'91 KACC 1991. 10. 22~24

GENERALIZATION OF A COMPLEX-SYSTEM BY EQUIVALENT TRANSFORM IN THE DISCRETE SENSE

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ABSTRACT

The states, inputs, outputs and parameters of a complex-system are all complex values. The introduction of such complex systems makes it more suitable to treat not only the robust control but also the pole assignment in the separate regions. The relation called "equivalence in the discrete sense" is introduced to define a complex-system corresponding to a real-system with real-axis poles as well as complex conjugate poles. The relation between the feedback-control laws of the equivalent systems in the discrete sense are derived so that their closed-loop systems should hold the equivalence in the discrete sense.

1. INTRODUCTION

The real-system has been considered as the mathematical model of a controlled plant. The system having complex parameters is sometimes more easily handled such as in robust analysis[1]. They established not only Hurwitz stability but also Shur stability of a set of disk polynomials with complex coefficients. This paper is concerned with deriving a complex-system from a real-system and analyzing and designing control systems based on the complex-system. In the paper[3] we proposed a complex-system which corresponds to a real-system under the assumption that the real-system has n pairs of distinct complex conjugate poles and no real-axis poles. In this paper we consider a generalization of the complex-system to deal with more large class real-system.

Consider two systems shown in Fig.1.1(a) and Fig.1.1(b). Figure 1.2(a) and Fig.1.2(b) show discretized versions of the continuous-time systems and Fig.1.3 shows the unit step responses of the continuous-time systems in Fig.1.1(a) and Fig.1.1(b). The pulse transfer function of the discretized system in Fig.1.2(a) (when the sampling period $T = 1$) is $G(z) = (1-e^{-1})/(z-e^{-1})$. And this is identical with the pulse transfer function of the discretized system in Fig.1.2(b). It is known that the introduction of sampling brings about uncontrollable or unobservable states and the sampled response of a system becomes the same as that of a lower-order system when the sampling period T assumes certain values[2]. Taking advantage of the property, we introduce a relation called "equivalence in the discrete sense" and define a complex-system corresponding to a real-system with real-axis poles as well as complex conjugate poles.

This paper is organized in the following way. In the next section the relation called "equivalence in the discrete" sense is defined. In Section 3 the relation between the feedback-control laws of the equivalent systems so that the closed-loop systems should become equivalent in the discrete sense is derived. Using the relation equivalence in the discrete sense, we introduce a complex system corresponding to a real-system which has real-axis poles as well as complex conjugate poles in Section 4, while conclusions are given in Section 5.

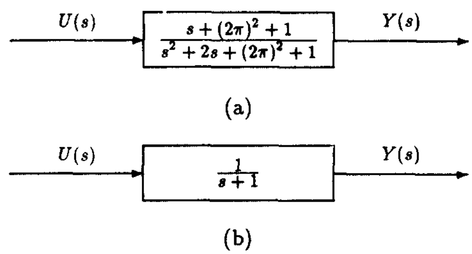

Fig.1.1 Continuous-time systems

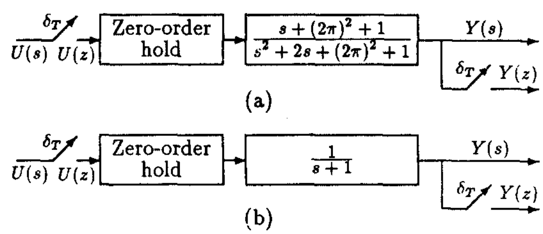

Fig.1.2 Discretized version of the systems

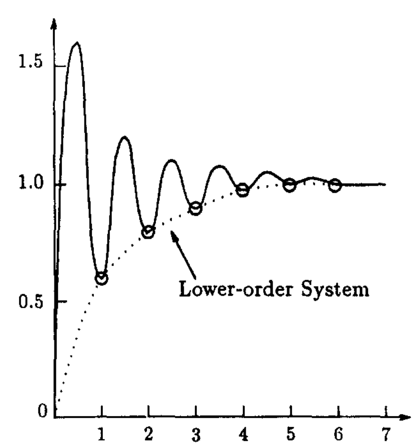

Fig.1.3 Unit step response

2. EQUIVALENCE IN THE DISCRETE SENSE

Consider a linear time-invariant controllable system with single input and single output described in the Jordan canonical form:

