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A Formal Linearization Method via Cubic Splines and its Applications

Hitoshi Takata* and Katsuhiro Narikiyo**

*Department of Electric, Electronic and computer Engineering
Kyushu Institute of Technology, Tobata, Kitakyushu 804, Japan

**Department of Electronic Control Engineering
Hiroshima National College of Maritime Technology
Higashino, Toyota, Hiroshima 725-02, Japan

Abstract

To solve the nonlinear system problems, many methods have been proposed. Generally those methods however need long processing time because of their complicated algorithms. On the other hand, some simple linearization methods also have been studied. In this paper, a new linearization method using cubic splines[1] is proposed. The approximated linear system obtained by this method we can apply the conventional simple linear system theories such as Kalman filter[2,3] for the estimation problem.

1. Introduction

In the case of the extended Kalman filter, an approximated linear system of the first order approximation by Taylor expansion is used instead of using original nonlinear system. In this paper, the new way to convert the nonlinear systems into approximated linear systems is described. For the one dimensional nonlinear equation, aproximate it using cubic spline, take the square and cube of the original nonlinear equation, approximate themusing cubic splines too, treat the first, second and third order terms as new variables, and the approximated three dimensional linear equation is obtained. For the two dimensional nonlinear system, approximated fifteen dimensional linear equation is derived similarly.

The nonlinear observer and filter algorithms are obtained by adapting conventional linear algorithms to these approximated linear systems.

2. Formal linearization method via cubic splines for one dimensional system

Consider the following nonlinear system:

$$X_{k+1} = f(X_k) \tag{1}$$
$$(X_k \in R,\ X_k = X(t_k))$$

where X_k is a state variable and $f(\cdot)$ is any nonlinear function.

Let $\mathfrak{D}$ be the domain of state variable X_k. Set the interval $[\xi_0, \xi_n)$ such that

$$X_k \in [\xi_0, \xi_n) \subset \mathfrak{D} \tag{2}$$

and divide this interval into n subintervals, so that there will be n-1 points:

$$\xi_1, \xi_2, \ldots, \xi_{n-1} \quad (\xi_0 < \xi_1 < \xi_2 < \ldots < \xi_{n-1} < \xi_n). \tag{3}$$

Next approximate equation(1) by using cubic spline on the interval $[\xi_0, \xi_n)$.

Thus we have the form of the approximated cubic polynomial:

$$X_{k+1} = a_{i0} + a_{i1}X_k + a_{i2}X_k^2 + a_{i3}X_k^3 \quad X_k \in [\xi_i, \xi_{i+1}) \tag{4}$$

where a_{i0}, a_{i1}, a_{i2} and a_{i3} are coefficients of the cubic spline.

Then square both sides of equation(1)

$$X_{k+1}^2 = f^2(X_k) \equiv g(X_k) \tag{5}$$

and again approximate $g(X_k)$ by using cubic spline.

Setting interval for the domain of X_k,

$$X_k \in [\zeta_0, \zeta_p) \subset \mathfrak{D} \tag{6}$$

and dividing it into p subintervals, there will be p-1 points:

$$\zeta_1, \zeta_2, \ldots, \zeta_{p-1} \quad (\zeta_0 < \zeta_1 < \zeta_2 < \ldots < \zeta_{p-1} < \zeta_p). \tag{7}$$

And then approximate $g(X_k)$ by using cubic spline

$$X_{k+1}^2 = b_{j0} + b_{j1}X_k + b_{j2}X_k^2 + b_{j3}X_k^3 \quad X_k \in [\zeta_j, \zeta_{j+1}). \tag{8}$$

Furthermore take the cube of equation(1):

$$X_{k+1}^3 = f^3(X_k) \equiv h(X_k) \tag{9}$$

then approximate $h(X_k)$ using cubic spline.

