No.	(a) $[Hz]$	(b) $[Hz]$	the error [%]
1	1.3924	1.4620	12.2
2	14.3102	15.7466	10.0
3	29.1769	31.0399	6.4
4	45.1166	47.4217	5.0

rived the equations of motion using the Hamillton's principle. Then regarding the tension of beam as constant, we characterized the eigenfrequencies and the feature of corresponding eigenfunctions. Finally, we compared the eigenfrequencies obtained based on the two types of Southwell's approximation mathod with the exact ones. As a result, it is indicated for the hung beam that the second Southwell's approximation is better than the first one in order to evaluate the eigenfrequencies. However the corresponding mode shapes do not need the exact ones.

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