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Mixed H_2/H_∞ Control with Pole Placement : A Convex Optimization Approach

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Abstract

In this paper, we consider the synthesis of mixed H_2/H_∞ controllers such that the closed-loop poles are located in a specified region in the complex plane. Using solution to a generalized Riccati equation and a change of variable technique, it is shown that this synthesis problem can be reduced to a convex optimization problem over a bounded subset of matrices. This convex programming can be further reduced to Generalized Eigenvalue Minimization Problem where Interior Point method has been recently developed to efficiently solve this problem.

1 Introduction

Mixed H_2/H_∞ control theory offers a way of combining disturbance attenuation system which is guaranteed by H_∞-norm of a certain closed-loop transfer function, and quadratic performance which is measured by H_2-norm of another transfer function[2,3,5,6]. The mixed-norm theory, however, does not directly deal with the desired dynamic characteristic of the closed-loop system which is commonly expressed in terms of transient response. This characteristic is generally described with the aid of conditions imposed on the spectrum of the closed-loop system[14-16]. For many practical problems, exact eigenvalue assignment may not be necessary; it suffices to locate closed-loop eigenvalues in prescribed subregions in the left half plane, for example by employing a criterion for root clustering[11-14].

The problem of determining controllers which minimize a quadratic performance subject to an H_∞ constraint such that the closed-loop eigenvalues are located in a class of regions in the left half plane, is recently presented by Bambang *et al*[8]. In that paper, an equivalent optimization problem is formulated and necessary condition is derived. This condition involves highly coupled nonlinear equations which do not have closed-form solutions, and thus iterative algorithm is required to solve such equations. In the present paper we consider the same problem, but we take alternative approach based on the convex optimization techniques. A generalized Riccati equation is employed in order to satisfy H_∞ and pole placement constraint as well as to obtain an upperbound on the H_2 cost. Using a solution to the generalized Riccati equation and a change of variable technique, we show that mixed H_2/H_∞ synthesis problem with regional pole constraint can be reduced to the convex optimization problem over a bounded subset of symmetric real matrices. For computational purpose, we will reduce the resulting convex optimization problem into a Generalized Eigenvalue Minimization Problem where an attractive Interior Point method has been recently developed to efficiently solve this problem[9].

Now, let us outline the content of the present paper. In Section 2, we formulate mixed H_2/H_∞ control with pole placement. In this section we present some preliminaries which are useful in deriving some of the results of this paper. In Section 3 we will show that this synthesis problem can be reduced to a convex optimization problem over a bounded subset of matrices, via a solution to generalized Riccati equation and change of variable technique. For computational purpose, we will reduce this convex programming to Generalized Eigenvalue Minimization Problem and present Interior Point Method to find its solution. Finally, in Section 5 we present some concluding remarks.

2 Problem Formulation And Preliminaries

2.1 Problem Formulation

In this subsection, we formulate mixed H_2/H_∞ control problem with root clustering. Consider linear time-invariant systems described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_2u(t) + B_1w(t) \\ z_1(t) &= C_1x(t) + D_1u(t) \\ z_2(t) &= C_2x(t) + D_2u(t), \end{aligned} \tag{2.1}$$

where $w(t) \in \mathbf{R}^{n_w}$ is the disturbance, $u(t) \in \mathbf{R}^{n_u}$ is the control; $x(t) \in \mathbf{R}^{n_x}$ is the state; $z_1(t) \in \mathbf{R}^{n_{z_1}}$ and $z_2(t) \in \mathbf{R}^{n_{z_2}}$ are H_∞ and H_2 performance variables, respectively. Assume the pair $[A, B_2]$ is stabilizable.

For the plant given by (2.1), determine state feedback controller described by

$$u(t) = Kx(t) \tag{2.2}$$

such that the following design criteria are satisfied,

1. If $w(t)$ is L_2 deterministic signal, the closed-loop transfer function from $w(t)$ to $z_1(t)$ satisfies

$$\|T_{z_1w}(s)\|_\infty < \gamma \tag{2.3}$$
$$\|T_{z_1w}(s)\|_\infty := \|(C_1 + D_1K)(sI - A - B_2K)^{-1}B_1\|_\infty$$

2. If $w(t)$ is a white noise signal with unit strength, H_2 performance criterion defined by

$$J := \lim_{T\to\infty} \mathcal{E}\{\frac{1}{T}\int_0^T [x'(t)R_1x(t) + u'(t)R_2u(t)]dt\} \quad (2.4)$$

is minimized, where $R_1 := C_2'C_2 \geq 0$, $R_2 := D_2'D_2 > 0$, and $\mathcal{E}$ denotes the expectation;

3. The spectrum of the closed-loop system matrix are located in a class of regions

$$\sigma(A + B_2K) \subset \mathcal{D}, \quad (2.5)$$

where $\mathcal{D}$ is a subregion in the open left half plane.

Implicit in the above objectives is the requirement that the closed-loop system is asymptotically stable. We suppose that $\mathcal{D}$ is a circle region in the open left half plane. This circle region can be considered as an approximation to the region which is formed by intersecting a sector region and a straight line parallel to imaginary axis. The latter region puts a lower bound on both exponential decay rate and the damping ratio of the closed-loop response, and thus is very common in practical control design[15].

