A Robust Generalized Predictive Controls

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ABSTRACT

In this paper, a new GPC (Generalized Predictive Control) algorithm which is robust to disturbances is proposed. This controller minimizes the LQ cost function when the disturbance maximizes this cost function. The solution is obtained from the min-max problem which can be solved by differential game theory and has the non-recursive form which does not use the Riccati equation. Its another solution for state space models is investigated.

1. Introduction

Recently H_∞ control has been one of the most attractive and famous modern controls [1]. In H_∞ control theory, disturbance is used in representing the unmodeled dynamics of the system and external effects to the system and it can be modeled as deterministic signals, where only an upper bound of power is known [2]. There are several methods which solve the problem of obtaining the H_∞ controller which minimizes the maximum effects of disturbance on the system performances. One of these methods is using the LQ differential game theory, in which the disturbance is modeled as a player that intends to maximize, while the controller tries to minimize the performances of the system [3]. The link between the H_∞ optimization and the LQ differential game theories is showed in continuous cases and discrete cases for state space models [4].

But these approaches have not been applied for discrete time I/O models in time domain, where the optimal cost minimization is demanded. Especially in another control area as Predictive Control, there are few results which deal with this robust concept.

Predictive control is widely used in process application because this scheme has good tracking performances and nice ability to manipulate the constraint. GPC is a representative predictive control for I/O models which uses the same cost function as RHTC(Receding Horizon tracking Control) which is the finite time receding horizon LQ tracking control[5],[6]. However, the stability and the robustness of this algorithm are investigated not by analytical methods but by simulations or experiences [7].

In this paper, we will present a new GPC algorithms for discrete time I/O models which is robust to unmodeled disturbances, in the sense that this controller minimizes the cost function for the case of that disturbance maximizes the cost function. This means that this controller intends to achieve the good performances when the unmodeled dynamics of system or external effects give the worst influences on the system performances, even if this controller gives less performances than the original GPC with no disturbance. The basic formulation of the predictor is originated from the framework of Clarke's GPC [6]. But deterministic disturbance is included in model system and prediction equations. The solution is obtained from the well known results of LQ differential game theories, but its solution does not demand the Riccati equation and can have the tracking form.

In section 2, the modeled system and predictors are introduced and the Robust Generalized Predictive Control for I/O models is derived by the results of LQ

Game theory. In section 3, robust GPC for state space models is proposed and some remarks are presented. Section 4 is the conclusion of this paper.

2. Robust Generalized Predictive Controls

Consider a linear time-invariant discrete-time system in an I/O model described by,

$$A_p(q^{-1})y(t) = B_p(q^{-1})u(t-1) + D_p(q^{-1})w(t) \tag{2.1}$$

$$A_p(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_n q^{-n}$$
$$B_p(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_m q^{-m}$$
$$D_p(q^{-1}) = d_0 + d_1 q^{-1} + \ldots + d_p q^{-p},$$

where q^{-1} is the unit delay operator, u(t) and y(t) are the control input and output respectively, and the order of $A_p(q^{-1})$ is the system order, n. w(t) is a unpredictable disturbance to system which must be taken into account. For convenience, $D_p(q^{-1})$ is assumed to be identity, which means that the disturbance sequences are uncorrelated and the current disturbance w(t) gives influence only the current state.

To derive a j-step ahead predictor, let's introduce the following Diophantine equation.

$$1 = E_j(q^{-1})A_p(q^{-1}) + q^{-j}F_j(q^{-1}) \tag{2.2}$$

$$E_j(q^{-1}) = e_0^j + e_1^j q^{-1} + \ldots + e_{j-1}^j q^{-j+1}$$
$$F_j(q^{-1}) = f_0^j + f_1^j q^{-1} + \ldots + f_{n-1}^j q^{-n+1},$$

where the coefficients e_i's and f_i's are determined uniquely from the above equation.

