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Linear Quadratic Control Problem of Delay Differential Equation

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Abstract

In this paper we are concerned with optimal control problems whose costs are quadratic and whose states are governed by linear delay equations and general boundary conditions. The basic new idea of this paper is to introduce a new class of linear operators in such a way that the state equation subject to a starting function can be viewed as an inhomogeneous boundary value problem in the new linear operator equation. In this way we avoid the usual semigroup theory treatment to the problem and use only linear operator theory.

1. Introduction

Let $\Re$ be the field of all real numbers and let $\Re^n$ be the Euclidean real Hilbert Space of finite dimension n(n ≥ 1 integer). For given $0 < \tau < t_1 < \infty$, $[-\tau, t_1]$ be an compact interval. Also for given integer q, let $\aleph_q$ denote the Hilbert Space of $x : [-\tau, t_1] \rightarrow \Re^n$ such that $|x| = \left(\int_{-\tau}^{t_1} x^*(t)x(t)\,dt\right)^{\frac{1}{2}} < \infty$ The inner product $\langle \cdot, \cdot \rangle$ of $\aleph_q$ is denoted by $\langle x, y \rangle = \int_{-\tau}^{t_1} x^*(t)y(t)\,dt$. For $\alpha \in \Re^n$, $\|\alpha\| = (\alpha^*\alpha)^{\frac{1}{2}}$. Let Let $-\infty < a < b < \infty$ be real numbers. $L_2^n[a,b]$ will denote the space of equivalence class of all square integrable functions from [a,b] into $\Re^n$. **AC**[a,b] will denote the space of the absolutely continuous functions from [a,b] into $\Re^n$. $x_{[a,b]}$ denote the restriction of x to [a,b].

Consider a linear delay differential equation

$$(1\text{-}1) \quad \dot{x}(t) = A_1(t)x(t) + A_1(t)x(t-\tau) + f(t), \quad t \in [0, t_1]$$
$$x(t) = f(t), \quad t \in [-\tau, 0]$$

where $A_1(t)$, $A_2(t)$ are $n \times n$ real matrix valued functions whose columns are in $\aleph_n$ and $f \in \aleph_n$. Equation (1-1) is the simple delay-differential equation. However, analogous properties to those listed below can be derived for more general types of equations having time-varing delay, multiple delays, and so on.

It is well known that there exists a unique real continuous solution which satisfy (1-1) a.e on $[-\tau, t_1]$ when the given initial function $x(t) = f(t)$ is continuous on $[-\tau, 0]$. In Halany[6] he discussed about the solution of equation(1-1) when the initial function is continuous, and found the solution in terms of fundamental matrix solution utilizing the adjoint systems. But our equation is different from[4] in that the initial function is in $L_2^n([-\tau,0]; \Re^n)$, $1 \le p \le \infty$.

In section 2 we introduce a new linear operator in such a way that the state equation subject to a starting function can be viewed as an inhomogenious boundary problem, and derive the adjoint operator of new operator, and then define the formal adjoint operator which will be play an important role in the charaterization of the optimal control.

In section 3 we discuss about fundamental matrix solution of (1-1) and adjoint system, and find the relation between two matrices. And then we discuss the solution of delay operator equations. Also we characterize the fundamental matrix which will be useful in practice.

In section 4 we consider the optimal control over a closed convex subset of $L_2^m[-\tau,t_1]$ and develop the necessary and sufficient conditions for an optimal-response pair in terms of adjoint equations and inclusions.

2. Delay differential operator

Define the delay differential operator $\Im : Dom\,\Im \to \aleph_n$ by

$$(\Im x)(t) = \begin{cases} \dot{x}(t) - A_1(t)x(t) - A_2(t-\tau), & t \in [0,t_1] \\ x(t), & t \in [-\tau,0] \end{cases}$$

where

i) $Dom\,\Im = \left\{ x \in \aleph_n \mid x_{[0,t_1]} \in AC[0,t_1],\ \dot{x}_{[0,t_1]} \in L_2^n[0,t_1] \right\}$

ii) $A_1(t)$ and $A_2(t)$ are $n \times n$ real matrix valued functions whose columns are in $\aleph_n$.

We see that $\Im$ is a linear operator.

Theorem 2.1 Let be defined as in the above. Then the adjoint operator $\Im^* : Dom\ \Im^* \to \aleph_n$ is

$$(\Im^* y)(t) = \begin{cases} y(t) - A_2^*(t+\tau)y(t+\tau), & t \in [-\tau,0) \\ -\dot{y}(t) - A_1^*(t)y(t) - A_2^*(t+\tau)y(t+\tau), & t \in [0,t_1-\tau) \\ -\dot{y}(t) - A_1^*(t)y(t), & t \in [t_1-\tau,t_1] \end{cases}$$

where Dom $\Im^*$ =
$\left\{ y \in \aleph_n \mid y(t_1) = y(t) = 0, y_{[0,t_1]} \in AC[0,t_1],\ \dot{y}_{[0,t_1]} \in L_2^n[0,t_1] \right\}$.

