'91 KACC 1991. 10. 22~24

A Study of Parameter Estimation of Stochastic Volatility Model

Makiko Tsukui and Katsuhisa Furuta

Department of Control Engineering
Tokyo Institute of Technology, Tokyo, Japan

Abstract

The theory of stock option pricing has, recently, attracted attention of many researchers interested not only in finance but also in statistics and control theory. In this field, the problem of estimating stock return volatility is, above all, of great importance in calculating actual stock option value.

In this paper, we assume that the stock market is represented by the stochastic volatility model which is the same as that of Hull and White. Then, we propose an approximation function of option value. It is a type of Black–Sholes option formula in which the first and the second order moments of logarithmic stock value are modified in a special form from the original model.

Finally, an algorithm of estimating the parameters of the stochastic volatility model is given, and parameters are estimated by using Nikkei 225 index option data.

1 Introduction

An option can be defined as a security for the right to buy or sell some commodity. Since origin is old, there is a long history of studying the option. However in 1973, Black and Sholes first derived the option value formula under non-arbitrage condition which was a break-through to the theoretical way of studying this problem.[3] , where the non-arbitrage condition means that the market does not allow free lunch.

Their result was, however, based on many strict assumptions. Especially, the one that the stock volatility is time-constant contradicts real phenomena of stock market. Here, the stock volatility is a standard deviation of stock value and it is the most important parameter in the option value function.

Hull and White, then, derived the stock option value with a more general stochastic volatility model.[7] They also proposed an approximation function of option value by using a Taylor series expansion. Its expression was , however, rather complex.

In this paper, we assume that the stock market is represented by the stochastic volatility model which is the same as that of Hull and White. Then, we propose an approximation function of option value. It is a type of Black-Sholes option formula in which the first and the second order moments of logarithmic stock value are modified in a special form from the original model. It is simpler in expression than the approximation function of Hull and White.

The contents of this paper is as follows. In Section 2, we introduce a stochastic volatility model as a stock market model, and investigate the characteristics of the stock distribution under this model. In Section 3, we introduce the option value under non-arbitrage condition which was derived by Hull and White[7]. In Section 4, we propose an approximation function of option value and test its reliability. We also compare our approximation function with the one which was proposed by Hull and White. In Section 5, an algorithm of estimating parameters of stochastic volatility model is given, and parameters are estimated by using Nikkei 225 index option data. Concluding remarks are given in Section 6.

2 Security market model

2.1 Security market model

First, we shall assume the following assumptions for our security market:

Assumption 2.1: The market is complete.

Assumption 2.2: The short–term interest rate is known and constant within a specified period of time.

Assumption 2.3: The stock volatility follows log–normal distribution, and the stock value follows a random walk process under this condition.

Assumption 2.4: There is no dividend and division of the stock.

Assumption 2.5: The exercise of an option is not allowed except for on the maturity date.

Assumption 2.6: The fund can be borrowed or lent with the short–term interest rate without any restriction.

Assumption 2.7: The proceeds of short selling of stock or option can be re–invested without any cost.

Assumption 2.8: All investors in the market have risk aversion preference.

Let B and S be, respectively, the bond and the stock traded in the market. Then, under the assumptions above, the expected earning rate of these assets can be represented by the following Markov stochastic differential equation.

$$\frac{dB}{B} = rdt \tag{2.1}$$

$$\frac{dS}{S} = \eta dt + \sigma dw \tag{2.2}$$

$$\frac{dV}{V} = \mu dt + \xi dz \tag{2.3}$$

Where r is a short-term interest rate, σ, the stock volatility, and $V = \sigma^2$. The variables, dw and dz, follow standard Gaussian distribution and they are independent each other. The parameters, η, μ, and ξ, are supposed to be time-invariant constants.

2.2 The stochastic characteristics of the stock process

Here we investigate the stochastic characteristics of the stock process represented by the equations (2.1), (2.2), (2.3). First, we try to derive the first moment of the stock process. We have the following theorem from the above conditions.

Theorem 2.1 *The stock process, $S(T)$, can be represented as follows:*

$$E[S(T)|S(t),V(t)] = e^{\eta(T-t)}S(t) \tag{2.4}$$

Where, the states $S(t)$,$V(t)$, and $\sigma(u)$ are given under current date t,and $T \geq t$.