Similarly setting the interval

$$X_k \in [\mu_0, \mu_q) \subset \mathfrak{D} \tag{10}$$

and dividing this interval into q subintervals, there will be q-1 points:

$$\mu_1, \mu_2, \ldots, \mu_{q-1} \quad (\mu_0 < \mu_1 < \mu_2 < \ldots < \mu_{q-1} < \mu_q) \tag{11}$$

then equation(9) results to

$$X_{k+1}^3 = c_{m0} + c_{m1}X_k + c_{m2}X_k^2 + c_{m3}X_k^3 \quad X_k \in [\mu_m, \mu_{m+1}) \tag{12}$$

Let X_k, X_k^2, X_k^3 be $\phi_1(X_k), \phi_2(X_k), \phi_3(X_k)$ respectively, and introduce $\Phi_k(X_k)$ as

$$\Phi_k(X_k) = [\phi_1(X_k) \ \ \phi_2(X_k) \ \ \phi_3(X_k)]^T. \tag{13}$$

Combining equations (4),(3) and (12), we have

$$\Phi_{k+1} = A(X_k)\Phi_k + B(X_k) \tag{14}$$

where

$$A(X_k) = \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} \\ b_{j1} & b_{j2} & b_{j3} \\ c_{m1} & c_{m2} & c_{m3} \end{bmatrix}$$

$$B(X_k) = [a_{i0} \ \ b_{j0} \ \ c_{m0}]^T.$$

Equation(14) is the approximated linear form of equation(1).

3. Formal linearization method via bicubic splines for two dimensional system

Consider the following nonlinear system:

$$X_{k+1} = f(X_k, Y_k) \quad (15)$$

$$Y_{k+1} = g(X_k, Y_k) \quad (16)$$

$$(X_k, Y_k \in R,\ X_k = X(t_k), Y_k = Y(t_k))$$

where X_k and Y_k are state variables, $f(\cdot)$ and $g(\cdot)$ are nonlinear functions.

Let $\mathfrak{D}$ be the domain of state variables X_k and Y_k in the XY-plane. Consider the appropreate region:

$$[\xi_0, \xi_m) \times [\zeta_0, \zeta_n) \subset \mathfrak{D}. \quad (17)$$

Divide this region into $m \times n$ meshes with mesh points:

$$(\xi_i, \zeta_j) \quad i=0,1,\ldots,m,\ j=0,1,\ldots,n. \quad (18)$$

Approximate equation(15) by using bicubic spline function[4].

For the point:

$(X_k, Y_k) \in [\xi_i, \xi_{i+1}) \times [\zeta_j, \zeta_{j+1})$,

the approximated equation(15) is represented as

$$X_{k+1} = \sum_{p=0}^{3} \sum_{q=0}^{3} \alpha_{pq}^{ij} X_k^p Y_k^q \quad (19)$$

where α_{pq}^{ij} are coefficients of bicubic spline.

Introducing $\phi_{ki}(X_k, Y_k)$ as

$\phi_{k0} = 1$ $\quad \phi_{k1} = Y_k$ $\quad \phi_{k2} = Y_k^2$

$\phi_{k3} = Y_k^3$ $\quad \phi_{k4} = X_k$ $\quad \phi_{k5} = X_k Y_k$

$\phi_{k6} = X_k Y_k^2$ $\quad \phi_{k7} = X_k Y_k^3$ $\quad \phi_{k8} = X_k^2$

$\phi_{k9} = X_k^2 Y_k$ $\quad \phi_{k10} = X_k^2 Y_k^2$ $\quad \phi_{k11} = X_k^2 Y_k^3$

$\phi_{k12} = X_k^3$ $\quad \phi_{k13} = X_k^3 Y_k$ $\quad \phi_{k14} = X_k^3 Y_k^2$

$\phi_{k15} = X_k^3 Y_k^3$,

equation(19) can be rewritten as

$$\phi_{k+1\ 4} = \sum_{p=0}^{3} \sum_{q=0}^{3} \alpha_{pq}^{ij} \phi_{k\ 4p+q}. \quad (20)$$