Now let's define the formal adjoint operator, which will be useful to the characterization of optimal control,
$\Im^+ : Dom\,\Im^+ \to \aleph_n$ by

$$(\Im^+ y)(t) = \begin{cases} y(t) - A_2^*(t+\tau)y(t+\tau), & t \in [-\tau,0) \\ -\dot{y}(t) - A_1^*(t)y(t) - A_2^*(t+\tau)y(t+\tau), & t \in [0,t_1-\tau) \\ -\dot{y}(t) - A_1^*(t)y(t), & t \in [t_1-\tau,t_1] \end{cases}$$

where

$$Dom\,\Im^+ = \left\{ y \in \aleph_n \mid y_{[0,t_1]} \in AC[0,t_1],\ \dot{y}_{[0,t_1]} \in L_2^n[0,t_1] \right\}.$$

Note that $\Im^* \subset \Im^+$.

3. Matrix solution of Delay differential operatot equation

Define a $n \times n$ matrix valued functions $X(t,s)$ and $Y(s,t)$ on $\Re \times \Re$ as follows: for each $s \in \Re$ fixed,

i)
$$X(t,t) = I_n$$
$$X(t,s) = 0, \qquad t > s$$
$$\frac{\partial}{\partial t}X(t,s) = A_1^*(t)X(t,s) + A_2^*(t)Y(t-\tau,s), \quad t \geq s$$

ii)
$$Y(t,t) = I_n$$
$$Y(s,t) = 0, \qquad s > t$$
$$\frac{\partial}{\partial s}Y(s,t) = A_1^*(s)Y(s,t) + A_2^*(s+\tau)Y(s+\tau,t), \quad s \leq t$$

Then we have the following theorem.

Theorem 3.1

$$X(t,s) = Y^*(s,t), \text{ for all } t, s \in \Re.$$

The following theorem characterize the fundamental matrix solution $X(t,s)$ of (1-2) in terms of the fundamental matrix solution $\Phi(t)$ of $\dot{x}(t) = A_1(t)x(t)$.

Theorem 3.2. Let $\Phi(t)$ be the $n \times n$ fundamental matrix solution of $\dot{x}(t) = A_1(t)x(t)$. Then

$$X(t,s) = \Phi(t)\sum_{i=0}^{\ell} H_i(t,s)\Phi^{-1}(s), \quad t > s$$

where

i) $\ell \in N$ such that $t \in [s+\ell\tau, s+(\ell+1)\tau]$

ii) $H_0(t,s) = I$.

iii) for $j = 1,2,\cdots,\ell$

$$H_j(t,s) = \begin{cases} \int_{s+(j-1)\tau}^{t-\tau} h(\alpha)H_{j-1}(\alpha,s)\,d\alpha, & s+j\tau \leq t \leq s+(j+1)\tau \\ 0, & \text{ortherwise} \end{cases}$$

here

$$h(\alpha) = \Phi^{-1}(\alpha)A_2(\alpha+\tau)\Phi(\alpha+\tau).$$

Proof. We construct $X(t,s)$ using "step by step" method.
Since for $t < s+\tau$,

$$X(t-\tau, s) = 0,$$

for $s \leq t < s+\tau$, i) becomes

$$\frac{\partial}{\partial t}X(t,s) = A_1(t)X(t,s)$$
$$X(s,s) = I.$$

Now $X(t,s)$ is the matrix solution of $\dot{x}(t) = A_1(t)x(t)$.
Thus $X(t,s) = \Phi(t)C(s)$.
But $X(s,s) = I$. Therefore $C(s) = \Phi^{-1}(s)$.
That is,

$$X(t,s) = \Phi(t)\Phi^{-1}(s), \text{ for } s \leq t < s+\tau.$$

For $s+\tau \leq t \leq s+2\tau$, ,

$$\frac{\partial}{\partial t}X(t,s) = A_1(t)X(t,s) + A_2(t)X(t-\tau,s)$$

with $X(s+\tau,s) = \Phi(s+\tau)\Phi^{-1}(s)$.

Therefore

$$X(t,s) = \Phi(t)\Phi^{-1}(s+\tau)X(s+\tau,s) + \int_{s+\tau}^{t}\Phi(t)\,\Phi^{-1}(\alpha)A_2(\alpha)X(\alpha-\tau,s)\,d\alpha$$

$$= \Phi(t)\,\Phi^{-1}(s+\tau)\Phi(s+\tau)\,\Phi^{-1}(s) + \int_{s}^{t-\tau}\Phi(t)\,\Phi^{-1}(\alpha+\tau)A_2(\alpha+\tau)\Phi(\alpha)\,\Phi^{-1}(s)\,d\alpha$$

$$= \Phi(t)\left[I + \int_{s}^{t-\tau}\Phi^{-1}(\alpha+\tau)A_2(\alpha+\tau)\Phi(\alpha)\,d\alpha \right]\Phi^{-1}(s)$$

Let

$$H_1(t,s) \equiv \int_s^{t-\tau} \Phi^{-1}(\alpha+\tau)A_2(\alpha+\tau)\Phi(\alpha)\,d\alpha .$$

Then

$$X(t,s) = \Phi(t)(I + H_1(t,s))\Phi^{-1}(s).$$

Now we show " by induction" that for $s+\ell\tau \le t \le s+(\ell+1)\tau$,

$$X(t,s) = \Phi(t)\sum_{i=0}^{\ell-1} H_i(t,s)\Phi^{-1}(s).$$

where

i) $H_0(t,s) = I$.

ii) $H_i(t,s) = \int_{s+(i-1)\tau}^{t-\tau} h(\alpha)H_{i-1}(\alpha,s)\,d\alpha,\ i = 1,2,\cdots,\ell$

here $h(\alpha) = \Phi^{-1}(\alpha+\tau)A_2(\alpha+\tau)\Phi(\alpha)$.