Proof)

By applying Ito's lemma [6], we obtain from the equation (2.2) that

$$S(T) = S(t)e^{\int_t^T(\eta-\frac{\sigma^2(u)}{2})du+\int_t^T\sigma(u)dw(u)} \tag{2.5}$$

Then,

$$E[S(T)|S(t),V(t)] = S(t)e^{\eta(T-t)}E[e^{-\frac{1}{2}\int_t^T\sigma^2(u)du}e^{\int_t^T\sigma(u)dw(u)}|V(t)] \tag{2.6}$$

This can be written with the conditional expectation of $\{\sigma(u)|0 \leq u \leq T\}$ that

$$E[S(T)|S(t),V(t)] = S(t)e^{\eta(T-t)}E[E[e^{-\frac{1}{2}\int_t^T\sigma^2(u)du}e^{\int_t^T\sigma(u)dw(u)}|\{\sigma(u)\}]|V(t)] \tag{2.7}$$

Here, $\int_t^T\sigma(u)dw(u)$ follows a normal distribution since $\int_t^T\sigma(u)dw(u)$ can be represented as the sum of stochastic variables $dw(u)$ which follow a normal distribution $N(0,u)$. $e^{\frac{1}{2}\int_t^T\sigma^2(u)du}$ and $e^{\int_t^T\sigma(u)dw(u)}$ are independent each other when $\sigma(u)$ is given. Moreover, the mean of $\int_t^T\sigma dw(u)$ is 0 as the mean of $dw(u)$ is 0. By the Ito's isometry [9], the variance of $\int_t^T\sigma dw(u)$ is given as follows.

$$E[\{\int_t^T\sigma(u)dw(u)\}^2|V(t)] = E[\int_t^T\sigma^2(u)du|V(t)] \tag{2.8}$$

Under the condition of $\{\sigma(u)|0 \leq u \leq T\}$, the variance of equation (2.8) becomes $\int_t^T\sigma^2(u)du$. From the equation (2.8), equation (2.7) can be represented as

$$E[S(T)|S(t),V(t)] = S(t)e^{\eta(T-t)}E[E[e^{-\frac{1}{2}\int_t^T\sigma^2(u)du}e^{\frac{1}{2}\int_t^T\sigma^2(u)du}|\sigma(u)]] = S(t)e^{\eta(T-t)}$$

□

Next, we shall prove the following theorem which tells that the second moment of the stock process diverges.

Theorem 2.2 *If $0 \leq t \leq T$, $\xi \neq 0$, and $\mu \geq 0$, then the second moment of the stock process, $E[S^2(T)|S(t),V(t)]$, will diverge. Here the states $S(t)$, $V(t)$, and $\sigma(u)$ are given under current date t.*

Proof)

As in the Theorem 2.1, the second moment of the stock process can be represented as

$$E[S^2(T)|S(t),V(t)] = S^2(t)e^{2\eta(T-t)}E[e^{\int_t^T\sigma^2(u)du}|V(t)] \tag{2.9}$$

By Theorem 2.1, the solution process of V can be given by

$$\sigma^2(u) = V(t)e^{(\mu-\frac{\xi^2}{2})(u-t)}e^{\xi\{z(u)-z(t)\}} \tag{2.10}$$

Applying the equation (2.10), $E[e^{\int_t^T\sigma^2(u)du}|V(t)]$can be represented by Taylor expansion as

$$\begin{aligned} &E[e^{\int_t^T\sigma^2(u)du}|V(t)] \\ &= E[\lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{n!}(\int_0^{T-t}V(t)e^{(\mu-\frac{\xi^2}{2})v}e^{\xi z(v)}dv)^n|V(t)] \end{aligned} \tag{2.11}$$

Now, we divide the domain of integration $[0,T-t]$ into k intervals, and define $G(T,k)$ as the discreet function of the equation (2.11) for these intervals. Here k is a finite integer and $v_i = i\frac{T-t}{k}$,$v_0 = 0$,$\Delta v_i = v_i - v_{i-1}$. We then have

$$G(T,k) = E[\lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{n!}(V(t)\frac{T-t}{k})^n(\sum_{i=1}^{k}\prod_{j=1}^{i}e^{(\mu-\frac{\xi^2}{2})\Delta v_j+\xi z(\Delta v_j)})^n|V(t)]$$