Remark 2.1
We would like to emphasize here that in the above synthesis problem, we interpret the H_∞-norm of a transfer function matrix as the maximal output energy for all inputs in L_2 with unit energy. Thus, H_∞-norm constraint (2.3) provides a prespecified level of *disturbance attenuation*. In this interpretation, our controller will minimize the H_2 performance while simultaneously provide a prespecified level of disturbance attenuation and the containment of closed-loop poles in region $\mathcal{D}$. As is well known H_∞-norm also provide robust stability for feedback system under the presence of unstructured uncertainty. However, in the later interpretation our design may not guarantee simultaneous robust stability and pole placement, although still provide a nominal performance in H_2-norm sense.

The closed-loop system is given by

$$\begin{aligned} \dot{x} &= \tilde{A}x + \tilde{B}_1 w \\ z_1 &= \tilde{C}_1 x \\ z_2 &= \tilde{C}_2 x, \end{aligned} \quad (2.6)$$

where

$$\tilde{A} = A + B_2K,\ \ \tilde{B}_1 = B_1$$
$$\tilde{C}_1 = C_1 + D_1K,\ \ \tilde{C}_2 = C_2 + D_2K.$$

Suppose that the closed-loop system (2.6) is internally stable. Then, it is well known that H_2 performance in (2.4) can be equivalently expressed as

$$J = \|T_{z_2w}\|_2^2 = tr\,[\tilde{C}_2'\tilde{C}_2P], \quad (2.7)$$

where P is positive definite solution to Lyapunov equation

$$\tilde{A}P + P\tilde{A}' + \tilde{B}_1\tilde{B}_1' = 0. \quad (2.8)$$

2.2 Preliminaries

In this subsection, we collect some results which are useful in developing convex programming associated with the above synthesis control problem.

The following well-known lemma characterizes H_∞-norm bound of a transfer function matrix in terms of a Riccati equation.

Lemma 2.1 *Let transfer function matrix $G(s) := C(sI-A)^{-1}B$ be given. Then, A is stable matrix and $\|G(s)\|_\infty < \gamma$ if, and only if, there exists $X = X' > 0$ satisfying*

$$AX + XA' + \gamma^{-2}XC'CX + BB' < 0. \quad (2.9)$$

As stated above, for pole placement objective we consider a circle region $\mathcal{D}$ that is contained in the open left half plane. The next lemma characterizes the pole placement constraint in terms of solution to a generalized Lyapunov equation[11-14].

Lemma 2.2 *Let $G(s) := C(sI - A)^{-1}B$ be given. Suppose that $\mathcal{D}$ is a circle region with center $(-q, 0)$ and radius r which is contained in the open left half plane. Then,*

$$\sigma(A) \in \mathcal{D}, \quad (2.10)$$

if, and only if given any $Z > 0$ there exists positive definite $Y = Y' > 0$ such that

$$\gamma_{10}AY + \gamma_{01}YA' + \gamma_{11}AYA' + \gamma_{00}Y + Z = 0, \quad (2.11)$$

where $\gamma_{00} = (q^2 - r^2)r^2$, $\gamma_{01} = \gamma_{10} = qr^2$ and $\gamma_{11} = r^2$.

The following proposition gives necessary and sufficient condition for the satisfaction of H_∞ constraint (2.3), as well as the containment of closed-loop poles in the given region (2.5).

Proposition 2.1 *Consider the closed-loop system described in (2.6). Then, the following conditions are equivalent*

1. *The closed-loop system is stable and satisfies*

$$\|T_{z_1w}(s)\|_\infty < \gamma,\ \ and\ \ \sigma(A + B_2K) \subset \mathcal{D}, \quad (2.12)$$

where $\mathcal{D}$ is a circle region in the left half plane with center $(-q, 0)$ and radius r.

2. *There exist $\tilde{P} = \tilde{P}' > 0$, $\mu > 0$(sufficiently small) and $Q > 0$ such that*

$$\begin{aligned} &\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \gamma^{-2}\tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \alpha_1\tilde{A}\tilde{P}\tilde{A}' + \alpha_2\tilde{P} \\ &\quad + \tilde{B}_1\tilde{B}_1' + \tfrac{\mu}{\gamma_{10}}Q = 0 \end{aligned} \quad (2.13)$$

with $\alpha_1 = \frac{\gamma_{11}}{\gamma_{10}}$, $\alpha_2 = \frac{\gamma_{00}}{\gamma_{10}}$.