Not only equation(15), also approximate the followng equations by using bicubic splines.

$$\phi_{k+1\ 4p+q} = f(X_k, Y_k)^p \cdot g(X_k, Y_k)^q \quad (21)$$

$$(P=0,1,2,3,\ q=0,1,2,3)$$

Introduce $\Phi_k(X_k, Y_k)$ as

$$\Phi_k(X_k, Y_k) = \begin{bmatrix} \phi_{k1}(X_k, Y_k) \\ \phi_{k2}(X_k, Y_k) \\ \cdots\cdots \\ \cdots\cdots \\ \phi_{k15}(X_k, Y_k) \end{bmatrix} \quad (22)$$

Using (22), (21) can be represented as

$$\Phi_{k+1} = A(X_k, Y_k)\Phi_k + B(X_k, Y_k) \quad (23)$$

where the elements of A and B are coefficients of bicubic splines. Equation(23) is the approximated fifteen dimensional linear form of equations(15) and (16).

4. Nonlinear filter for one dimensional system

The preceding result can be used in deriving a nonlinear filter algorithm.

Consider the following one dimensional system and observation equations

$$X_{k+1} = F(X_k) \quad (24)$$

$$Y_k = G(X_k) + V_k \quad (25)$$

where X_k is a state variable, Y_k is an observation value and V_k is Gaussian random noise such that

$$E V_k = m_k,\ E(V_k - m_k)^2 = w_k \quad (26)$$

and both $F(\cdot)$, $G(\cdot)$ are nonlinear functions.

Linearize equation(24) using the method described in section 2. Similarly approximate equation(25) using cubic spline.

Since $\mathfrak{D}$ is the domain of X_k, set the interval

$$X_k \in [\nu_0, \nu_r) \subset \mathfrak{D} \quad (27)$$

and divide this interval into r subintervals such that there will be r-1 points

$$\nu_1, \nu_2, \ldots, \nu_{r-1} \quad (28)$$
$$(\nu_0 < \nu_1 < \nu_2 < \ldots < \nu_{r-1} < \nu_r).$$

Using the spline function on $G(X_k)$ gives the following cubic polynomial

$$G(X_k) = d_{n0} + d_{n1}X_k + d_{n2}X_k^2 + d_{n3}X_k^3 \quad (29)$$
$$X_k \in [\nu_n, \nu_{n+1}).$$

Consequently, equations(24) and (25) are transformed into the following linear forms.

$$\Phi_{k+1} = A(X_k)\Phi_k + B(X_k) \quad (30)$$
$$Y_k = C(X_k)\Phi_k + D(X_k) + V_k \quad (31)$$

where

$$A(X_k) = \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} \\ b_{j1} & b_{j2} & b_{j3} \\ c_{m1} & c_{m2} & c_{m3} \end{bmatrix}$$

$$B(X_k) = [a_{i0} \quad b_{j0} \quad c_{m0}]^T$$
$$C(X_k) = [d_{n1} \quad d_{n2} \quad d_{n3}]$$
$$D(X_k) = d_{n0}$$

Equations (30) and (31) are linear with respect to Φ_k to which conventional linear filter theory can be applied.

<Nonlinear filter algorithm>

$$\tilde{\Phi}_k = A\hat{\Phi}_{k-1} + B$$
$$M_k = AP_{k-1}A^T$$
$$P_k = M_k - M_kC^T(W_k + CM_kC^T)^{-1}CM_k$$
$$\hat{\Phi}_k = \tilde{\Phi}_k + P_kC^TW_k^{-1}(Y_k - C\tilde{\Phi}_k - D - m_k)$$
$$\hat{X}_k = [1 \ 0 \ 0]\hat{\Phi}_k$$

given initial values $\hat{\Phi}_0$, P_0

5. Nonlinear observer for two dimensional system

The preceding result in section 3 can be used in deriving a nonlinear observer algorithm.