Let $X_j \equiv e^{(\mu-\frac{\xi^2}{2})\Delta v_j+\xi z(\Delta v_j)}$,$\bar{n}_i \equiv \sum_{j=i}^{k}n_j$, then,

$$G(T,k) = \lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{n!}(V(t)\frac{T-t}{k})^n\sum_{\sum n_i=n,n_i\geq 0}E[\prod_{i=1}^{k}X_i^{\bar{n}_i}|V(t)]$$

Since every X_i is independent to each other, we have

$$\begin{aligned} G(T,k) &= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{n!}(V(t)\frac{T-t}{k})^n\sum_{\sum n_i=n,n_i\geq 0}\prod_{i=1}^{k}E[X_i^{\bar{n}_i}|V(t)] \\ &= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{n!}(V(t)\frac{T-t}{k})^n\sum_{\sum n_i=n,n_i\geq 0}e^{\sum_{i=1}^{k}\{\mu\bar{n}_i+\frac{\xi^2}{2}(\bar{n}_i^2-\bar{n}_i)\}\frac{T-t}{k}} \end{aligned} \tag{2.12}$$

Here,$\sum_{i=1}^{k}\bar{n}_i = \sum_{i=1}^{k}\sum_{j=i}^{k}n_i = \sum_{i=1}^{k}in_i$ and $\sum_{i=1}^{k}(\bar{n}_i^2 - \bar{n}_i) \geq \sum_{i=1}^{k}in_i(n_i-1)$.

Since $n-i$ can be any combinations which satisfy $\sum n_i = n, n_i \geq 0$, there must exist certain n_i such that:

$$n_i = \begin{cases} n & (i = k) \\ 0 & (i \neq k) \end{cases}$$

Then, the inequalities, $\sum_{i=1}^{k}\bar{n}_i \geq kn, \sum_{i=1}^{k}(\bar{n}_i^2 - \bar{n}_i) \geq kn(n-1)$, must hold for the equation (2.12). Since $\xi^2 \geq 0$,$T-t \geq 0$,$\mu \geq 0$, and $V(t) \geq 0$ in the equation (2.12), $G(T,k)$ is bounded from below such that

$$\begin{aligned} G(T,k) &\geq \lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{n!}(V(t)\frac{T-t}{k})^n e^{\{\mu kn+\frac{\xi^2}{2}kn(n-1)\}\frac{T-t}{k}} \\ &\geq \lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{n!}(V(t)\frac{T-t}{k})^n e^{an+bn(n-1)} \\ &> \lim_{N\to\infty}\frac{1}{N!}(V(t)\frac{T-t}{k})^N e^{bN(N-1)} \end{aligned} \tag{2.13}$$

where $a = \mu(T-t) > 0, b = \frac{\xi^2}{2}(T-t) > 0$.

Applying Stirling's formula to the equation above, we can obtain the following result.

$$\begin{aligned} G(T,k) &> \lim_{N\to\infty}\frac{(V(t)\frac{T-t}{k})^N e^{bN(N-1)}}{(2\pi)^{\frac{1}{2}}N^{N+\frac{1}{2}}e^{-N}} \\ &= \lim_{N\to\infty}(\frac{e^{bN}}{N})^N\frac{(V(t)\frac{T-t}{k})^N e^N}{(2\pi)^{\frac{1}{2}}N^{\frac{1}{2}}} \end{aligned} \tag{2.14}$$

The right hand side of the equation (2.14) diverges for any sufficiently large k. On the other hand, $E[e^{\int_t^T\sigma^2(u)du}|V(t)]$ in the equation (2.9), can be obtain as $E[e^{\int_t^T\sigma^2(v)dv}|V(t)] = \lim_{k\to\infty}G(T,k)$ when it converges for each finite k. Therefore, $E[e^{\int_t^T\sigma^2(v)dv}|V(t)] = +\infty$.This implies that the value of the equation (2.9) diverges.

□

As mentioned above, the variance of $S(T)$ does not exist. This result seems to be reasonable since the realistic stock process is based on the following general assumptions [13],[14].

1. The daily fluctuation of the stock price follows an i.i.d stable palate distribution, t-distribution, and etc.

2. The daily fluctuation of the stock price follows a normal distribution whose variance depends on its history.

We, then, expect that a more reasonable option value can be obtained with this model.