Proof
Sufficiency
Suppose that there exists $\tilde{P} = \tilde{P}' > 0$ satisfying (2.13). Then considering the fact that $[\alpha_1\tilde{A}\tilde{P}\tilde{A}' + \alpha_2\tilde{P} + \frac{\mu}{\gamma_{10}}Q] > 0$, the existence of solution $\tilde{P} = \tilde{P}' > 0$ to (13) implies that

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \gamma^{-2}\tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \tilde{B}_1\tilde{B}_1' < 0.$$

By Lemma 2.1, we conclude that $\tilde{A}$, i.e. the closed-loop system matrix, is stable and $\|T_{z_1w}(s)\|_\infty < \gamma$. Now, by the fact that $[\gamma^{-2}\tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \tilde{B}_1\tilde{B}_1'] \geq 0$, we have

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \alpha_1\tilde{A}\tilde{P}\tilde{A}' + \alpha_2\tilde{P} + \frac{\mu}{\gamma_{10}}Q \leq 0.$$

Defining $Q_1 \geq 0$ as

$$\frac{\mu}{\gamma_{10}} Q_1 := -[\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \alpha_1 \tilde{A}\tilde{P}\tilde{A}' + \alpha_2 \tilde{P} + \frac{\mu}{\gamma_{10}} Q]$$

we have

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \alpha_1 \tilde{A}\tilde{P}\tilde{A}' + \alpha_2 \tilde{P} + \frac{\mu}{\gamma_{10}} Q + Q_1 = 0.$$

But since $\frac{\mu}{\gamma_{10}}[Q + Q_1] > 0$ then invoking Lemma 2.2 we conclude that $\sigma(A + B_2 K) \subset \mathcal{D}$.

Necessity:

Suppose that $\tilde{A}$ is a stable matrix, $\|T_{z_1 w}(s)\|_\infty < \gamma$, *and* $\sigma(A + B_2 K) \subset \mathcal{D}$. Then, by Lemma 2.1, there exists $P_1 = P_1' > 0$ such that

$$\tilde{A}P_1 + P_1\tilde{A}' + \gamma^{-2} P_1 \tilde{C}_1' \tilde{C}_1 P_1 + \tilde{B}_1 \tilde{B}_1' < 0, \tag{2.14}$$

and by Lemma 2.2, there exists $P_2 = P_2' > 0$ and $R > 0$ such that

$$\tilde{A}P_2 + P_2\tilde{A}' + \alpha_1 \tilde{A}P_2\tilde{A}' + \alpha_2 P_2 + \frac{1}{\gamma_{10}} R = 0,$$

or equivalently, there exist $P_2 = P_2' > 0$ and ϵ(sufficiently small) such that

$$\tilde{A}P_2 + P_2\tilde{A}' + \alpha_1 \tilde{A}P_2\tilde{A}' + \alpha_2 P_2 + \frac{\epsilon}{\gamma_{10}} R < 0. \tag{2.15}$$

Add equation (2.14) to (2.15) and rearrange terms to obtain

$$\begin{aligned} &\tilde{A}(P_1 + P_2) + (P_1 + P_2)\tilde{A}' + \gamma^{-2}(P_1 + P_2)\tilde{C}_1'\tilde{C}_1(P_1 + P_2) \\ &+\alpha_1 \tilde{A}(P_1 + P_2)\tilde{A}' + \alpha_2(P_1 + P_2) + \tilde{B}_1\tilde{B}_1' + \tfrac{\epsilon}{\gamma_{10}} R \\ &-[\gamma^{-2}(P_1\tilde{C}_1'\tilde{C}_1 P_2 + P_2\tilde{C}_1'\tilde{C}_1 P_1 + P_2\tilde{C}_1'\tilde{C}_1 P_2) \\ &+\alpha_1 \tilde{A}P_1\tilde{A}' + \alpha_2 P_1] < 0. \end{aligned}$$

Noting that $[\gamma^{-2}(P_1\tilde{C}_1'\tilde{C}_1 P_2 + P_2\tilde{C}_1'\tilde{C}_1 P_1 + P_2\tilde{C}_1'\tilde{C}_1 P_2) + \alpha_1 \tilde{A}P_1\tilde{A}' + \alpha_2 P_1] > 0$, we have

$$\begin{aligned} &\tilde{A}(P_1 + P_2) + (P_1 + P_2)\tilde{A}' + \gamma^{-2}(P_1 + P_2)\tilde{C}_1'\tilde{C}_1(P_1 + P_2) \\ &+\alpha_1 \tilde{A}(P_1 + P_2)\tilde{A}' + \alpha_2(P_1 + P_2) + \tilde{B}_1\tilde{B}_1' + \tfrac{\epsilon}{\gamma_{10}} R < 0. \end{aligned}$$

Defining symmetric matrix $\tilde{P} = \tilde{P}' := P_1 + P_2 > 0$ and real positive number $\mu := \epsilon$(sufficiently small) we obtain

$$\begin{aligned} &\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \gamma^{-2}\tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \alpha_1 \tilde{A}\tilde{P}\tilde{A}' + \alpha_2 \tilde{P} \\ &+\tilde{B}_1\tilde{B}_1' + \tfrac{\mu}{\gamma_{10}} R < 0. \end{aligned}$$

Let $Q_2 > 0$ be defined by

$$\begin{aligned} \frac{\mu}{\gamma_{10}} Q_2 \;:=\; &-[\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \gamma^{-2}\tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \alpha_1 \tilde{A}\tilde{P}\tilde{A}' + \alpha_2 \tilde{P} \\ &+\tilde{B}_1\tilde{B}_1' + \frac{\mu}{\gamma_{10}} R]. \end{aligned}$$

Then we have

$$\begin{aligned} &\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \gamma^{-2}\tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \alpha_1 \tilde{A}\tilde{P}\tilde{A}' + \alpha_2 \tilde{P} \\ &+\tilde{B}_1\tilde{B}_1' + \tfrac{\mu}{\gamma_{10}}(R + Q_2) = 0. \end{aligned}$$

Defining $Q := (R + Q_2) > 0$, the last equation implies that there exist $\tilde{P} = \tilde{P}' > 0$, $\mu > 0$(sufficiently small) and $Q > 0$ such that equation (2.13) is satisfied. This completes the proof.