Suppose that for $s+(\ell-1)\tau \le t \le s+\ell\tau$,

$$X(t,s) = \Phi(t)\sum_{i=0}^{\ell-1} H_i(t,s)\Phi^{-1}(s),$$

Then for $s+\ell\tau \le t \le s+(\ell+1)\tau$,

$$\begin{aligned}
X(t,s) &= \Phi(t)\Phi^{-1}(s+\ell\tau)X(s+\ell\tau,s) \\
&\qquad + \int_{s+\ell\tau}^{t}\Phi(t)\,\Phi^{-1}(\alpha)A_2(\alpha)X(\alpha-\tau,s)\,d\alpha \\
&= \Phi(t)\Phi^{-1}(s+\ell\tau)\Phi(s+\ell\tau)\sum_{i=0}^{\ell-1}H_1(s+\ell\tau,s)\Phi^{-1}(s) \\
&\quad + \Phi(t)\int_{s+(\ell-1)\tau}^{t-\tau}\Phi^{-1}(\alpha+\tau)\,A_2(\alpha+\tau)\Phi(\alpha)\sum_{i=0}^{\ell-1}H_i(\alpha,s)\Phi^{-1}(s)\,d\alpha \\
&= \Phi(t)\Bigg[\,I + \left(\int_s^{s+(\ell-1)\tau}h(\alpha)d\alpha + \int_{s+(\ell-1)\tau}^{t-\tau}h(\alpha)d\alpha\right) \\
&\qquad + \left(\int_{s+\tau}^{s+(\ell-1)\tau}h(\alpha)H_1(\alpha,s)d\alpha + \int_{s+(\ell-1)\tau}^{t-\tau}h(\alpha)H_1(\alpha,s)d\alpha\right) \\
&\qquad\qquad \cdots \qquad \cdots \qquad \cdots \\
&\qquad + \left(\int_{s+(\ell-2)\tau}^{s+(\ell-1)\tau}h(\alpha)H_{\ell-2}(\alpha,s)d\alpha + \int_{s+(\ell-1)\tau}^{t-\tau}h(\alpha)H_{\ell-2}(\alpha,s)d\alpha\right) \\
&\qquad + \int_{s+(\ell-1)\tau}^{t-\tau}h(\alpha)H_{\ell-1}(\alpha,s)\,d\alpha\Bigg]\Phi^{-1}(s)
\end{aligned}$$

Thus we have

$$X(t,s) = \Phi(t)\sum_{i=0}^{\ell}H_i(t,s)\Phi^{-1}(s).$$

In the following theorem we state the solution of delay differential operator equation.

Theorem 3.3 Let $f \in \mathbf{L}_2^n[-\tau,t_1]$. Then

$$(\mathfrak{I}x)(t) = f(t),\ \text{for a.a.}\ t \in [-\tau,t_1]$$

if and only if

i) $x(t) = f(t)$, for a.a. $t \in [-\tau,0)$

ii) $x(t) = X(t,0)x_0 + \int_{-\tau}^{0}X(t,s+\tau)A_2(s+\tau)x(s)\,ds + \int_0^t X(t,s)f(s)\,ds,\ t \in [0,t_1]$

where $x_0 \in \Re^n$ is a constant.

Now we state the corollary which will be useful to characterize the optimal control.

Corollary 3.4 Let $f \in \mathbf{L}_2^n[0,t_1+\tau]$. Then

$$(\mathfrak{I}^{*}y)(t) = f(t),\ t \in [-\tau,t_1]$$

if and only if

$$y(t) = \begin{cases} A_2^{*}(t+\tau)y(t+\tau) + f(t), & t \in [-\tau,0] \\ Y(t_1,t)\beta + \int_t^{t_1} Y(s,t)f(s)ds, & t \in [0,t_1] \end{cases}$$

where $\beta \in \Re^n$ is given.

4. Optimal control over $\mathbf{L}_2^n[-\tau,t_1]$

For $i = 1,2$ let $\mathbf{F}_i : \aleph_n \otimes \aleph_n \to \Re^d$ be defined by

$$\mathbf{F}_i(u,x) = \int_{-\tau}^{t_1}\left(f_{i1}^{*}(t)x(t) + f_{i2}^{*}u(t)\right)dt$$

where f_{i1}, f_{i2} are $n \times d$, $m \times d$ real valued matrices whose columnes are in $\aleph_n$, $\aleph_m$ respectively. Let

$$\mathbf{J}(u,x) = \int_{-\tau}^{t_1}\left(|Uu|^2 + |Wx|^2\right)dt + |F_1(u,x)|^2,$$

for $u \in \aleph_m$, $x \in Dom\,\mathfrak{I}$, where $|\cdot|$ is the Euclidean norm.

