Generalized Digital Prolate Spheroidal Sequences in Digital Distance Protection

Sae-Young Chung ° Jong-Keun Park
Dep. of Electrical Engineering
Seoul National University

***Abstract* -- A generalization to the Digital Prolate Spheroidal Sequences is investigated, so that the Generalized Digital Prolate Spheroidal Sequences (GDPSS's) can be used as a powerful time window with many desirable properties.**

The GDPSS's in the form of the time window has 3 parameters with which we can control the shape of the window freely. Hence, GDPSS's can be used as the very useful time windows especially for digital distance protection.

I. INTRODUCTION

Between time and frequency representations of signals, there are some kinds of inherent uncertainty relations which make it impossible to measure a time-frequency position of a signal with arbitrary precision. This phenomena is known as the Uncertainty Principle. It sometimes determines the lower bounds of the signal resolutions or sometimes the upper bounds of the energy concentrations in the time-frequency domain.

Each type of the uncertainty relation corresponds to its own uncertainty measure. With this measure, we can determine a wave packet having the minimum uncertainty, which is frequently used as an optimal window.

Since there is no single optimality condition, we must choose one of the optimal senses by their properties.

The Prolate Spheroidal Wave Functions (PSWF's) [1] play very important roles in signal processing. Time windows using the PSWF's have a property of the maximum energy concentration in a given frequency range. Discrete time versions of the PSWF's are called the Digital Prolate Spheroidal Sequences (DPSS's) [1].

Discrete time windows using the DPSS's have two parameters, the time duration and interesting frequency range where the maximum energy concentration is to be obtained. But two is not enough to control the shapes of the windows in the applications as the digital distance relay.

This paper shows that the effective time duration can be adopted to the DPSS's as the third parameter. As a result, we can control the shapes of the windows more freely using the Generalized DPSS's (GDPSS's).

Finally, a strategy is stated to determine the three parameters of the GDPSS's under some specifications of design parameters as in the digital distance relay.

II. FORMULATION OF THE GDPSS'S

Let's first define the Fourier transform pair of the finite-duration (-M to M) discrete-time sequence $\{y_n\}$.

$$Y(\omega) = \sum_{n=-M}^{M} y_n e^{-\iota n\omega To} \qquad (1)$$

$$y_n = \frac{1}{\omega o} \int_{-\frac{\omega o}{2}}^{\frac{\omega o}{2}} Y(\omega) e^{\iota n\omega To} d\omega \qquad (2)$$

$$(\omega o \, To = 2\pi)$$

The optimization problem related to the GDPSS's is as follows.

$$\max \ \beta = \int_{-\Omega}^{\Omega} |Y(\omega)|^2 d\omega \qquad (3)$$

$$(0 < \Omega < \frac{\omega o}{2})$$

,where the constraints are

$$\frac{1}{\omega o} \int_{-\frac{\omega o}{2}}^{\frac{\omega o}{2}} |Y(\omega)|^2 d\omega = 1 \qquad (4)$$

or equivalently $$\sum_{n=-M}^{M} |y_n|^2 = 1, \qquad (4')$$

$$\sum_{n=-M}^{M} n^2 |y_n|^2 = \alpha^2, \qquad (5)$$

$$(0 < \alpha < M)$$

and finally we require that y_n 's are real.

Since β has the upper limit determined by the Uncertainty Principle, this problem is well defined, thus has a solution.

There are three parameters in this problem, i.e. M,Ω, and α. M determines the time duration of the sequence. Ω determines the interesting frequency region where the optimization is performed. α determines the effective time duration of the sequence.

Now let's solve the problem. First, we define β' as follows using two Lagrangian multipliers -2λ and -2μ. (-2 factor for mathematical convinience)

$$\beta' = \int_{-\Omega}^{\Omega} |Y(\omega)|^2 d\omega + (-2\lambda) \sum |y_n|^2 + (-2\mu) \sum n^2 |y_n|^2 \qquad (6)$$

Then, we calculate $\frac{\partial \beta'}{\partial y_n}$'s for all n=-M to M, and let them be zero to get the solution.

$$\frac{\partial \beta'}{\partial y_n} = \int_{-\Omega}^{\Omega} 2\mathrm{Re} \left[Y^*(\omega) \frac{\partial Y}{\partial y_n}(\omega) \right] d\omega - 4y_n (\lambda + \mu n^2) = 0 \qquad (7)$$

Calculating the integral in (7) using (1), we obtain

$$\sum_{m=-M}^{M} y_m \,\mathrm{sinc}\, [\Omega To\,(m-n)] = y_n (\lambda + \mu n^2) \qquad (8)$$

Now, the optimization problem is converted into the algebraic equations (8), (4') and (5), where 2M+3 equations and 2M+3 unknowns are involved. For convinience, let's use matrix notations.

$$A_{ij} = \mathrm{sinc}\,[\Omega To\,(i-j)],$$
$$N_{ij} = i^2 \delta_{ij} ,$$

where i,j is varied from -M to M.

Using these elements, we can define A,N,y. Rewritting (8),(4'), and (5) using A,N,y, we obtain

$$Ay = \lambda y + \mu Ny \qquad (9)$$
$$y^T y = 1 \qquad (10)$$
$$y^T N y = \alpha^2 \qquad (11)$$

Let's modify (9) as

$$(A - \mu N) y^{(i)}(\mu) = \lambda^{(i)}(\mu) y^{(i)}(\mu) \quad (12)$$

, where index i represents the i-th eigen value and the i-th eigen vector and index μ represents that y and λ are functions of μ.