□

For convenience in stating some of the results of this paper, let us define

$$\begin{aligned} R(\tilde{P}) \;:=\; &\tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \gamma^{-2}\tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \alpha_1 \tilde{A}\tilde{P}\tilde{A}' + \alpha_2 \tilde{P} \\ &+\tilde{B}_1\tilde{B}_1' + \frac{\mu}{\gamma_{10}} Q \end{aligned} \tag{2.16}$$

Suppose that the condition in Proposition 2.1 is satisfied. Then, the following conditions can be easily verified[3,6],

$$0 \leq P \leq \tilde{P} \leq \hat{P}, \tag{2.17}$$

$$J \leq \tilde{J} := tr[\tilde{C}_2'\tilde{C}_2\tilde{P}], \tag{2.18}$$

where $\hat{P}$ denotes any real symmetric solution to the generalized Riccati inequality $R(\hat{P}) < 0$, with R defined by (2.16). Furthermore by similar argument as the proof of Theorem 7.3 of [14], it can be shown that both J and $\tilde{J}$ is finite. Note that $\tilde{J}$, which is given in terms of solution to generalized Riccati equation $R(\tilde{P}) = 0$, is an upperbound to the quadratic cost J. Instead of minimizing the quadratic cost it self, we will minimize this upperbound in our optimization problem defined later.

The following results can be established in the same way as that of [3, Lemma 2.1], the different being that in the present paper Riccati equation involved in the definition of the upper bound involves an additional linear and quadratic terms.

Lemma 2.3 *Consider the closed-loop system described in (2.6) and let T_{zw} denote the transfer function matrix from w to $z = (z_1, z_2)$. Suppose that*

$$\|T_{z_1 w}(s)\|_\infty < \gamma, \;\; \textit{and} \;\; \sigma(A + B_2 K) \subset \mathcal{D}.$$

Then,

$$\begin{aligned} J(T_{zw}) \;=\; &inf\{tr(\tilde{C}_2'\tilde{C}_2\tilde{P}) : \tilde{P} = \tilde{P}' > 0 \;\; such \;\; that \\ &R(M, \tilde{P}) < 0\}. \end{aligned} \tag{2.19}$$

Now, let the transfer function of the plant (2.1) be denoted by $\mathcal{P}$. The transfer function of the overall closed-loop system will be denoted by

$$T_{zw} := \begin{bmatrix} T_{z_1 w} \\ T_{z_2 w} \end{bmatrix}.$$

We call a controller K ***admissible*** if K internally stabilizes the plant $\mathcal{P}$. Introduce the following sets :

$$\begin{aligned} \mathcal{A}(\mathcal{P}) \;&:=\; \{K : K \text{ is admissible}\} \\ \mathcal{A}_{\infty,\mathcal{D}}(\mathcal{P}) \;&:=\; \{K \in \mathcal{A}(\mathcal{P}) : \;\|T_{z_1 w}(s)\|_\infty < \gamma, \;\; \textit{and} \\ &\qquad \sigma(A + B_2 K) \subset \mathcal{D}\}. \end{aligned} \tag{2.20}$$

In view of Proposition 2.1 and Lemma 2.3, we consider the following synthesis problem which may be considered as an extension of "suboptimal H_2/H_∞ controller synthesis" introduced by Khargonekar and Rotea[3] to the mixed H_2/H_∞ control problem with pole palacement in a specified region.

Synthesis Problem: "Compute the mixed performance measure

$$\theta_m(\mathcal{P}) := inf\{J(T_{zw}) : K \in \mathcal{A}_{\infty,\mathcal{D}}(\mathcal{P})\}, \tag{2.21}$$

and, given any $\theta > \theta_m$, find a controller $K \in \mathcal{A}_{\infty,\mathcal{D}}(\mathcal{P})$ such that $J(T_{zw}) < \theta$".

3 Convex Optimization Approach

In this section we will develop a convex optimization approach for solving the controller synthesis problem introduced above. Motivated by the result of [3], where it is proved that all memoryless state feedback mixed H_2/H_∞ controllers cannot be improved upon by the use of dynamic "full information" controllers, we are interested in the computation of constant state feedback matrices for the minimization of $J(\mathcal{P},K)$. The set of such controllers will be denoted by

$$\mathcal{A}_{\infty,\mathcal{D},m}(\mathcal{P}) := \{K \in \mathcal{A}_{\infty,\mathcal{D}}(\mathcal{P}) : K \in \mathbf{R}^{n_u \times n_x}\}. \quad (3.1)$$

It will be shown that the optimal performance $\theta_m(\mathcal{P})$ defined in (2.21) is the value of (finite dimensional) convex optimization problem. Further, given any $\theta > \theta_m$, one can find K such that $J(\mathcal{P},K) < \alpha$ by solving a convex programming problem.