3 The derivation of option value under non-arbitrage condition

In this section, we , first, introduce the asset value function under non-arbitrage condition as Hull and White did, and then, the option value [7]. First, we introduce the differential equation which must be satisfied by some asset value function, F, under non-arbitrage condition.[7]

Theorem 3.1 *The differential equation that an asset value function, F, must satisfy under non-arbitrage condition can be represented as follows.*

$$F_t + rSF_S + \tilde{\mu}VF_V + \frac{1}{2}F_{SS}\sigma^2S^2 + \frac{1}{2}F_{VV}\xi^2V^2 - rF = 0 \tag{3.15}$$

Where $F_t \equiv \frac{\partial F}{\partial t}$, $F_S \equiv \frac{\partial F}{\partial S}$, $F_V \equiv \frac{\partial F}{\partial V}$, $F_{SS} \equiv \frac{\partial^2 F}{\partial S^2}$, $F_{VV} \equiv \frac{\partial^2 F}{\partial V^2}$.

Next, we introduce the asset value function under non-arbitrage condition. Let $\tilde{S}$ and $\tilde{V}$ be the stock process and volatility process in which each drift term of S and V are r and $\tilde{\mu}$ under the original measure. Each $\tilde{S}$ and $\tilde{V}$ on the initial date t_0 are equal to $S(t_0)$ and $V(t_0)$,respectively. dw and dz follow the standard normal distribution.

Theorem 3.2
The following asset value function, $\tilde{F}(S(t), V(t), t)$, which can be represented by $\tilde{S}$ and $\tilde{V}$ satisfies the non-arbitrage condition. Let $F(T) \equiv F(\tilde{S}(T), \tilde{V}(T), T)$, $p(T;t) \equiv p(\tilde{S}(T), \tilde{V}(T), T; S(t), V(t), t)$, we have

$$\tilde{F}(S(t), V(t), t) = e^{-r(T-t)}E\{F(\tilde{S}(T), \tilde{V}(T), T)|S(t), V(t)$$

$$= e^{-r(T-t)}\int_{\tilde{S}(T)\in R_+} F(T)p(T;t)dS(T) \tag{3.16}$$

, where $p(\tilde{S}(T), \tilde{V}(T), T; S(t), V(t), t)$is the transition probability density function from $(\tilde{S}(t), \tilde{V}(t), t)$to $(\tilde{S}(T), \tilde{V}(T), T)$.

At last, we introduce the option value under non-arbitrage condition which is the special case of the previous section which was derived by Hull and White [7]. Here we show only the call-option value which concerns the security for the right to buy the stock. Let C be the call-option value function. Then, it must satisfy the following boundary condition on the maturity date.

Condition 3.3 *The call-option value function, C, must satisfy the following boundary condition on the maturity date.*

$$C(S(T), V(T), T) = max[S(T) - K, 0] \tag{3.17}$$

Then, the call-option value under non-arbitrage condition is given in the following theorem.

Theorem 3.4 *The call-option value function under the stochastic volatility model given by the equations (2.1), (2.2), and (2.3) can be represented as follows.*

$$C(S(t), V(t), t) = \int_{\bar{V}\in R_+} C_{BS}(S(t), \bar{V}, t)p(\bar{V}|V(t), t)d\bar{V} \tag{3.18}$$

, where $C_{BS}(S(t), \bar{V}, t)$is the option value of Black-Sholes formula when the volatility is given by following $\bar{V}$.

$$\bar{V}(t,T) \equiv \frac{1}{T-t}\int_t^T \tilde{\sigma}^2(u)du \equiv \bar{V} \tag{3.19}$$

The thorem above shows the way to evaluate the option price precisely. However, this valuation formula is difficalt to solve. Therfore it's not efficient to apply this formula directly for the practical use. It, then, seems to be necessary to obtain a simpler apporximation function of the option value, although it may give a little less precise value.

4 Proposal of an approximation function of the option value

In this section, we propose an approximation function of the option value. It is a type of Black-Sholes option formula in which the first and the second order moments of logarithmic the stock value are modified in a special form from the original model. We, then, test its reliability and compare it with the one which was proposed by Hull and White.

4.1 Proposal of an option approximation function

Here, we propose an approximation function of the option value.

The call-option value function, $C(S(t), V(t), t)$, can be approximated as

$$C(S(t), V(t), t) \simeq e^{-r(T-t)}E[max(\hat{S}(T) - K, 0)] \tag{4.20}$$

where $\ln \hat{S}(T)$ follows a normal distribution.