$$\dot{x} = Ax + bu \quad (2.1)$$
$$y = cx \quad (2.2)$$

where $x \in \mathbf{R}^{n\times 1}, u \in \mathbf{R}, y \in \mathbf{R}$ and

$$A = \mathrm{diag}(\alpha_1, \ldots, \alpha_p, \lambda_{p+1}, \bar{\lambda}_{p+1}, \ldots, \lambda_{p+q}, \bar{\lambda}_{p+q}) \in \mathbf{C}^{n\times n},$$
$$b = \begin{bmatrix} t_1 & \ldots & t_p & t_{p+1} & \bar{t}_{p+1} & \ldots & t_{p+q} & \bar{t}_{p+q}\end{bmatrix}^T \in \mathbf{C}^{n\times 1},$$
$$c = \begin{bmatrix} w_1 & \ldots & w_p & w_{p+1} & \bar{w}_{p+1} & \ldots & w_{p+q} & \bar{w}_{p+q}\end{bmatrix} \in \mathbf{C}^{1\times n};$$
$$w_i t_i = \begin{cases} \eta_i \in \mathbf{R} & i = 1, \ldots, p; \\ (\bar{w}_i \bar{t}_i)^* = \tau_i & i = p+1, \ldots, p+q \end{cases}$$

under the assumption that the system has distinct p real-axis poles $\{\lambda_i \mid \lambda_i = \alpha_i \in \mathbf{R},\ i = 1, \ldots, p\}$ and q pairs of distinct complex conjugate poles $\{\lambda_j \mid \lambda_j = \alpha_j + \beta_j j,\ \beta_j > 0,\ j = p+1, \ldots, p+q\} \cup \{\bar{\lambda}_j \mid \bar{\lambda}_j = \lambda_j^*,\ j = p+1, \ldots, p+q\}$, where $(p + 2q = n)$ and * designates complex conjugate.

When the input u is sampled and fed to a zero-order hold before being applied to the system, the discrete-time system at sampling period T_1 is

$$x_{k+1} = \Phi x_k + \Gamma u_k \quad (2.3)$$
$$y_k = c x_k \quad (2.4)$$

where

$$\Phi = \mathrm{diag}(\Phi_1, \Phi_2, \ldots, \Phi_{p+q}) \in \mathbf{C}^{n\times n},$$
$$\Gamma = \begin{bmatrix} \Gamma_1^T & \Gamma_2^T & \ldots & \Gamma_{p+q}^T \end{bmatrix}^T \in \mathbf{C}^{n\times 1},$$
$$\Phi_i = \begin{cases} \begin{bmatrix} e^{\alpha_i T_1} \end{bmatrix} & i = 1, \ldots, p \\ \begin{bmatrix} e^{\lambda_i T_1} & \\ & e^{\bar{\lambda}_i T_1} \end{bmatrix} & i = p+1, \ldots, p+q \end{cases},$$
$$\Gamma_i = \begin{cases} \begin{bmatrix} \frac{t_i(e^{\alpha_i T_1} - 1)}{\alpha_i} \end{bmatrix} & i = 1, \ldots, p \\ \begin{bmatrix} \frac{t_i(e^{\lambda_i T_1} - 1)}{\lambda_i} \\ \frac{\bar{t}_i(e^{\bar{\lambda}_i T_1} - 1)}{\bar{\lambda}_i} \end{bmatrix} & i = p+1, \ldots, p+q \end{cases}$$

And the pulse transfer function of the discrete-time system becomes

$$G(z) = \sum_{i=1}^{p} \frac{w_i t_i (e^{\alpha_i T_1} - 1)}{\alpha_i (z - e^{\alpha_i T_1})} + \sum_{i=p+1}^{p+q} \left\{ \frac{w_i t_i (e^{\lambda_i T_1} - 1)}{\lambda_i (z - e^{\lambda_i T_1})} + \frac{\bar{w}_i \bar{t}_i (e^{\bar{\lambda}_i T_1} - 1)}{\bar{\lambda}_i (z - e^{\bar{\lambda}_i T_1})} \right\} \quad (2.5)$$

As stated in section 1, the introduction of sampling brings about uncontrollable or unobservable states and the sampled response of a system becomes the same as that of a lower-order system when the sampling period T assumes certain values[2]. We now define a relation called equivalence in the discrete sense in the following definition.