Simple manipulations with the above equations yield:

$$\begin{aligned} y(t+j) &= E_j(q^{-1})B_p(q^{-1})u(t+j-1) + F_j(q^{-1})y(t) + E_j(q^{-1})w(t+j) \\ &= G_j(q^{-1})u(t+j-1) + H(q^{-1})u(t-1) + F_j(q^{-1})y(t) + E_j(q^{-1})w(t+j), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} H(q^{-1})u(t-1) &= h_0^j u(t-1) + h_1^j u(t-2) + \ldots h_m^j u(t-m-1) \\ G_j(q^{-1})u(t+j-1) &= g_0^j u(t+j-1) + g_1^j u(t+j-2) + \ldots + g_{j-1}^j u(t), \end{aligned} \tag{2.4}$$

and h_i^j and g_i^j are calculated from $E_j(q^{-1})B_p(q^{-1})$. As the g_i^j's in the equation (2.4) are the impulse responses of the system[6], $g_i^j = g_i$ for i<j. $H(q^{-1})$ is of degree m and $G_j(q^{-1})$ is of degree j-1.

As a result, future output y(t+j) is composed of $f(t+j) \triangleq H(q^{-1})u(t-1) + F_j(q^{-1})y(t)$, $G_j(q^{-1})u(t+j-1)$ and $E_j(q^{-1})w(t+j)$. The first depends on past information for inputs and outputs and corresponds to the zero-input response. The next depends on the future undetermined controls and corresponds to zero-state response. The last is unpredictable term by which the future disturbances will affect on the future output y(t+j). The predicted output at time t+j is given by,

$$\begin{aligned} y(t+j) &= f(t+j) + g_0 u(t+j-1) + g_1(t+j-2) + \ldots + g_{j-1}u(t) \\ &+ e_0^j w(t+j) + e_1^j w(t+j-1) + \ldots + e_{j-1}^j w(t+1). \end{aligned} \tag{2.5}$$

This predicted outputs are obtained from j=1 to j=N. Where the output horizon N is used in cost function and the control horizon N_u are existing concept which enables the gain matrix to be simple [6]. From the concept of control horizon, it is assumed that the future control inputs for [$t+N_u$, t+N-1] are zero. The relations of the vector forms of outputs and inputs are given by the following matrix equation.

$$Y = GU + DW + f \tag{2.6}$$

where the input, output and disturbance vectors are defined by,

$$Y = \begin{bmatrix} y(t+1) \\ y(t+2) \\ \cdot\cdot \\ y(t+N) \end{bmatrix}, U = \begin{bmatrix} u(t) \\ u(t+1) \\ \cdot\cdot \\ u(t+N_u-1) \end{bmatrix}, \quad W = \begin{bmatrix} w(t+1) \\ w(t+2) \\ \cdot\cdot \\ w(t+N) \end{bmatrix}, \tag{2.7}$$

and the gain matrix G, D and the vector f are given by,

$$G=\begin{bmatrix} g_0 & 0 & \cdot\cdot & 0 \\ g_1 & g_0 & \cdot\cdot & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & 0 \\ g_{N_u-1} & g_{N_u-2} & \cdot\cdot & g_0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ g_{N-1} & g_{N-2} & \cdot\cdot & g_{N-N_u} \end{bmatrix},\ D=\begin{bmatrix} e_0^1 & 0 & \cdot\cdot & 0 \\ e_1^2 & e_0^2 & \cdot\cdot & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ e_{N-1}^N & e_{N-2}^N & \cdot\cdot & e_0^N \end{bmatrix}, \quad (2.8)$$

$$f=[f(t+1)\,f(t+2)\ldots f(t+N)]' \quad (2.9)$$

Consider the following cost function:

$$\begin{aligned} J_{LQ} &= \sum_{i=1}^{N}\{[y(t+i)-y_r(t+i)]' \\ &\quad [y(t+i)-y_r(t+i)] \\ &\quad + ru(t+i-1)'u(t+i-1)\} \\ &= (Y-Y_r)'(Y-Y_r)+rU'U \end{aligned} \quad (2.10)$$

where $y_r(t+i)$ is the desired output at time t+i and Y_r are given as,

$$Y_r=[y_r(t+1)\,y_r(t+2)\cdots y_r(t+N)]' \quad (2.11)$$

Then the problem of obtaining the predictive controller which is robust to the unpredictable disturbances is:

$$\min_U \max_W [J_{LQ}-\gamma W'W]$$
$$J = J_{LQ}-\gamma W'W$$

This cost J can be rewritten by (2.6) and (2.10):

$$\begin{aligned} J &= (Y-Y_r)'(Y-Y_r) + rU'U-\gamma W'W \\ &= U'(G'G+rI)U+U'G'(f-Y_r) \\ &\quad +(f-Y_r)'GU+(f-Y_r)'(f-Y_r) \\ &\quad +U'G'DW+W'D'GU \\ &\quad +(f-Y_r)'DW \\ &\quad +W'D(f-Y_r)+W'D'DW-\gamma W'W \end{aligned}$$