The mean and the variance of $\ln \hat{S}(T)$ are approximated by the first and the second moment of $\ln \tilde{S}(T)$ respectively.Then, the following relation holds:

$$E[\ln \hat{S}(T)] \simeq rt - \frac{V(t)}{2\mu}(e^{\mu t} - 1) \tag{4.21}$$

$$V[\hat{S}(T)] = E[\hat{S}^2(T)] - E[\hat{S}(T)]^2$$
$$\simeq \frac{1}{\mu}(e^{\mu t} - 1)V(0) + \frac{V(0)^2}{2}[\frac{e^{\Theta_2 t}}{(\mu+\xi^2)\Theta_2} - \frac{e^{2\mu t}}{4\mu^2} + \frac{\mu - \xi^2}{2\mu^2\Theta_1} - \frac{\xi^2}{2\Theta_2\xi^2}] \tag{4.22}$$

Where, $\Theta_1 \equiv \mu + \xi^2$, and $\Theta_2 \equiv 2\mu + \xi^2$.

Where, $\Theta_1 \equiv \mu + \xi^2$, and $\Theta_2 \equiv 2\mu + \xi^2$.

4.2 The approximation function of Hull and White

The approximation function which was proposed by Hull and White is represented by the following C_{HW}.

$$C_{HW} = C(\bar{V}) + \frac{1}{2}\frac{\partial^2 C}{\partial \bar{V}^2}|_{\bar{V}} Var(\bar{V}) + \frac{1}{6}\frac{\partial^3 C}{\partial \bar{V}^3}|_{\bar{V}} Skew(\bar{V})$$

where, $Var(\bar{V})$ and $Skew(\bar{V})$ are the second and third central momnets of $\bar{V}$.

4.3 The test of the approximation functions

In this section, we test the reliability of our approximation function and compare it with the one proposed by Hull and White.

4.3.1 Testing Algorithm

We employ a reliability test is as follows. We, first, compute the numerical value of the equation (3.18) by a simulation, and then, check whether the errors of approximation values to the precise values are within 5%.

To calculate the numerical value of the equation (3.18), we need the discrete type of the option value function such that

$$C \simeq \frac{1}{N}\sum_{h=1}^{N} C_{BS}(S(t), \bar{V}_h(t,T), t) \equiv \frac{1}{N}\sum_{h=1}^{N} \bar{C}_h \qquad (4.23)$$

where N is a sufficiently large integer and $C \equiv C(S(t), V(t), t)$.

For the calculation of this equation, we also need the discrete function of $\bar{V}$ which will be obtained by dividing the domain of integration $[0, T-t]$ into k intervals. We then have

$$\bar{V} \simeq \frac{\hat{V}(t)}{k}\sum_{j=1}^{k} e^{(\bar{\mu}-\frac{\xi^2}{2})u_j} e^{\xi \sum_{j=1}^{i} z(\Delta u_j)} \qquad (4.24)$$

where the definition of $\Delta u_i$is the same as Δv_iin Theorem 2.2.

Now, if we generate a series of stochastic variable $z(\Delta u_i)$which follows the Gaussian distribution $N(0, \Delta u_i)$, we can calculate the discrete function of the option value from the above relations (4.23) and (4.24).

The 5% reliability interval of the option value will be obtained as follows.

$$\underline{\mu_C} - 0.05\mu_C \leq \mu_C \leq \overline{\mu_C} + 0.05\mu_C \qquad (4.25)$$

where $\alpha = 1.96$, $\mu_C \equiv \frac{1}{N}\sum_{i=1}^{N}\bar{C}_i$, $\sigma_C \equiv \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(\bar{C}_i - \mu_C)^2}$, $\underline{\mu_C} \equiv \frac{1}{N}\sum_{i=1}^{N}\bar{C}_i - \frac{\sigma_C}{\sqrt{N}}\alpha$, and $\overline{\mu_C} \equiv \frac{1}{N}\sum_{i=1}^{N}\bar{C}_i + \frac{\sigma_C}{\sqrt{N}}\alpha$. To apply the above mentioned method, we shall test the reliability of the approximation function in the following.