Let Ξ denote the set of $n_x \times n_x$ real symmetric matrices, and define

$$\Omega := \{(X,\tilde{P}) \in \mathbf{R}^{n_u \times n_x} \times \Xi : \tilde{P} > 0\}. \quad (3.2)$$

Observe that Ω is an open strictly convex subset of $\mathbf{R}^{n_u \times n_x} \times \Xi$. Given $(X,\tilde{P}) \in \Omega$, define

$$f(X,\tilde{P}) := tr[(C_2\tilde{P} + D_2X)\tilde{P}^{-1}(C_2\tilde{P} + D_2X)'] \quad (3.3)$$

and, for $(X,\tilde{P}) \in \mathbf{R}^{n_u \times n_x} \times \Xi$, let

$$\begin{aligned}\hat{R}(X,\tilde{P}) := {} & A\tilde{P} + \tilde{P}A' + B_2X + X'B_2' + B_1B_1' + \tfrac{\mu}{\gamma_{10}}Q \\ & +\gamma^{-2}(C_1\tilde{P} + D_1X)'(C_1\tilde{P} + D_1X) \\ & +\alpha_1(A\tilde{P} + B_2X)\tilde{P}^{-1}(A\tilde{P} + B_2X)' + \alpha_2\tilde{P}. \quad (3.4)\end{aligned}$$

Define also the set of real matrices:

$$\Phi(\mathcal{P}) := \{(X,\tilde{P}) \in \Omega : \hat{R}(X,\tilde{P}) < 0\}, \quad (3.5)$$

and consider the optimization problem

$$\tau(\mathcal{P}) := inf\{f(X,\tilde{P}) : (X,\tilde{P}) \in \Phi(\mathcal{P})\}. \quad (3.6)$$

Theorem 3.1 *Consider the plant $\mathcal{P}$ defined in (2.1). Let T_{zw} denote its transfer matrix, and $\mathcal{A}_{\infty,\mathcal{D},m}(T_{zw})$ denote the set of controllers defined in (3.1). Let $\Phi(\mathcal{P})$ be given by (3.5). Let θ_m and $\tau(\mathcal{P})$ be as defined in (2.20) and (3.6), respectively. Then,*

$$\mathcal{A}_{\infty,\mathcal{D},m}(\mathcal{P}) \neq \emptyset \quad (3.7)$$

if, and only if,

$$\Phi(\mathcal{P}) \neq \emptyset \quad (3.8)$$

with $\emptyset$ denote empty set. In this case,

$$\theta_m(\mathcal{P}) = \tau(\mathcal{P}). \quad (3.9)$$

Furthermore, given any $\alpha > \theta_m(\mathcal{P})$, there exists $(X,\tilde{P}) \in \Phi(\mathcal{P})$ such that the state feedback gain $K := X\tilde{P}^{-1}$ satisfies

$$K \in \mathcal{A}_{\infty,\mathcal{D},m}(\mathcal{P}) \text{ and } J(T_{zw},K) \leq f(X,\tilde{P}) < \alpha. \quad (3.10)$$

Proof

First, we will show that if $\Phi(\mathcal{P}) \neq \emptyset$ then $\mathcal{A}_{\infty,\mathcal{D},m}(\mathcal{P}) \neq \emptyset$, and $\theta_m(\mathcal{P}) \leq \tau(\mathcal{P})$. Suppose that $\epsilon > 0$ is given. From the definition of $\tau(\mathcal{P})$, it follows that there exists a $(X,\tilde{P}) \in \Omega$ such that

$$f(X,\tilde{P}) \leq \tau(\mathcal{P}) + \epsilon, \quad \hat{R}(X,\tilde{P}) < 0,$$

where $f(X,\tilde{P})$ and $\hat{R}((X,\tilde{P})$ are defined in (3.3) and (3.4), respectively. We will construct state feedback matrix $K \in \mathcal{A}_{\infty,\mathcal{D},m}(\mathcal{P})$ such that $J(T_{zu},K) \leq f(X,\tilde{P}) < \alpha$. Define the real matrix $K := X\tilde{P}^{-1}$, then closing the loop, we have the closed-loop system described in (2.6). Using standard algebraic manipulation, it can be easily verified that

$$\begin{aligned}\hat{R}(X,\tilde{P}) = {} & \tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \gamma^{-2}\tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \alpha_1\tilde{A}\tilde{P}\tilde{A}' \\ & +\alpha_2\tilde{P} + \tilde{B}_1\tilde{B}_1' + \tfrac{\mu}{\gamma_{10}}Q = R(\tilde{P}), \quad (3.11)\end{aligned}$$

and that

$$f(X,\tilde{P}) = tr(\tilde{C}_2'\tilde{C}_2\tilde{P}). \quad (3.12)$$

Since $\tilde{P} > 0$ satisfies $\hat{R}(X,\tilde{P}) < 0$, it can be verified from the properties of Lyapunov equation that $\tilde{A}$, i.e. closed-loop system matrix, is asymptotically stable. Furthermore, by Proposition 2.1, we conclude that $\|T_{z_1w}(s)\|_\infty < \gamma$, and that $\sigma(A + B_2K) \subset \mathcal{D}$. Therefore, $K \in \mathcal{A}_{\infty,\mathcal{D},m}(\mathcal{P})$. In view of Lemma 2.3, we have $J(T_{zw},K) \leq tr(\tilde{C}_2'\tilde{C}_2P)$. Hence,

$$\theta_m(\mathcal{P}) \leq J(T_{zw},K) \leq f(X,\tilde{P}) \leq \tau(\mathcal{P}) + \epsilon.$$

Since ϵ is arbitrary positive real number, then $\theta_m(\mathcal{P}) \leq \tau(\mathcal{P})$.