Let $\gamma \in \Re^d$ be given and $\mathbf{U}$ be a convex subset of $\aleph_m$. we consider the following problem :

Minimize $\mathbf{J}$ over all $\{u,x\}$ such that

i) $u \in \mathbf{U}$, $x \in Dom\,\mathfrak{I}$

ii) $(\mathfrak{I}x)(t) = B(t)u(t),\ t \in [-\tau,t_1]$

iii) $\mathbf{F}_2(u,x) = \gamma$.

Here $B(t)$ is a $n \times m$ real valued matrix whose columns are in $\aleph_m$.

Let $\mathbf{D} = \{\{u,x\} \mid u, x \text{ satisfy i),ii) and iii)}\}$. An element $\{u,x\} \in \mathbf{D}$ is called a succesful control response pair. Throughout this paper, we assume that $\mathbf{D}$ is non empty. When $\{u^{+},x^{+}\}$ minimize $\mathbf{J}$ over $\mathbf{D}$, it is called an optimal-response pair.

The above cost functional is different from the usual one in that $U(t)$ and $W(t)$ are defined on $[-\tau,t_1]$. Futhermore we minimize $\mathbf{J}$ over $u \in \mathbf{U} \subset \aleph_m$ by considering the initial function as a part of the control function. In here, we are using the general boundary condition. we can change this boundary condition into two point boundary condition by choosing f_{i1} and f_{i2} properly.

Remark : Since the initial function $B(t)u(t)$ is not continuous but squre integrable in $[-\tau,0)$, there is no unique solution unless we specify the value of $x(t)$ at $t=0$.

For $t \in [-\tau, t_1]$,

$$X_1(t,s) = \begin{cases} X(t,s+\tau)A_2(s+\tau), & s \in [-\tau,0) \\ X(t,s), & s \in [0, t_1] \end{cases}$$

and define $\mathbf{T}_1, \mathbf{T}_2 : \aleph_m \to \aleph_n$ by

$$(\mathbf{T}_1 u)(t) = \int_{-\tau}^{t_1} X_1(t,s)B(s)u(s)\,ds$$

and

$$(\mathbf{T}_2 u)(t) = B_1(t)u(t)$$

$$\text{where } B_1(t) = \begin{cases} 0, & t \in [0, t_1] \\ B(t), & t \in [-\tau,0) \end{cases}.$$

Set $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$. Then the state function $x(t)$ is expressed by

$$x(t) = X(t,0)x_0 + (\mathbf{T}u)(t), \quad t \in [-\tau, t_1].$$

For $i = 1, 2$, let

$$\mathbf{Q}_i = \int_0^{t_1} f_{i1}^*(t)X(t,0)\,dt,$$

$$m_i(t) = \begin{cases} \left(\int_{t+\tau}^{t_1} f_{i1}^*(s)X(s,t+\tau)ds A_2(t+\tau) + f_{i1}^*(t)\right)B(t) + f_{i2}^*(t), & t \in [-\tau,0) \\ \int_t^{t_1} f_{i1}^*(t)X(s,t)\,ds\, B(t) + f_{i2}^*(t), & t \in [0, t_1] \end{cases}$$

and define $\mathbf{M}_i u : \aleph_m \to \Re^d$ by

$$\mathbf{M}_i u = \int_{-\tau}^{t_1} m_i(t)u(t)\,dt.$$

Then, for $i = 1, 2$,

$$\mathbf{F}_i(u,x) = \mathbf{Q}_i x(0^+) + \mathbf{M}_i u.$$

Thus

$$\mathbf{F}_2(u,x) = \gamma \text{ iff } \mathbf{Q}_2 x(0^+) = \gamma - \mathbf{M}_2 u.$$

Let's consider $\mathbf{Q}_2$ as an operator from $\Re^n$ to $\Re^d$ and assume that

$$\mathbf{U}_a = \{ u \in \mathbf{U} \mid \gamma - \mathbf{M}_2 u \in \mathit{Range}\,\mathbf{Q}_2 \}$$

be non-empty so that $\mathbf{D}$ is not empty. Now pick an arbitrary, but fixed algebraic operator part of $\mathbf{Q}_2$, say it $\mathbf{Q}_2^{\#}$. Then $\mathbf{U}_a$ is a convex subset of $\aleph_m$ and

$x(0^+) = \mathbf{Q}_2^{\#}(\gamma - \mathbf{M}_2 u) + q$, for some $q \in \mathit{Null}\,\mathbf{Q}_2$

Thus

$$\mathbf{F}_1(u,x) = \mathbf{Q}_1(\mathbf{Q}_2^{\#}(\gamma - \mathbf{M}_2 u) + q) + \mathbf{M}_1 u.$$

Hence we have the following theorem.

Theorem 4.1 $\{u, x\} \in \mathbf{D}$ if and only if

i) $u \in \mathbf{U}_a$

ii) $x = X(\cdot,0)(\mathbf{Q}_2^{\#}(\gamma - \mathbf{M}_2 u) + q) + \mathbf{T}u$, for some $q \in \mathit{Null}\,\mathbf{Q}_2$.