4.3.2 Simulational test

Here, we show the results of the numerical test on the condition that option parameters are $k = 100, N = 500$, short-term interest rate,$r = 0.055$,and $V(t) = 0.1$. There are three cases of the test, that is, the cases of varying ξ, varying μ, and varying $T - t$. For each of these cases, we tested the three cases of "out-the-money", "at-the-money", and "in-the-money". The result is shown in Table 4_1_1 -4_3_3, where the cases of varying ξ in Table 4_1_1 - 4_1_3, varying μ in Table 4_2_1 - 4_2_3, and varying $T - t$ in Table 4_3_1 - 4_3_3.

These results tell us that the approximation function $\tilde{C}$ is sufficiently reliable since all parameters are within the 5% error interval.

5 Parameter estimation using the daily data

Here, we shall estimate parameters of the stochastic volatility model, $\tilde{mu}$ and σ. We, first, show the estimation algorithm, then the results obtained by using it.

5.1 Estimation algorithm

The algorithm of parameter estimation is as follows.

1. We set appropriate initial values to the parameters, $\tilde{\mu}$ and ξ.
2. We calculate the volatility of the day using our approximation function. There are several kinds of options for a stock traded in the market within a day. They are different depending on the exercise prices and the lengths of time till the maturity date, respectively. We, then, obtain several implied-volatilities for these different options by using our approximation function. We calculate the volatility of the day by averaging these daily implied-volatilities.
3. We estimate μ and ξ as the unbiasedness estimates of the drift term and variance of the series of these daily volatility obtained in 2.
4. By repeating the procedure from 2 to 3, we estimate the fixed points of μ and ξ.

We used the Brent-method in the above computation of implied-volatility.[11]

5.2 The results of parameter estimation

For this simulation, we used Nikkei 225 index option data from June 6,1989 to April 3, 1990. The fluctuation of the stock price is shown in Fig 5.2.

The fluctuation of the volatilities which are estimated by our approximation method is shown in Fig 5.4.

The estimated values of parameters are $\tilde{\mu} = 0.00584$,$\xi = 0.250310$.

6 Concluding remarks

In this paper, we assume that the stock market is represented by a stochastic volatility model which is the same as that of Hull and White. Then, we have proposed an approximation function of option value. It is a type of Black-Sholes option formula in which the first and the second order moments of logarithmic stock value receive special modification from the original model. Although it may give a little less precise value, it is simpler in expression than the approximation function of Hull and White. And the results of reliability test tell us that our approximation function is sufficiently reliable since all parameters are within the 5% error interval. Finally, we shall estimate parameter of the stochastic volatility model, $\tilde{\mu}$ and σ, with an algorithm of estimating parameters of stochastic volatility model.

We hope that by extending the results in this paper, it will become useful for the evaluation of new financial comodity in the market and for the application to the optimal portfolio theory.

References

[1] John.C.Cox and Mark Rubinstein,"Options Market",Prentice-Hall,Inc, 1985

[2] K.Ohmura and M.Shimizu,"The Stock Option", Kinyu-Zaiseijijyo Kenkyukai, 1987,(in Japanese)

[3] F.Black and M.Scholes,"The Pricing of Options and Corporate Liabilities", Journal of Political Economy 81(May-June 1973)

[4] R.C.Merton, "Option Pricing When Underlying Stock Returns Are Discontinuous" Journal of Financial Economics 3, pp.125-144, 1975

[5] H.Kunida,"The Estimation of Stochastic Process",Sangyo-Tosho,1976,(in Japanese)

[6] H.J.Kushner,"Stochastic Stability and Control",Academic Press,1967

[7] J.Hull and A.White, "The Pricing of Options on Assets with Stochastic Volatilities",The Journal of Finance,vol XLII,No.2,June 1987

[8] Y.Sunahara,"Stochastic System Theory",EIC,(in Japanese)

[9] "Stochastic Differential Equations",Springer-Verlag

[10] I.Karatzas and S.E.Shreve,"Brownian Motion and Stochastic Calculus", Springer-Verlag

[11] "Turbo C Library of Numerical Computation",HBJ-Shuppan-Sha,(in Japanese)

[12] S.Karlin, "A First Course in Stochastic Processes", Academic Press,Inc. 1969

[13] K.Kishimoto,"The Study of Positive Subordination of Stock Fluctuation",ORSJ, Abstract of The 1989 Spring National Conference of ORSJ,(in Japanese)

[14] K.Kishimoto and K.Yagi,"The Study of Stock Fluctuation Distribution",ORSJ Abstract of The 1990 Spring National Conference of ORSJ,(in Japanese)