From (3), we obtain

$$\int_{-\Omega}^{\Omega} |Y(\omega)|^2 d\omega = 2 \Omega (\lambda + \mu \alpha^2). \quad (13)$$

(13) means that we should choose the index i which gives the maximum value of the eigen value for a given μ.

Therefore, we can easily set up the procedure for finding unknowns in (9),(10) and (11).

1. Let the initial value of $\mu = 0$.
2. From (12), calculate the maximum eigenvalue and the corresponding normalized eigenvector.
3. Calculate $y^T N y - \alpha^2$.
4. Find next μ by the bisection algorithm using the sign of the result in step 3 and return to step 2 if not reached the solution.

III. EXAMPLES

It is sometimes required for a time window to distinguish the fundamental frequency (60Hz) and the next dominant frequency (for example, 120Hz). This requirement corresponds to the maximum energy concentration in the frequency range of about -30 to 30Hz. Therefore, Ω can be choosen as $\frac{2\pi * 60}{2}$. Let To $= \frac{1}{60*12}$ for 12 samples in one period.

Example 1)
$\alpha = 0.5$
energy concentration $\beta = 0.36$

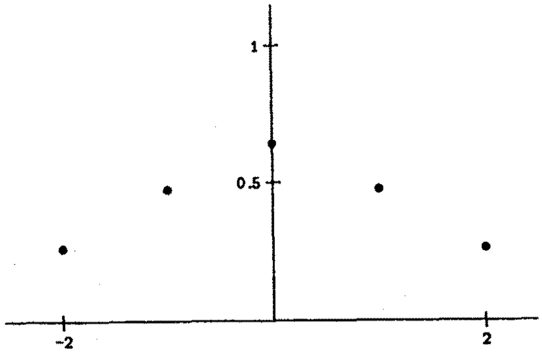

fig. 1 Time sequence of example 1

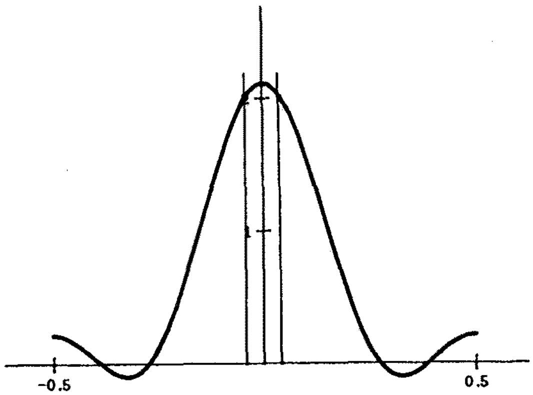

fig. 2 Fourier Transform of fig. 1

At the same condition original DPSS gives
$\alpha = 0.7$ and
energy concentration $\beta = 0.39$.
Therefore, we can conclude that the GDPSS gives a more desirable solution.

Example 2)
$\alpha = 0.58$
energy concentration $\beta = 0.74$

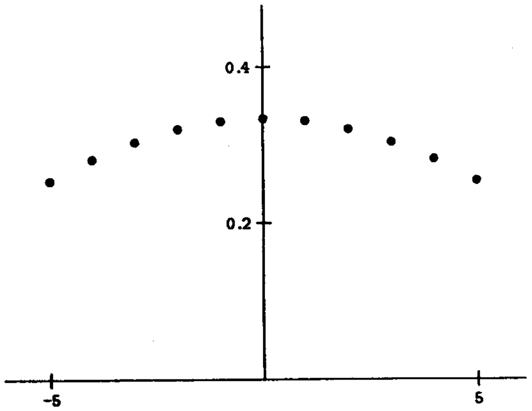

fig. 3 Time sequence of example 2

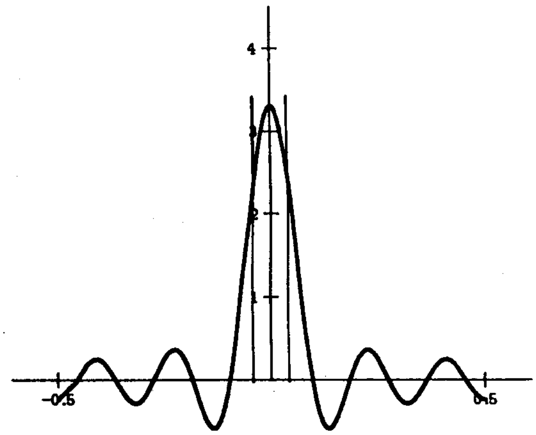

fig. 4 Fourier Transform of fig. 3

IV. STRATEGY TO DETERMINE M,Ω,α

Physical meanings of the parameters of GDPSS's can be understood by thinking the strategy to determine the parameters. As a concluding remark, a simple strategy is introduced here.

1. Amount of calculations available for windowing.
   => determine M
2. Acceptable time delay by windowing
   => determine α
   (Effective time delay is α + M)
   If inconsistent with M, reduce M.
3. Frequency boundary to distinguish the fundamental frequency and the next dominant frequency.
   => determine Ω
4. Satisfactory energy concentration in the frequency region? If not, increase α and go to step 2.

References

[1] D. Slepian, "Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty - V:The Discrete Case", BSTJ v.57, no.5, pp. 1371-1430.

[2] B. Jeyasurya, "Identification of a best algorithm for digital distance protection of transmission lines", IEEE tr. on PAS, v.PAS-102, no.10, Oct. 1983, pp.3358-3369.