To obtain solution of the above problem, we use the results of the LQ game theory. In LQ game theory, the necessary conditions of the existence of optimal solution U^* and W^* are:

$$\begin{aligned} A:&\ \frac{\partial J}{\partial W}\Big|_{W=W^*}=0 \qquad \frac{\partial^2 J}{\partial W^2}\Big|_{W=W^*}\le 0 \\ B:&\ \frac{\partial J}{\partial U}\Big|_{U=U^*}=0 \qquad \frac{\partial^2 J}{\partial U^2}\Big|_{U=U^*}\ge 0 \end{aligned} \quad (2.12)$$

To obtain the worst disturbance W^* which maximize the cost function, the above condition A is applied to cost function.

$$\frac{\partial J}{\partial W} = 2[D'D-\gamma I]W + 2D'(f-Y_r)+2D'GU = 0$$

Then

$$W^*=[\gamma I-D'D]^{-1}D'[GU+f-Y_r] \quad (2.13)$$

and this result also satisfies the following condition:

$$\frac{\partial^2 J}{\partial W^2}\Big|_{W=W^*} = D'D-\gamma I \le 0 \quad (2.14)$$

Then if let $\Xi = \gamma I - D'D$ and the disturbance W^* is really applied to system, the above cost function becomes

$$\begin{aligned} J\big|_{W=W^*} &= U'(G'G+rI)U+U'G'(f-Y_r) \\ &\quad +(f-Y_r)'GU+(f-Y_r)'(f-Y_r) \\ &\quad +U'G'D\Xi^{-1}D'GU+U'G'D\Xi^{-1}D'GU \\ &\quad +U'G'D\Xi^{-1}D'(f-Y_r) \\ &\quad +(f-Y_r)'D\Xi^{-1}D'GU+\ldots \\ &\quad +(f-Y_r)'D\Xi^{-1}D'GU \\ &\quad +U'G'D\Xi^{-1}D'(f-Y_r)+\ldots \\ &\quad -[GU+f-Y_r]'D\Xi^{-1}D'[GU+f-Y_r]. \end{aligned}$$

To obtain the optimal control which minimizes the cost function when the worst case of disturbance is applied to system, the condition B is used as follows:

$$\frac{\{\partial J|_{W=W^*}\}}{\partial U} = 2[G'G+rI+G'D\Xi^{-1}D'G]U + 2G'[I+D\Xi^{-1}D']^{-1}f = 0$$

Then the optimal control vector U^* is obtained by

$$\begin{aligned} U^* &= [G'G+rI+G'D\Xi^{-1}DG]^{-1} \\ &\quad \cdot G'[I+D\Xi^{-1}D'](Y_r-f) \\ &= [G'(I+D\Xi^{-1}D')G+rI]^{-1} \\ &\quad \cdot G'[I+D\Xi^{-1}D'](Y_r-f) \\ &= [G'(I-\tfrac{1}{\gamma}DD')^{-1}G+rI]^{-1} \\ &\quad \cdot G'(I-\tfrac{1}{\gamma}DD')^{-1}(Y_r-f) \end{aligned}$$

The robust GPC law is described by the following control which uses the receding horizon concepts:

$$\begin{aligned} u(t) &= [1\ 0\ \ldots\ 0][G'(I-\tfrac{1}{\gamma}DD')^{-1}G+rI]^{-1} \\ &\quad \cdot G'(I-\tfrac{1}{\gamma}DD')^{-1}(Y_r-f) \end{aligned} \quad (2.15)$$

