Special Session 3

ANALYSIS OF ROBUSTNESS IN FUZZY CONTROL

Hung T. Nguyen*
LIFE Chair of Fuzzy Theory
Tokyo Institute of Technology
Department of Systems Science
4259 Nagatsuta, Midori-Ku,
Yokohama 227 - JAPAN

Extended Abstract. This lecture is about an investigation into a desired property of fuzzy systems when degrees of uncertainty involved are uncertain. We characterize the robustness of fuzzy logic operators by their moduli of continuity. Theoretical results for design methodology are presented and a case study is discussed.

1. MOTIVATION

From a practical viewpoint, the acquisition of a knowledge base for the design of a fuzzy logic controller is the most delicate phase. The flexibility in choosing components of a control, such as membership functions of fuzzy concepts, combining operators, defuzzification procedures, etc., is beneficial but requires also justifications. Thus, it is desirable that a chosen design is insensitive to small pertubations of its components. This is similar to Neural Networks: small variations of connection weights should not affect much of the output as well as the universal approximation property (see [2]).

In the following, we will investigate the robustness of components of fuzzy controllers viewed as functions of several variables. Since we are concerned with comparison of continuous functions in terms of their variations, we will characterize their robustness property by their moduli of continuity (in the *Theory of Approximation of Functions*, see e.g. [12]).

2. FUZZY LOGIC CONTROLLERS

To be concrete, let us specify a typical class of fuzzy control systems. If we fix Mamdani's implication and aggregation operators (see e.g. [3], [23]), then to design a control, we need to choose elements of the following sets:

$\mathcal{M}$: a class of memberships functions;
$\mathcal{L}$: a class of fuzzy logic connectives;
$\mathcal{D}$: a class of defuzzification procedures.

The above classes have to be chosen so that the resulting systems are universal approximators. Specifically, for each integer $n \geq 1$ (dimension of the input variable), the input-output map $f : \mathbf{R}^n \to \mathbf{R}$, say, in the class of controls $\mathcal{C}(\mathcal{M}, \mathcal{L}, \mathcal{D})$ has the following property: For any compact set K of $\mathbf{R}^n$, the restriction of $\mathcal{C}$ to K is dense (in the sup-norm) in the

*On leave from Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, U.S.A.

space of continuous functions on K. The following two examples (for details, see [15]) are modifications of results in [21] and [10],

$$\text{(i)} \quad \begin{array}{rcl} \mathcal{M} &=& \{\text{Gaussians membership functions}\}, \\ \mathcal{L} &=& \text{min operator for conjunction}, \\ \mathcal{D} &=& \text{centroid defuzzification}. \end{array}$$

$$\text{(ii)} \quad \begin{array}{rcl} \mathcal{M} &=& \{\text{continuous functions with compact support}\}, \\ \mathcal{L} &=& \{\text{continuous } t\text{-norms and } t\text{-conorms}\}, \\ \mathcal{D} &=& \{\text{arbitrary defuzzification procedures}\}. \end{array}$$

3. ROBUSTNESS OF OPERATORS

To spell out the rationale behind our approach to the characterization of robustness, let's look at $\mathcal{L}$. Elements in $\mathcal{L}$ are functions from $[0,1]^n$ to $[0,1]$ ($n = 1$ corresponds to negation operators, $n = 2$ for logical connectives like "and", "or", extended to $n > 2$ by associativity). Let's consider t-norms (see e.g. [5]) as binary operators for connective "and". Of course, it is obvious that at least robustness reflects continuity, so that we need to focus only on continuous t-norms.

In the class $\mathcal{A}$ of all continuous t-norms, can we (and how) single out a continuous one which is, roughly speaking, "more robust" (if we don't want to say "more continuous"!) than all others? Now, in comparing the variations of f, $g \in \mathcal{A}$, we can proceed as follows. For each $\delta \in [0,1]$, look at the quantities

$$r_f(\delta) = Max\,|f(x,y) - f(x',y')| \text{ over all } (x,y),$$

$$(x',y') \text{ such that } |x - x'| \leq \delta \text{ and } |y - y'| \leq \delta;$$

and the corresponding $r_g(\delta)$.

As in standard Decision Theory, we say that f is more robust than g if $r_f(\delta) \leq r_g(\delta)$ for all $\delta \in [0,1]$ and there is some δ_0 such that $r_f(\delta_0) < r_g(\delta_0)$.

Note that the (continuous) function r_f is precisely the modulus of continuity of f (see [12]). This "measure of continuity" is used in the *Theory of Approximation of Functions* for another purpose, namely, as information for the estimation of the degree of approximation of a continuous function by trigonometric or algebraic polynomials.

It is easy to see that

$$r_f(\delta) = Min\{\alpha : |f(x,y) - f(x'y')| \leq \alpha \text{ whenever } |x - x'| \leq \delta \text{ and } |y - y'| \leq \delta\}.$$

As far as robustness of systems is concerned, only small values of δ are of interest. f is said to be locally more robust than g if $r_f(\delta) < r_g(\delta)$, for $\delta < \alpha$, for some $\alpha > 0$.

Examples (for details, see [14], [16]).

(i) $f(x) = 1 - x$ is the most robust among the class of all negation operators.

(ii) $f(x,y) = Min(x,y)$ is the most robust among the class of all t-norms.

(iii) $f(x,y) = Max(x,y)$ is the most robust among the class of all t-conorms.

Remarks.

a) The above most robust operators are unique in their respective classes.

b) They are "really" robust in that they are strictly less other candidates in some neighborhood of zero.

c) Let $\mathcal{N}$ denote the class of normalization operators of the form $f : [0, \beta] \to [0, 1]$ with f being increasing and $f(0) = 0$, $f(\beta) = 1$ (for β fixed). Then there is no (global) most robust element in $\mathcal{N}$. However, $f(x) = x/\beta$ is most locally robust.

4. ROBUSTNESS OF OTHER COMPONENTS AND RESEARCH ISSUES

(i) An empirical study was carried in [17] to check our approach to the mathematical formulation of robustness with positive results. Incidentally, the same study indicated that the fuzzy controller so designed is also "robust" in another sense, namely it continues to perform well within a larger class of controlled plants (i.e. when the dynamics class changes). The choice of a robust fuzzy logic, as well as other robust components in fuzzy control, is recommended when robustness property is of major concern in the design.

(ii) As mentioned in the Motivation, modeling of fuzzy knowledge is the most delicate step. In general, the meaning of a fuzzy concept might guide us in this art. Robust modeling of membership functions can be obtained in each specific class of functions (for more information, see again [16]). More works are needed.

(iii) In [23], general defuzzification procedures are described. The robustness of these procedures is more complex because the domain of these operators is some function space. As a first step, we have compared two well-known procedures, namely centroid and center-of-maximum. It was found that ([16]) the centroid rule is continuous, while the other is not.

(iv) So far we are concerned only with control (or general systems) based upon common sense or experts' knowledge (expressed, say, in linguistic form). In many situations, we might also have statistical data. From an intuitive standpoint, more knowledge should lead to better systems. However, the process of combining statistical and linguistic data is still an art. Theoretical research is needed to point the way to a theory of data fusion which guarantees that adding experts' knowledge to numerical data can only enhance the performance of systems.

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