Consider the following two dimensional system and observation equations

$$X_{k+1} = F(X_k, Y_k) \quad (32)$$
$$Y_{k+1} = G(X_k, Y_k) \quad (33)$$

$$Z_k = H(X_k, Y_k) \quad (34)$$

$$(X_k, Y_k, Z_k \in R,$$
$$X_k = X(t_k), Y_k = Y(t_k), Z_k = Z(t_k))$$

where X_k, Y_k are state variables, Z_k is an obser-vation value, and $F(\cdot), G(\cdot), H(\cdot)$ are non-linear functions.

Linearize equations(32) and (33) by using the method described in section 3. Similarly approximate equation(34) by using bicubic spline

$$H(X_k, Y_k) = \sum_{p=0}^{3}\sum_{q=0}^{3} \alpha_{pq}^{ij}\phi_{k\ 4p+q}. \quad (35)$$

Equations(32),(33) and (34) are transformed into the following linear forms

$$\Phi_{k+1} = A(X_k, Y_k)\Phi_k + B(X_k, Y_k) \quad (36)$$
$$Z_k = C(X_k, Y_k)\Phi_k + D(X_k, Y_k). \quad (37)$$

Equations (36) and (37) are linear with respect to Φ_k to which conventional linear observer theory can be applied.

<Nonlinear observer algorithm>

$$\hat{\Phi}_k = A(\hat{X}_{k-1}, \hat{Y}_{k-1})\hat{\Phi}_{k-1} + B(\hat{X}_{k-1}, \hat{Y}_{k-1})$$
$$+ K[Z_k - C(\hat{X}_{k-1}, \hat{Y}_{k-1})\{A(\hat{X}_{k-1}, \hat{Y}_{k-1})\hat{\Phi}_{k-1} + B(\hat{X}_{k-1}, \hat{Y}_{k-1})\}$$
$$- D(\hat{X}_{k-1}, \hat{Y}_{k-1})]$$

$$\hat{X}_k = [0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ ]\hat{\Phi}_k$$

$$\hat{Y}_k = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ ]\hat{\Phi}_k$$

where $\hat{\Phi}_0$ is given and K is an appropriate constant matrix.

6. Numerical experiments

The validity of this proposed method may be tested using numerical experiments.

6.1 Formal linearization

Consider the following nonlinear difference equation

$$X_{k+1} = \sin X_k. \tag{38}$$

Let the domain of X_k be $[0, 2\pi)$, the number of subintervals n=p=q=4, and the length of each subinterval be equal. Figure 1 shows the graph of sin X using the spline function approximation. For comparison, the first order approximation by Taylor expansion is likewise shown.

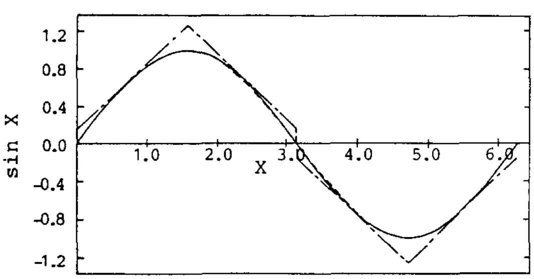

Fig.1 Approximation of sin X
———, True value
------, By cubic spline
—-—, By Taylor expansion

Figure 2 shows the trajectories of X_k with initial value $X_0=1.2$, domain $X_k \in [0,2)$, n=p=q=4 and with equal subinterval lengths.