Definition 1: *If the sampled output response of the given continuous-time systems for the same inputs are same, i.e., the pulse transfer functions of the given continuous-time systems are identical, these systems are called to be equivalent in the discrete sense and such a relation is called equivalence in the discrete sense.*

Consider an $n+p$-dimensional continuous-time real-system with single input and single output

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{b}u \quad (2.6)$$
$$y = \tilde{c}\tilde{x} \quad (2.7)$$

where

$$\tilde{A} = \mathrm{diag}\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_{p+q}\right) \in \mathbf{C}^{2(p+q)\times 2(p+q)}$$
$$\tilde{A}_i = \begin{bmatrix} \tilde{\lambda}_i & \\ & \bar{\tilde{\lambda}}_i \end{bmatrix} = \begin{cases} \begin{bmatrix} \alpha_i + 2k_{1i}\pi j & \\ & \alpha_i - 2k_{1i}\pi j \end{bmatrix} & i = 1, \ldots, p \\ \begin{bmatrix} \lambda_i & \\ & \bar{\lambda}_i \end{bmatrix} & i = p+1, \ldots, p+q \end{cases},$$
$$\tilde{b} = \begin{bmatrix} b_1 & \bar{b}_1 & \ldots & b_p & \bar{b}_p & b_{p+1} & \bar{b}_{p+1} & \ldots & b_{p+q} & \bar{b}_{p+q} \end{bmatrix}^T \in \mathbf{C}^{2(p+q)\times 1},$$
$$\tilde{c} = \begin{bmatrix} c_1 & \bar{c}_1 & \ldots & c_p & \bar{c}_p & c_{p+1} & \bar{c}_{p+1} & \ldots & c_{p+q} & \bar{c}_{p+q} \end{bmatrix} \in \mathbf{C}^{1\times 2(p+q)},$$

$$k_{1i} = kk_{10}, \quad k_{10} > \frac{1}{2\pi} \max_{j=p+1,p+q} |\mathrm{Im}(\lambda_j)|, \ i = 1, \ldots, p \quad (2.8)$$
$$b_i = t_i, \quad c_i = w_i, \quad i = p+1, \ldots, p+q \quad (2.9)$$

, and k is a positive integer. The following theorem gives the condition under which the $n+p$-dimensional system (2.6)–(2.7) becomes equivalent in the discrete sense to the n-dimensional system (2.1)–(2.2).

Theorem 1: *The system (2.6)–(2.7) is equivalent to the system (2.1)–(2.2) in the discrete sense if and only if*

$$\overrightarrow{c_i b_i} = \vec{h}_i + k\vec{h}_i^{\perp}, \quad k \in \mathbf{R} \quad (2.10)$$

where

$$\overrightarrow{c_i b_i} = (\mathrm{Re}(c_i b_i)\ ,\ \mathrm{Im}(c_i b_i)), \quad \vec{\tilde{\lambda}}_i = (\alpha_i\ ,\ 2k_{1i}\pi), \quad i = 1, \ldots, p \quad (2.11)$$

, $\vec{h}_i$ is the orthogonal projection vector of $\overrightarrow{c_i b_i}$ on the vector $\vec{\tilde{\lambda}}_i$, and $\vec{h}_i^{\perp}$ is the unit vector which is orthogonal to $\vec{h}_i$. Figure 2.1 shows the relation between $\overrightarrow{c_i b_i}$ and $\vec{\tilde{\lambda}}_i$.

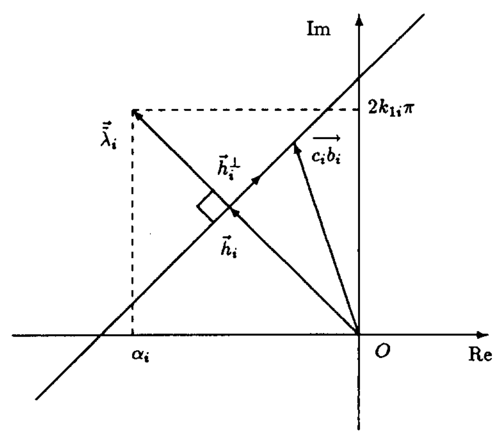

Fig.2.1 Relation between $\overrightarrow{c_i b_i}$ and $\vec{\tilde{\lambda}}_i$