Let $\mathbf{H} = \aleph_m \times \aleph_n \times \Re^d$ with the inner product

$$\langle \{u_1, x_1, p_1\}, \{u_2, x_2, p_2\} \rangle = \int_{-\tau}^{t_1} \left(u_1^*(t)u_2(t) + x_1^*(t)x_2(t)\right) dt + p_1^* p_2$$

and its norm

$$\|\{u, x, p\}\|^2 = \int_{-\tau}^{t_1} \left(|u(t)|^2 + |x(t)|^2\right) dt + |p|^2.$$

Then

$$\begin{aligned} \mathbf{J}(u,x) &= \| \{\mathbf{U}u, \mathbf{W}x, \mathbf{F}_2(u,x)\} \|^2 \\ &= \Big\| \{\mathbf{U}u, \mathbf{W}(\mathbf{T} - X(\cdot,0)\mathbf{Q}_2^{\#}\mathbf{M}_2)u, (\mathbf{M}_1 - \mathbf{Q}_1\mathbf{Q}_2^{\#}\mathbf{M}_2)u\} \\ &\quad + \{0, \mathbf{W}X(\cdot,0)q, \mathbf{Q}_1 q\} + \{0, \mathbf{W}X(\cdot,0)\mathbf{Q}_2^{\#}r, \mathbf{Q}_1\mathbf{Q}_2^{\#}r\} \Big\|^2 \\ &= \mathbf{J}_1(u,q), \text{ for some } q \in \mathit{Null}\,\mathbf{Q}_2. \end{aligned}$$

Theorem 4.2 $\{u^+, x^+\}$ is an optimal pair if and only if

i) $\{u^+, x^+\} \in \mathbf{D}$

ii) $\{u^+, q^+\}$ minimize $\mathbf{J}_1(u,q)$ over $\mathbf{U}_a \times \mathit{Null}\,\mathbf{Q}_2$

where

$$x^+ = X(\cdot,0)(\mathbf{Q}_2^{\#}(\gamma - \mathbf{M}_2 u^+) + q^+) + \mathbf{T}u^+$$

Proof. We know that $\{u^+, x^+\}$ is an optimal pair if and only if $\{u^+, x^+\} \in \mathbf{D}$ and $\mathbf{J}(u^+, x^+) \le \mathbf{J}(u,x)$ for all $\{u, x\} \in \mathbf{D}$. Assume that there exists $q^+ \in \mathit{Null}\,\mathbf{Q}_2$ such that

$$x^+ = X(\cdot,0)(\mathbf{Q}_2^{\#}(\gamma - \mathbf{M}_2 u^+) + q^+) + \mathbf{T}u^+.$$

Then $\mathbf{J}(u^+, x^+) \le \mathbf{J}(u^+, q^+)$ and $\mathbf{J}(u^+, q^+) \le \mathbf{J}(u,q)$ for all $\{u, q\} \in \mathbf{U}_a \times \mathit{Null}\,\mathbf{Q}_2$. This proves the theorem•

Theorem 4.3 Assume that $\mathbf{U}$ is closed and U is invertible. Then

$$\mathbf{K}_1 = \left\{ \{\mathbf{U}u, \mathbf{W}(\mathbf{T} - X(\cdot,0)\mathbf{Q}_2^{\#}\mathbf{M}_2)u, (\mathbf{M}_1 - \mathbf{Q}_1\mathbf{Q}_2^{\#}\mathbf{M}_2)u\} \mid u \in \mathbf{U}_a \right\}$$

is closed convex subset of $\mathbf{H}$.

Proof. Let's define $\mathbf{A} : \aleph_m \to \aleph_n$ by $(\mathbf{A}u)(t) = U(t)u(t)$, $t \in [-\tau, t_1]$. Then A is a bound operator . Note that $\mathbf{M}_3 = \mathbf{W}(\mathbf{T} - X(\cdot,0)\mathbf{Q}_2^{\#}\mathbf{M}_2)$ and $\mathbf{M}_4 = \mathbf{M}_1 - \mathbf{Q}_1\mathbf{Q}_2^{\#}\mathbf{M}_2$ are bounded operators. Thus $\mathbf{K}_1$ is a convex subset of $\mathbf{H}$. Let $\{a_n, b_n, c_n\}$ be sequence in $\mathbf{K}_1$ converge to $\{a, b, c\}$. We want to show $\{a, b, c\} \in \mathbf{K}_1$. Since U is invertible, $\mathbf{A}$ is one to one and *Range* $\mathbf{A}$ is closed. Therefore there exists a sequence $\{u_n\} \in \mathbf{U}_a$ converging to $\{u\} \in \mathbf{U}_a$ such that $U(\cdot)u_n(\cdot) = a_n$ and $U(\cdot)u(\cdot) = a$. Thus a_n converge to a. Let $\{a_n, b_n, c_n\} = \{\mathbf{U}u_n, \mathbf{M}_3 u_n, \mathbf{M}_4 u_n\}$. Then $\{a_n, b_n, c_n\}$ converge to $\{a, b, c\} = \{\mathbf{U}u, \mathbf{M}_3 u, \mathbf{M}_4 u\}$ as u_n converge to u. Thus $\mathbf{K}_1$ is closed•

Remark. If we let

$$\mathbf{K}_2 = \{\{0, \mathbf{W}X(\cdot,0)q, \mathbf{Q}_1 q\} \mid q \in \mathit{Null}\,\mathbf{Q}_2\}$$

then $\mathbf{K}_2$ is a closed convex subset of $\mathbf{H}$.