3. Robust GPC for state space models

Consider the following SISO state space model,

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + B_1 w(t) \\ y(t) &= Cx(t) \end{aligned} \qquad (3.1)$$

where $x(t) \in R^n, u(t), w(t)$ and $y(t)$ is state, control input, disturbance and output respectively. The future output $y(t+i)$ is given for $i=1$ to N,

$$y(t+1) = CAx(t) + CBu(t) + CB_1 w(t)$$

$$y(t+2) = CA^2x(t) + [\,CAB \;\; CB\,]\begin{bmatrix} u(t) \\ u(t+1) \end{bmatrix} + [\,CAB_1 \;\; CB_1\,]\begin{bmatrix} w(t) \\ w(t+1) \end{bmatrix}$$

$$\vdots$$

$$y(t+N) = CA^N x(t) + [\,CA^{N-1}B \;\; CA^{N-2}B \;\cdots\; CB\,]\begin{bmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+N-1) \end{bmatrix} + [\,CA^{N-1}B_1 \;\; CA^{N-2}B_1 \;\cdots\; B_1\,]\begin{bmatrix} w(t) \\ w(t+1) \\ \vdots \\ w(t+N-1) \end{bmatrix}$$

Its vector form is :

$$Y(t) = GU(t) + Hx(t) + DW(t) \qquad (3.2)$$

where

$$H = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^N \end{bmatrix} \quad G = \begin{bmatrix} CB & 0 & \cdot & 0 \\ CAB & CB & \cdot & 0 \\ & \cdots & & \\ CA^{N-1}B & CA^{N-2}B & \cdot & CB \end{bmatrix}$$

$$Y = \begin{bmatrix} y(t+1) \\ y(t+2) \\ \vdots \\ y(t+N) \end{bmatrix} \quad U = \begin{bmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+N-1) \end{bmatrix}$$

$$D = \begin{bmatrix} CB_1 & 0 & \cdot & 0 \\ CAB_1 & CB_1 & \cdot & 0 \\ & \cdots & & \\ CA^{N-1}B_1 & CA^{N-2}B_1 & \cdot & CB_1 \end{bmatrix} \quad W = \begin{bmatrix} w(t) \\ w(t+1) \\ \vdots \\ w(t+N-1) \end{bmatrix}$$

If we let $f \triangleq Hx(t)$, the above equation (3.2) has the same form as that of (2.6). Thus by the same procedure of section 2, the robust control problem for state space models can be solved. The another solution which is obtained by the Riccati equation is presented in [2].

The necessay condition of existence of solution in the LQ game problem is,

$$\gamma > \max_i \lambda_i(D_N' D_N) = \max_i \lambda_i(D_N D_N')$$

and the sufficient condition of the existence of control (2.15), even if it may not be the optimal solution, is as follows,

$$G'[\,I - \frac{1}{\gamma} D_N D_N'\,]^{-1} G + rI > 0 \qquad (3.3)$$

Optimal cost J^* which is obtained when both the worst case of disturbance W^* and the optimal control U^* are applied to system is

$$\begin{aligned} J^* &= J(U_N^*, W_N^*) \\ &= U_N^{*\prime}[G'\Sigma G + rI]U_N^* + U_N^{*\prime} G'\Sigma f + f'\Sigma G U_N + f'\Sigma f \\ &= f'\Sigma G(G\Sigma G + rI)^{-1} G'\Sigma f - f'\Sigma f \qquad (3.4) \\ &= f'[\Sigma - \Sigma G(G'\Sigma G + rI)^{-1} G'\Sigma]f \\ &= f'[\Sigma^{-1} + \frac{1}{r} GG']^{-1} f \\ &= f'[I - \frac{1}{\gamma} D_N D_N' + \frac{1}{r} GG']^{-1} f \end{aligned}$$

where

$$\Sigma = I + D_N(\gamma I - D_N' D_N)^{-1} D_N' = (I - \frac{1}{\gamma} D_N D_N')^{-1}$$

If the equation (3.3) is satisfied, the optimal cost (3.4) is positive for all f vector, which corresponds to having the positive definite solution of Riccati equation. The condition which guarantees the positive definitness of Riccati solution in [2] can be compared with (3.3)

4. Conclusion

In this paper, from I/O models which contains the disturbance effects, a new predicitor is introduced. The predicted future output is represented by the past information, the undetermined control and unpredictable disturbances for the finite future time. A new GPC (Generalized Predictive Control) algorithm which is designed to be robust to these disturbances is proposed. This controller minimizes the finite time LQ cost function when disturbances of finite future time maximize this cost function. The solution is obtained from the min-max problem which can be solved by differential LQ game theory and has the non-recursive form which does not use the Riccati equations. For same state space models in [2], a another solution of non-recursive form can be obtained, which is expected to be identical to the recursive solution of [2].

The further researches for the stability and the robustness bound of this control and the solution for the case of that $D(q^{-1})$ is not identity is demanded.

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