Next we will show that if $\mathcal{A}_{\infty,m}(\mathcal{P}) \neq \emptyset$, then $\Phi(\mathcal{P}) \neq \emptyset$, and $\theta_m(\mathcal{P}) = \tau(\mathcal{P})$. Let ϵ be given. From the definition of $\theta_m(\mathcal{P})$ in (2.21), there exists $K \in \mathcal{A}_{\infty,\mathcal{D},m}(\mathcal{P})$ such that

$$J(T_{zw},K) \leq \theta_m(\mathcal{P}) + \epsilon/2.$$

Using the controller K, the closed-loop system is given by (2.6). It follows from Lemma 2.3 that there exists $\tilde{P} = \tilde{P} > 0$ such that

$$\begin{aligned}& tr(\tilde{C}_2'\tilde{C}_2\tilde{P}) \leq J(T_{zw},K) + \epsilon/2 \leq \theta_m(\mathcal{P}) + \epsilon, \\ & R(\tilde{P}) = \tilde{A}\tilde{P} + \tilde{P}\tilde{A}' + \gamma^{-2}\tilde{P}\tilde{C}_1'\tilde{C}_1\tilde{P} + \alpha_1\tilde{A}\tilde{P}\tilde{A}' + \alpha_2\tilde{P} \\ & \quad +\tilde{B}_1\tilde{B}_1' + \tfrac{\mu}{\gamma_{10}}Q < 0\end{aligned}$$

Define $X := K\tilde{P}$. Then,

$$(X,\tilde{P}) \in \Omega \text{ and } \hat{R}(X,\tilde{P}) = R(\tilde{P}) < 0.$$

It follows that $(X,\tilde{P}) \in \Phi(\mathcal{P})$ and from (3.3), $f(X,\tilde{P}) = tr(\tilde{C}_2'\tilde{C}_2\tilde{P})$. Then, we have

$$f(X,\tilde{P}) \leq \theta_m(\mathcal{P}) + \epsilon.$$

Again, since ϵ is arbitrary positive number, we conclude that $\theta_m(\mathcal{P}) \geq \tau$.

The last part of this theorem follows immediately from the definitions and the construction for K.

□

From Theorem 3.1, it follows that the computation of $\tau(\mathcal{P})$ involves a search over the set $\Phi(\mathcal{P})$, where X, and $\tilde{P}$ serve as the decision variables. On the other hand $\theta_m(\mathcal{P})$ is computed by solving nonlinear programming problem with only the real matrix K as the decision variable. Furthermore, the set of feasible static feedback gains, $\mathcal{A}_{\infty,\mathcal{D},m}(\mathcal{P})$ does not necessarily convex, and therefore the original optimization problem for mixed H_2/H_∞ controller synthesis does not necessarily convex. We will show that the optimization problem defined in (3.6) is indeed a *convex problem.*

Theorem 3.2 *Let f and Φ be as defined in (3.3) and (3.5), respectively, and consider the optimization problem (3.6). Then, the set Φ is convex and the function $f : \Phi \to \mathbf{R}^+$ is convex and real analytic. Consequently, the optimization problem defined in (3.6) is convex.*

Proof

The fact that f is a real analytic function on the open set Ω follows immediately from (3.3). The convexity of f can be established in the same manner as that of Theorem 4.1 of [3].

The convexity of Φ is derived by showing that $\hat{R}(X,\tilde{P}) : \mathbf{R}^{n_u \times n_x} \times \Sigma \to \Sigma$ is a convex mapping(with respect to the cone of positive semidefinite matrices). Let us rewrite $\hat{R}$ as

$$\hat{R}(X,\tilde{P}) = \Phi_1(X,\tilde{P}) + \Phi_2(X,\tilde{P}), \qquad (3.13)$$

where

$$\begin{aligned}\Phi_1(X,\tilde{P}) &:= A\tilde{P} + \tilde{P}A' + B_2X + X'B_2' + B_1B_1' + \frac{\mu}{\gamma_{10}}Q \\ &\quad +\gamma^{-2}(C_1\tilde{P} + D_1X)'(C_1\tilde{P} + D_1X) \\ \Phi_2(X,\tilde{P}) &:= \alpha_1(A\tilde{P} + B_2X)\tilde{P}^{-1}(A\tilde{P} + B_2X)' + \alpha_2\tilde{P}.\end{aligned}$$

The convexity of Φ_1 has been established in [3]. Therefore, to prove the convexity of Φ it remains to show that Φ_2 is also convex in the domain $(X,\tilde{P})$.