For the sake of convenience let

$$\mathbf{P}(u,q) = \{\mathbf{U}u, \mathbf{W}(\mathbf{T} - X(\cdot,0)\mathbf{Q}_2^{\#}\mathbf{M}_2)u, (\mathbf{M}_1 - \mathbf{Q}_1\mathbf{Q}_2^{\#}\mathbf{M}_2)u\} + \{0, \mathbf{W}X(\cdot,0)q, \mathbf{Q}_1 q\}$$

and

$$\beta = \{0, WX(\cdot,0)Q_2^{\#}\gamma, Q_1Q_2^{\#}\gamma\}.$$

Then

$$\mathbf{K} = \mathbf{K}_1 \oplus \mathbf{K}_2 = \{\mathbf{P}(u,p) \mid u \in \mathbf{U}_a, \ \mathit{Null}\,Q_2\}$$

is a closed convex subset of $\mathbf{H}$.

Theorem 4.4 (Existence)

Assume that $\mathbf{U}$ is closed and U is invertible. Then there exists a unique $\{u^+, x^+\} \in \mathbf{D}$ and $\mathbf{J}(u^+, x^+) \le \mathbf{J}(u,x)$, for all $\{u, x\} \in \mathbf{D}$.

Proof. Since $\mathbf{K}$ is a closed convex subset of $\mathbf{H}$ and $\beta \in \mathbf{H}$, by the projection theorem for the closed convex subset of Hilbert space, there exist a unique $\mathbf{P}(u^+, p^+) \in \mathbf{K}$ such that

$$\left\|\mathbf{P}(u^+, p^+) + \beta\right\|^2 \le \|\mathbf{P}(u,p) + \beta\|^2, \mathbf{P}(u,p) \in \mathbf{K}.$$

That is,

$$\mathbf{J}_1(u^+, p^+) \le \mathbf{J}_1(u,p), \{u, q\} \in \mathbf{U}_\alpha \times \mathit{Null}\,Q_2.$$

Now note that $\mathbf{J}_1(u^+, p^+) = \mathbf{J}(u^+, x^+)$ and $\mathbf{J}_1(u,p) = \mathbf{J}(u,x)$. Thus there exists a unique $\{u^+, x^+\} \in \mathbf{D}$ such that

$$\mathbf{J}(u^+, x^+) \le \mathbf{J}(u,x), \ \{u, x\} \in \mathbf{D}.$$

Theorem 4.5 (Neccessary and sufficient condition)

Assume that $\mathbf{U}$ is closed and U is invertible. Let $\{u^+, x^+\} \in \mathbf{D}$. Then $\{u^+, x^+\}$ is an optimal pair if and only if there exists $\eta \in \mathit{Dom}\ \mathfrak{I}^+$ and $\delta_i \in \mathit{Dom}\ \mathfrak{I}^+ (i=1,2)$ columnwise such that

i) $(\mathfrak{I}^+\eta)(t) = W^*(t)W(t)x^+(t), \ t \in [-\tau, t_1]$

$\eta(t_1) = 0$

ii) for $j = 1,2,\ldots,d$

$(\mathfrak{I}^+\delta_{ij})(t) = f_{i1_j}(t), \ t \in [-\tau, t_1]$

$\delta_{ij}(t_1) = 0$

iii) $\langle U^*Uu^+ + B^*\eta + (B^*\delta_1 + f_{12})\mathbf{F}_1(u^+, x^+) - (B^*\delta_2 + f_{22})(Q_2^{\#})^*(X(\cdot,0)W^*Wx^+ + Q_1^*)\mathbf{F}_1(u^+, x^+), u^+ - u\rangle \le 0$, for all $u \in \mathbf{U}_a$

and

$$X^*(\cdot,0)W^*Wx^+ + Q_1^*\mathbf{F}_1(u^+, x^+) \in (\mathit{Null}\,Q_2)^\perp.$$

Proof. By theorem 1.4, there exists $\mathbf{P}(u^+, p^+) \in \mathbf{K}$ such that

$$\left\|\mathbf{P}(u^+, p^+) + \beta\right\|^2 \le \|\mathbf{P}(u,p) + \beta\|^2, \mathbf{P}(u,p) \in \mathbf{K}.$$

Futhermore,

$$\langle \mathbf{P}(u^+, p^+) + \beta, \mathbf{P}(u^+, p^+) - \mathbf{P}(u,p)\rangle \le 0, \ \mathbf{P}(u,p) \in \mathbf{K}.$$

That is,

$$\Big\langle\{(Uu^+, Wx^+, \mathbf{F}_1(u^+, x^+))\}, \{U(u^+ - u), WX(\cdot,0)(q^+ - q) + \mathbf{M}_3(u^+ - u), Q_1(q^+ - q) + \mathbf{M}_4(u^+ - u)\}\Big\rangle \le 0, \ \text{for all } \{u, q\} \in \mathbf{U}_\alpha \times \mathit{Null}\,Q_2$$

where $\mathbf{M}_3 = W(\mathbf{T} - X(\cdot,0)Q_2^{\#}\mathbf{M}_2)$ and $\mathbf{M}_4 = \mathbf{M}_1 - Q_1Q_2^{\#}\mathbf{M}_2$.