A Tables

Table 4.1.1 The case of out-the-money ($S = 34000, K = 38000$)

ξ	0.1	0.25	0.5	1.0
C_{HW_2}	145.5971	145.8159	146.6214	150.2589
C_{HW_3}	145.5967	145.8017	146.3777	145.0578
$\hat{C}$	145.5563	145.5591	145.5697	145.6172
μ_C	145.2455	146.3355	146.0206	150.4207
$[\underline{\mu_C}, \overline{\mu_C}]$	[144.3262 , 146.1648]	[144.1648 , 148.5063]	[141.4714 , 150.5697]	[140.9411 , 159.9004]
5%error interval	[137.0639 , 153.4271]	[136.8480 , 155.8231]	[134.1704 , 157.8707]	[133.4200 , 167.4214]

Table 4.1.2 The case of at-the-money ($S = 34000, K = 34000$)

ξ	0.1	0.25	0.5	1.0
C_{HW_2}	1478.0517	1477.2280	1474.1960	1460.5044
C_{HW_3}	1478.0521	1477.2435	1474.4611	1466.1597
$\hat{C}$	1478.2084	1478.2138	1478.2338	1478.3241
μ_C	1478.2732	1479.1625	1477.7072	1458.5834
$[\underline{\mu_C}, \overline{\mu_C}]$	[1476.6855 , 1479.8610]	[1475.3867 , 1482.9384]	[1469.8288 , 1485.5855]	[1443.5459 , 1473.6209]
5%error interval	[1402.77187 , 1553.7746]	[1401.4285 , 1556.8965]	[1395.9434 , 1559.4709]	[1370.6167 , 1546.5500]

Table 4.1.3 The case of in-the-money ($S = 34000, K = 28000$)

ξ	0.1	0.25	0.5	1.0
C_{HW_2}	6760.0312	6760.0930	6760.3204	6761.3475
C_{HW_3}	6760.0313	6760.0951	6760.3570	6762.1282
$\hat{C}$	6760.0197	6760.0205	6760.0234	6760.0365
μ_C	6760.0214	6760.0865	6760.3265	6761.8023
$[\underline{\mu_C}, \overline{\mu_C}]$	[6760.0100 , 6760.0328]	[6760.0533 , 6760.1197]	[6760.2375 , 6760.4155]	[6761.1565 , 6762.4480]
5%error interval	[6422.0090 , 7098.0339]	[6422.0489 , 7098.1240]	[6422.2212 , 7098.4318]	[6423.0664 , 7100.5381]

Table 4.2.1 The case of out-the-money ($S = 34000, K = 38000$)

μ	0.001	0.005	0.01
C_{HW_2}	146.3751	146.6214	146.9298
C_{HW_3}	146.1316	146.3777	146.6858
$\hat{C}$	145.3214	145.5697	145.8805
μ_C	144.8048	147.3045	146.3315
$[\underline{\mu_C}, \overline{\mu_C}]$	[140.2666 , 149.3429]	[142.9699 , 151.6391]	[141.7742 , 150.8887]
5%error interval	[133.0263 , 156.5832]	[135.6047 , 159.0044]	[134.4577 , 158.2053]

Table 4.2.2 The case of at-the-money ($S = 34000, K = 34000$)

μ	0.001	0.005	0.01
C_{HW_2}	1473.7625	1474.1960	1474.7384
C_{HW_3}	1474.0270	1474.4611	1475.0041
$\hat{C}$	1477.7952	1478.2338	1478.7827
μ_C	1474.3592	1475.3899	1478.3681
$[\underline{\mu_C}, \overline{\mu_C}]$	[1466.3267 , 1482.3917]	[1467.5100 , 1483.2698]	[1470.8290 , 1485.9072]
5%error interval	[1392.6087 , 1556.1096]	[1393.7406 , 1557.0393]	[1396.9106 , 1559.8256]

Table 4.2.3 The case of in-the-money ($S = 34000, K = 28000$)

μ	0.001	0.005	0.01
C_{HW_2}	6760.3158	6760.3204	6760.3262
C_{HW_3}	6760.3523	6760.3570	6760.3630
$\hat{C}$	6760.0202	6760.0234	6760.0274
μ_C	6760.3873	6760.3131	6760.2712
$[\underline{\mu_C}, \overline{\mu_C}]$	[6760.2720 , 6760.5026]	[6760.2110 , 6760.4152]	[6760.1807 , 6760.3617]
5%error interval	[6422.2526 , 7098.5220]	[6422.1954 , 7098.4309]	[6422.1671 , 7098.3752]