Let us define $\tilde{X} := (A\tilde{P} + B_2X)$. Then

$$\Phi_2(X,\tilde{P}) = \alpha_1\tilde{X}'\tilde{P}^{-1}\tilde{X} + \alpha_2\tilde{P}.$$

In view of Proposition E.7.f on p. 459 of[1], the mappings from $(\tilde{X},\tilde{P}) \to \alpha_1\tilde{X}'\tilde{P}^{-1}\tilde{X}$ and $(\tilde{X},\tilde{P}) \to \alpha_2\tilde{P}$ are convex. The map $(X,\tilde{P}) \to \tilde{X}$ is obviously convex due to linearity of $\tilde{X}$ in the variables X and $\tilde{P}$. Therefore, $\Phi_2(X,\tilde{P})$ is a convex mapping, and the convexity of Φ follows from the convexity of Φ_1 and Φ_2.

Now, since the set Ω defined in (3.2) is convex, from the fact that the level sets of a convex mapping are convex, it follows that the set Φ defined in (3.5) is convex. Since the objective function $f(.)$ is convex on $\Omega \supset \Phi$, we conclude that the optimization problem defined in (3.6) is a convex problem.

□

Remark 3.1

Under certain condition that is counterpart of that of Lemma 4.6 in [3], we can show that the set Φ defined in (3.5) is bounded. This condition is useful in guaranteeing that a numerical algorithm can be effectively used to solve (3.6).

Let us consider again mixed H_2/H_∞ control synthesis with pole placement for the state feedback plant $\mathcal{P}$. Suppose that $\alpha > 0$ is given. From Theorems 3.1 and 3.2, we know that there exists $K \in \mathcal{A}_{\infty,\mathcal{D},m}(\mathcal{P})$ such that $J(T_{zw},K) < \alpha$ if only if there exists $(X,\tilde{P}) \in \Phi$ such that $f(X,\tilde{P}) < \alpha$. And in this case, the real matrix $K := X\tilde{P}^{-1}$ is a solution to the sub-optimal synthesis problem. The problem of finding $(X,\tilde{P}) \in \Phi$ such that $f(X,\tilde{P}) < \alpha$ is a convex *feasibility program* which is a (nonsmooth) convex optimization problem[12].

4 Interior Point Method

In this section, we will show that the optimization problem defined in (3.6) can be reduced to Generalized Eigenvalue Minimization Problem(GEMP) and describe an Interior Point Method for solving the problem[12]. GEMP is the problem of minimizing the maximum generalized eigenvalue of a(symmetric, symmetric positive-definite) pair of matrices that depend affinely on a variable x that is subject to some constraints. In [12], a fast and attractive algorithm based on Interior Point Method has been applied to solve efficiently GEMP.

In the general case, GEMP with variables $x \in \mathbf{R}^m$ and $\lambda \in \mathbf{R}$ takes the form

$$\begin{array}{cc} \min & \lambda \\ \lambda G(x) - F(x) > 0 & \\ G(x) > 0 & \\ H(x) > 0 & \end{array} \qquad (4.1)$$

or equivalently,

$$\begin{array}{cc} \min & \lambda_{max}(F(x),G(x)). \\ G(x) > 0 & \\ H(x) > 0 & \end{array} \qquad (4.2)$$

where λ_{max} denotes the generalized maximum eigenvalue. This is a function defined on a pair of matrices X, Y by $\lambda_{max}(X,Y) := max\{\lambda \in \mathbf{R} | det(\lambda Y - X) = 0\}$. In (4.1) and (4.2), F ,G and H are symmetric matrices that depend affinely on $x \in \mathbf{R}^m$:

$$\begin{aligned} F(x) := F_0 + \textstyle\sum_{i=1}^m x_iF_i, \quad G(x) := G_0 + \sum_{i=1}^m x_iG_i, \\ H(x) := H_0 + \textstyle\sum_{i=1}^m x_iH_i, \end{aligned} \qquad (4.3)$$

where $F_i = F_i'$, $G_i = G_i' \in \mathbf{R}^{r\times r}$, and $H_i = H_i' \in \mathbf{R}^{s\times s}$. Matrices $F(x)$ and $G(x)$ may be complex Hermitian.

Let us turn our attention to the optimization problem defined in (3.6). For convenience, let us express the objective function (3.3) as:

$$\begin{aligned} f(X,\tilde{P}) &= tr(C_2\tilde{P}C_2' + C_2X'D_2' + D_2XC_2' \\ &\quad + D_2X\tilde{P}^{-1}X'D_2'). \end{aligned} \qquad (4.4)$$

The last term $\Theta(X,\tilde{P}) := D_2X\tilde{P}^{-1}X'D_2'$ in the above equation can be equivalently expressed as

$$\Theta(X,\tilde{P}) = \min_{\begin{bmatrix} S & D_2X \\ X'D_2' & \tilde{P} \end{bmatrix} > 0} tr(S).$$