That implies

$$\langle Uu^+, U(u^+ - u)\rangle + \langle Wx^+, WX(\cdot,0)(q^+ - q)\rangle + \langle Wx^+, \mathbf{M}_3(u^+ - u)\rangle + \langle \mathbf{F}_1(u^+, x^+), Q_1(q^+ - q)\rangle + \langle \mathbf{F}_1(u^+, x^+), \mathbf{M}_4(u^+ - u)\rangle \le 0,$$

for all $u \in \mathbf{U}_a$ and $q \in \mathit{Null}\,Q_2$.

Therefore,

(1-5) $\langle U^*Uu^+ + \mathbf{M}_3^*Wx^+ + \mathbf{M}_4^*\mathbf{F}_1(u^+, x^+), u^+ - u\rangle + \langle X^*(\cdot,0)W^*Wx^+ + Q_1^*\mathbf{F}_1(u^+, x^+), q^+ - q\rangle \le 0$,

for all $u \in \mathbf{U}_a$ and $q \in \mathit{Null}\,Q_2$.

Since $\mathit{Null}\,Q_2$ is a closed subspace of $\mathfrak{R}^n$, we have

(1-6) $\langle U^*Uu^+ + \mathbf{M}_3^*Wx^+ + \mathbf{M}_4^*\mathbf{F}_1(u^+, x^+), u^+ - u\rangle - \langle X^*(\cdot,0)W^*Wx^+ + Q_1^*\mathbf{F}_1(u^+, x^+), q^+ - q\rangle \le 0$,

for all $u \in \mathbf{U}_a$ and $q \in \mathit{Null}\,Q_2$.

By (1-5) and (1-6),

$$\langle U^*Uu^+ + \mathbf{M}_3^*Wx^+ + \mathbf{M}_4^*\mathbf{F}_1(u^+, x^+), u^+ - u\rangle \le 0, \ u \in \mathbf{U}_a$$

and

$$\langle X^*(\cdot,0)W^*Wx^+ + Q_1^*\mathbf{F}_1(u^+, x^+), q^+ - q\rangle = 0, \ q \in \mathit{Null}\,Q_2.$$

That implies

$$X^*(\cdot,0)W^*Wx^+ + Q_1^*\mathbf{F}_1(u^+, x^+) \in (\mathit{Null}\,Q_2)^\perp$$

Note that

$$\mathfrak{I}^*\nu = \begin{cases} B^*(t)\int_t^{t_1} X^*(s,t)\,\nu(s)\,ds, & t \in [0,t_1] \\ B^*(t)\left(\nu(t) + A_2^*(t+\tau)\int_{t+\tau}^{t_1} X^*(s,t+\tau)\,\nu(s)ds\right), & t \in [-\tau,0] \end{cases}$$

and

$$\mathbf{M}_i^*\,w = \begin{cases} \left(B^*(t)\int_t^{t_1} X^*(s,t)f_{i1}(s)\,ds + f_{i2}(t)\right) w(t), & t \in [0,t_1] \\ B^*(t)\left(f_{i1}(t) + A_2^*(t+\tau)\int_{t+\tau}^{t_1} X^*(s,t+\tau)f_{i1}(s)ds\right) w(t) + f_{i2}(t)\,w(t), & t \in [-\tau,0] \end{cases}$$

Now let

$$\eta(t) = \begin{cases} \int_t^{t_1} X^*(s,t)W^*(s)W(s)x^+(s)\,ds, & t \in [0,t_1] \\ W^*Wx^+(t) + A_2^*(t+\tau)\int_{t+\tau}^{t_1} X^*(s,t+\tau)W^*Wx^+(s)ds, & t \in [-\tau,0] \end{cases}$$

Then $\eta(t_1) = 0$ and $\eta(t)$ is the solution of

$$(\mathfrak{I}^+\eta)(t) = W^*(t)W(t)x^+(t), \ t \in [-\tau, t_1].$$

Similary, for $i = 1,2$, let

$$\delta_i(t) = \begin{cases} \int_t^{t_1} X^*(s,t)f_{i1}(s)\,ds, & t \in [0,t_1] \\ f_{i1}(t) + A_2^*(t+\tau)\int_{t+\tau}^{t_1} X^*(s,t+\tau)f_{i1}(s)ds, & t \in [-\tau,0] \end{cases}$$

Then, for $j = 1,2, \cdots, d$, $\delta_{ij}(t_1) = 0$ and $\delta_{ij}(t)$ is the solution of

$$\left(\mathfrak{T}^+\delta_{ij}\right)(t) = f_{i1_j}(t),\ t \in [-\tau, t_1].$$

Therefore

$$\left(\mathbf{T}^* W^* W x^+\right)(t) = B^*(t)\eta(t),\ t \in [-\tau, t_1]$$

and

$$\left(\mathbf{M}_i^* w\right)(t) = B^*(t)\delta_i(t) + f_{i2}(t), \quad t \in [0, t_1].$$