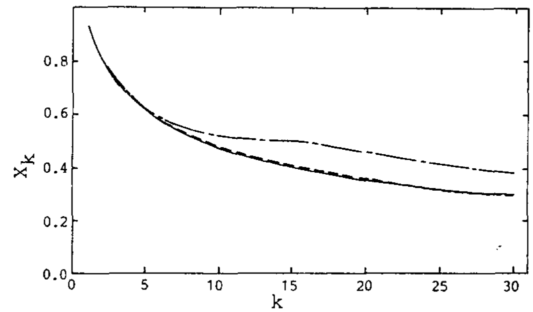

Fig.2 Trajectories of X_k
———, True value
------, By cubic spline
—-—, By Taylor expansion

6.2 Nonlinear filter

In order to test the validity of the proposed nonlinear filter algorithm, consider the following system.

$$X_{k+1} = \log X_k^2 + 2$$

$$y_k = X_k + V_k, \quad X_1 = 1.5$$

where the observation noise V_k is a Gaussian noise such that $N(V_k; 0.3, 0.01)$. The setting interval is $X_k \in [1, 5) \subset \mathcal{D}$, n=5, p=10, q=15 with equal subinterval lengths. The initial values are:

$$\hat{\Phi}_0 = [1.0\quad 1.0\quad 1.0]^T$$

$$P_0 = \begin{bmatrix} 0.100 & 0.000 & 0.000 \\ 0.000 & 0.010 & 0.000 \\ 0.000 & 0.000 & 0.001 \end{bmatrix}$$

Figure 3 shows the result of the state estimation.

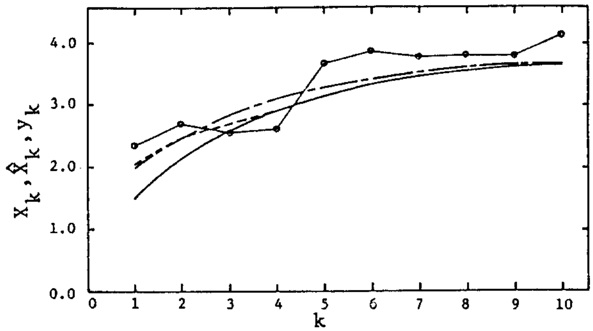

Fig.3 Estimation of X_k
———, True value
-------, By proposed filter
— - —, By extended Kalman filter
—•—, Observation

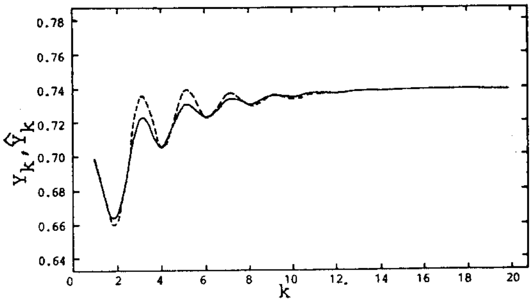

Fig.5 Estimation of Y_k
———, True value
-------, By proposed observer

6.3. Nonlinear observer

In order to test the validity of the proposed nonlinear observer algorithm, consider the following system.

$$X_{k+1} = \sin(X_k Y_k) \qquad X_0=0.3$$
$$y_{k+1} = \cos(X_k+Y_k) \qquad Y_0=0.5$$
$$Z_k = 4X_k+Y_k$$

The setting interval is $X_k, Y_k \in [0, 1)$, and the mesh size is 0.2. The initial values are $\hat{X}_0=0.5$ and $\hat{Y}_0=0.3$. Figures 4 and 5 show the result of the state estimation.

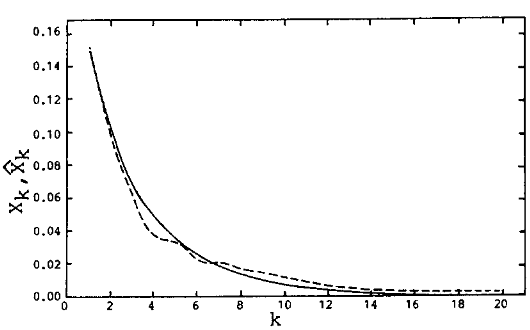

Fig.4 Estimation of X_k
———, True value
-------, By proposed observer

7. Conclusion

The approximation method by cubic spline was applied to the nonlinear system exemplified in the one or two dimensional difference equation. Based on this method, a new nonlinear filter and observer algorithms were derived. Finally the linearization method and the nonlinear filter and observer algorithms were validated through numerical experiments.

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