Table 4.3.1 The case of out-the-money ($S = 34000, K = 38000$)

$T-t$	0.3	0.5	0.8
C_{HW_2}	34.38748	146.6214	408.4462
C_{HW_3}	34.33555	146.3777	408.4231
$\hat{C}$	33.28221	145.56966	412.15720
μ_C	33.96749	147.30452	407.29146
$[\underline{\mu_C}, \overline{\mu_C}]$	[32.70286 , 35.23212]	[142.9699 , 151.6391]	[396.4653 , 418.1176]
5%error interval	[31.00448 , 36.93049]	[135.6047 , 159.0044]	[376.1007 , 438.4822]

Table 4.3.2 The case of at-the-money ($S = 34000, K = 34000$)

$T-t$	0.3	0.5	0.8
C_{HW_2}	1046.3538	1474.1960	2053.1523
C_{HW_3}	1046.4397	1474.4611	2053.8072
$\hat{C}$	1048.4026	1478.2338	2060.2586
μ_C	1046.6467	1475.3899	2058.4687
$[\underline{\mu_C}, \overline{\mu_C}]$	[1041.6380 , 1051.6554]	[1467.5100 , 1483.2698]	[2047.0653 , 2069.8720]
5%error interval	[989.3057 , 1103.9877]	[1393.7406 , 1557.0393]	[1944.1419 , 2172.7955]

Table 4.3.3 The case of in-the-money ($S = 34000, K = 28000$)

$T-t$	0.3	0.5	0.8
C_{HW_2}	6458.2511	6760.3204	7210.2186
C_{HW_3}	6458.2538	6760.3570	7210.3466
$\hat{C}$	6458.2338	6760.0234	7208.5666
μ_C	6458.2580	6760.3131	7209.7632
$[\underline{\mu_C}, \overline{\mu_C}]$	[6458.2505 , 6458.2655]	[6760.2110 , 6760.4152]	[7209.2559 , 7210.2705]
5%error interval	[6135.3376 , 6781.1784]	[6422.1954 , 7098.4309]	[6848.7677 , 7570.7586]

B Figures

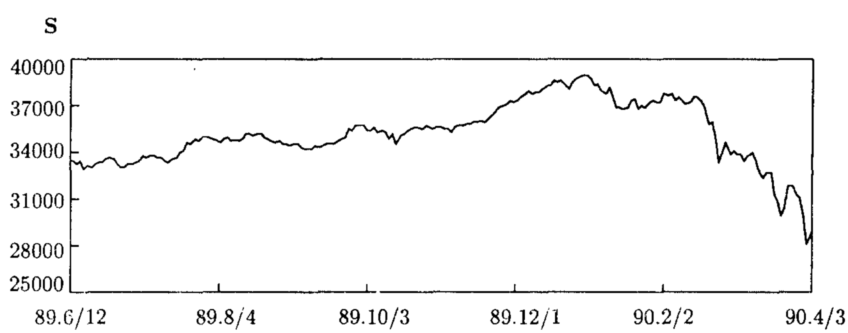

Fig. B.1 Stock Price

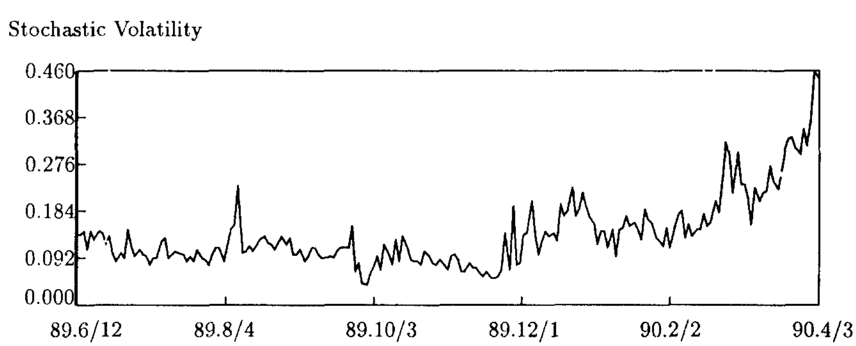

Fig. B.2 Implied volatility of Stochastic volatility model