Thus

$$\begin{aligned} &U^*Uu^+ + \mathbf{M}_3^* W x^+ + \mathbf{M}_4^* \mathbf{F}_1\left(u^+, x^+\right) \\ &= U^*Uu^+ + B^*\eta + \left(B^*(t)\delta_i(t) + f_{i2}(t)\right)\mathbf{F}_1\left(u^+, x^+\right) \\ &- \left(B^*\delta_2 + f_{22}\right)\left(\mathbf{Q}_2^{\#}\right)^*\left(X^*(\cdot,0)W^*Wx^+ + \mathbf{Q}_1^*\mathbf{F}_1\left(u^+, x^+\right)\right) \end{aligned}$$

This proves "only if part". The "if part" can be traced back•

Corollary 4.6 Assume that $\mathbf{U} = \mathbf{L}_2^m[-\tau, t_1]$ and U is invertible. Assume further that $\mathit{Null}\,\mathbf{Q}_2 = \{0\}$. Let $\{u^+, x^+\} \in \mathbf{D}$. Then $\{u^+, x^+\}$ is an optimal pair if and only if there exists $\eta \in \mathit{Dom}\ \mathfrak{T}^+$ and $\delta_i \in \mathit{Dom}\ \mathfrak{T}^+$ (i=1,2) columnwise such that

i) $\left(\mathfrak{T}^+\eta\right)(t) = W^*(t)W(t)x^+(t),\ t \in [-\tau, t_1]$

$\eta(t_1) = 0$

ii) for $j = 1,2, \ldots, d$

$\left(\mathfrak{T}^+\delta_{ij}\right)(t) = f_{i1_j}(t),\ \ t \in [-\tau, t_1]$

$\delta_{ij}(t_1) = 0$

iii)

$$\begin{aligned} &U^*Uu^+ + B^*\eta_1 + \left(B^*\delta_1 + f_{12}\right)\mathbf{F}_1\left(u^+, x^+\right) - \\ &\left(B^*\delta_2 + f_{22}\right)\left(Q_2^{\#}\right)^*\left(X(\cdot,0)W^*Wx^+ + Q_1^*\right)\mathbf{F}_1\left(u^+, x^+\right) = 0, \\ &\quad \text{for all } t \in [-\tau, t_1]. \end{aligned}$$

Proof. Since $\mathit{Null}\,\mathbf{Q}_2 = \{0\}$ and $\mathbf{U} = \mathbf{L}_2^m[-\tau, t_1]$, the admissible control space is trhe whole space. Therefore the inequality iii) of the above Theorem 2.5 becomes equality•

Let d=n and $f_{21}^*(t) = I_n$, $t \in [-\tau, t_1]$, and choose

$$f_{22}^*(t) = \begin{cases} \int_{t+\tau}^{t_1} X(s, t+\tau)ds A_2(t+\tau) + B(t), & t \in [-\tau, 0) \\ \int_t^{t_1} X(s,t)ds B(t), & t \in [0, t_1] \end{cases}$$

Then $\mathbf{F}_2(u,x) = \gamma$ becomes $x_0 = \left(\int_0^{t_1} Y(0,s)\,ds\right)^{-1}\gamma$. That is the general boundary condition becomes the initial condition. In the following we discuss the optimal control of the initial boundary condition.

Theorem 4.7 Let $x_0 \in \mathfrak{R}^n$ be given and **U** is a convex subset of $\mathbf{L}_2^m[-\tau, t_1]$. Cosider the problem of finding the optimal control u^+ minimizing

$$\mathbf{J}_0(u, x) = \int_{-\tau}^{t_1}\left(|Uu|^2 + |Wx|^2\right)dt$$

subject to

i) $u \in \mathbf{U}$, $x \in \mathit{Dom}\,\mathfrak{T}$

ii) $(\mathfrak{T}x)(t) = B(t)u(t)$, $t \in [-\tau, t_1]$

iii) $x(0) = x_0$.

Assume that U is invertible. Let $\{u^+, x^+\} \in \mathbf{D}$. Then $\{u^+, x^+\}$ is an optimal pair if and only if there exists $\eta \in \mathit{Dom}\ \mathfrak{T}^+$ such that

i) $\left(\mathfrak{T}^+\eta\right)(t) = W^*(t)W(t)x^+(t),\ t \in [-\tau, t_1]$

$\eta(t_1) = 0$

ii) $\left\langle U^*Uu^+ + B^*\eta, u^+ - u\right\rangle = 0$, for all $u \in \mathbf{U}$.

Proof. Referring to theorem 4.5, we see that $\mathbf{F}_1(u,x) = 0$, $\mathbf{Q}_2$ is a nonsingular matrix and $x_0 = \mathbf{Q}_2^{\#}\gamma$. For $i = 1,2$, $B^*\delta_i + f_{i2} = 0$. Thus we have the above theorem•

Corollary 4.8 Consider the problem of theorem 4.7. Assume that U is invertible and $\mathbf{U} = \mathbf{L}_2^m[-\tau, t_1]$. Let $\{u^+, x^+\} \in \mathbf{D}$. Then $\{u^+, x^+\}$ is an optimal pair if and only if there exists $\eta \in \mathit{Dom}\ \mathfrak{T}^+$ such that

i) $\left(\mathfrak{T}^+\eta\right)(t) = W^*(t)W(t)x^+(t),\ t \in [-\tau, t_1]$

$\eta(t_1) = 0$

ii) $u^+(t) = -\left(U^*U\right)^{-1}(t)B^*(t)\eta(t),\ t \in [-\tau, t_1]$.

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