Let us further define

$$\begin{aligned} L_1(\lambda,X,\tilde{P},S) &:= -tr(C_2\tilde{P}C_2' + C_2X'D_2' + D_2XC_2') \\ &\quad -tr(S) + \lambda \\ L_2(\lambda,X,\tilde{P},S) &:= \begin{bmatrix} L_{2a} & L_{2b} \\ L_{2c} & L_{2d} \end{bmatrix} \\ L_3(\lambda,X,\tilde{P},S) &:= \begin{bmatrix} S & D_2X \\ X'D_2' & \tilde{P} \end{bmatrix} \\ L(\lambda,X,\tilde{P},S) &:= diag(L_1,L_2,L_3), \end{aligned}$$

where

$$\begin{aligned} L_{2a} &= -(A\tilde{P} + \tilde{P}A' + B_2X + X'B_2' + \alpha_2\tilde{P} + B_1B_1' + \frac{\mu}{\gamma_{10}}Q) \\ L_{2b} &= [\gamma^{-1}(C_1\tilde{P} + D_1X)' \quad (A\tilde{P} + B_2X)] \\ L_{2c} &= L_{2b}' \\ L_{2d} &= \begin{bmatrix} I & 0 \\ 0 & \tilde{P} \end{bmatrix}. \end{aligned}$$

Note carefully that $L_1(\lambda,X,\tilde{P},S)$, $L_2(\lambda,X,\tilde{P},S)$ and $L_3(\lambda,X,\tilde{P},S)$ are affine matrix in the variables $(\lambda,X,\tilde{P})$.

Using the above constructions and employing the Schur complement formula which states that

$$\begin{bmatrix} Z_1 & Z_3 \\ Z_3' & Z_2 \end{bmatrix} > 0 \iff Z_2 > 0, \ \ and \ \ Z_1 - Z_3Z_2^{-1}Z_3' > 0,$$

our optimization problem (3.6) can now be represented as

$$\min_{L(\lambda,X,\tilde{P},S)>0} \lambda \quad (4.5)$$

which indeed is of the form (4.1). Represented in the form of (4.1), symmetric affine matrices $F(x)$, $G(x)$ and $H(x)$ for optimization problem (4.5) are given by

$$\begin{aligned} F(x) &:= diag([-tr(C_2\tilde{P}C_2' + C_2X'D_2' + D_2XC_2') - tr(S)], \\ &\quad L_2, L_3) \\ G(x) &:= diag(1,0,0,0) \\ H(x) &:= \tilde{P}. \end{aligned}$$

Vector x in (4.1) then contains the optimization variables which consist of the independent variables of $(\lambda, X, \tilde{P}, S)$.

The GEMP (4.1) can be effectively solved using Interior Point Method. The method is based on the notion of *analytic center* of an affine matrix inequality, say $D(x) = D_0 + \sum_{i=1}^{N} x_i D_i > 0$. Suppose that $\mathbf{X}$ denotes the feasible set

$$\mathbf{X} := \{x \in \mathbf{R}^N | D(x) > 0\}.$$

The analytic center x^* of the inequality $D(x) > 0$ is defined as

$$x^* = argmin_{x \in \mathbf{X}} log\, det D(x)^{-1}.$$

Starting with any feasible $x^{(0)}$, and a $\lambda^{(0)} = \lambda_{max}(A(x^{(0)}, B(x^{(0)}))$, the algorithm proceed as follows

$$\begin{aligned} \lambda^{i+1} &:= (1-\eta)\lambda_{max}(F(x^{(i)}, G(x^{(i)})) + \eta\lambda^{(i)} \\ x^{(i+1)} &:= \text{analytic center of } \lambda^{(i+1)}G(x) - F(x) > 0. \end{aligned}$$

In the above procedure $\eta \in (0,1)$ is a parameter which is typically small. It enables one to take $x^{(i)}$ as an initial guess for the Newton type method that finds the analytic center of inequality $\lambda^(i+1)G(x) - F(x) > 0$. Detailed analysis as well as the proof of convergence can be found in [9].

5 Conclusion

The problem of synthesizing mixed H_2/H_∞ with pole placement in a specified region in the complex plane has been presented for finite dimensional linear time-invariant systems. This synthesis problem is well motivated since in addition to providing a disturbance attenuation and nominal (quadratic) performance, it also guarantees a good transient response. The suboptimal synthesis problem has been reduced to convex optimization problem over a bounded subset of symmetric matrices via the use of solution to a Riccati equation and a change of variables technique. Due to convexities established in this paper, we can adopt any optimization method with global optimality properties. The resulting convex optimization problem can be in turn reduced to the Generalized Eigenvalue Minimization Problem where a powerful algorithm based on interior point method(analytic center) has been developed to find its solution[12]. This avoids solving highly coupled nonlinear equations.

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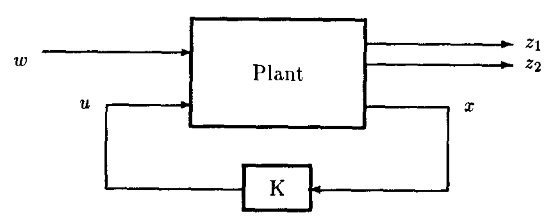

Figure 1: General framework for mixed H_2/H_∞ control design with pole placement