Proof: The transfer function of the system (2.6)–(2.7) is given by

$$\tilde{G}(s) = \sum_{i=1}^{p} \left(\frac{c_i b_i}{s - \alpha_1 - 2k_{1i}\pi j} + \frac{\bar{c}_i \bar{b}_i}{s - \alpha_1 + 2k_{1i}\pi j} \right)$$

$$+\sum_{i=p+1}^{p+q}\left(\frac{c_ib_i}{s-\lambda_i}+\frac{\bar{c}_i\bar{b}_i}{s-\bar{\lambda}_i}\right)$$

$$=\sum_{i=1}^{p}\frac{(c_ib_i+\bar{c}_i\bar{b}_i)(s-\alpha_i)+(c_ib_i-\bar{c}_i\bar{b}_i)2k_{1i}\pi j}{(s-\alpha_i)^2+4k_{1i}^2\pi^2}$$

$$+\sum_{i=p+1}^{p+q}\left(\frac{w_it_i}{s-\lambda_i}+\frac{\bar{w}_i\bar{t}_i}{s-\bar{\lambda}_i}\right) \quad (2.12)$$

Since the system (2.6)–(2.7) is a real-system, $c_ib_i-\bar{c}_i\bar{b}_i$ must be purely imaginary, i.e., $\bar{c}_i\bar{b}_i=(c_ib_i)^*$. When the input u is sampled and fed to a zero-order hold before being applied to the system, the discrete-time system at sampling period $T_1=\frac{1}{k_{10}}$ is

$$\tilde{x}_{k+1} = \tilde{\Phi}\tilde{x}_k+\tilde{\Gamma}u_k \quad (2.13)$$
$$y_k = \tilde{c}\tilde{x}_k \quad (2.14)$$

where

$$\tilde{\Phi}=\mathrm{diag}(\tilde{\Phi}_1,\tilde{\Phi}_2,\dots,\tilde{\Phi}_{p+q})\in \mathbf{C}^{2(p+q)\times 2(p+q)},$$

$$\tilde{\Gamma}=\begin{bmatrix}\tilde{\Gamma}_1^T & \tilde{\Gamma}_2^T & \cdots & \tilde{\Gamma}_{p+q}^T\end{bmatrix}^T\in \mathbf{C}^{2(p+q)\times 1},$$

$$\tilde{\Phi}_i=\begin{bmatrix}e^{\lambda_iT_1} & \\ & e^{\bar{\lambda}_iT_1}\end{bmatrix}=\begin{cases}\begin{bmatrix}e^{\alpha_iT_1} & \\ & e^{\alpha_iT_1}\end{bmatrix} & i=1,\dots,p\\ \begin{bmatrix}e^{\lambda_iT_1} & \\ & e^{\bar{\lambda}_iT_1}\end{bmatrix} & i=p+1,\dots,p+q\end{cases},$$

$$\tilde{\Gamma}_i=\begin{cases}\begin{bmatrix}\frac{b_i(e^{\alpha_iT_1}-1)}{\alpha_i+2k_{1i}\pi j}\\ \frac{\bar{b}_i(e^{\alpha_iT_1}-1)}{\alpha_i-2k_{1i}\pi j}\end{bmatrix} & i=1,\dots,p\\ \begin{bmatrix}\frac{b_i(e^{\lambda_iT_1}-1)}{\lambda_i}\\ \frac{\bar{b}_i(e^{\bar{\lambda}_iT_1}-1)}{\bar{\lambda}_i}\end{bmatrix} & i=p+1,\dots,p+q\end{cases}$$

And the pulse transfer function of the discrete-time system becomes

$$\tilde{G}(z)=\sum_{i=1}^{p}\frac{\{\alpha_i(c_ib_i+\bar{c}_i\bar{b}_i)-2k_{1i}\pi j(c_ib_i-\bar{c}_i\bar{b}_i)\}(e^{\alpha_iT_1}-1)}{(\alpha_i^2+4k_{1i}^2\pi^2)(z-e^{\alpha_iT_1})}$$

$$+\sum_{i=p+1}^{p+q}\left\{\frac{w_it_i(e^{\lambda_iT_1}-1)}{\lambda_i(z-e^{\lambda_iT_1})}+\frac{\bar{w}_i\bar{t}_i(e^{\bar{\lambda}_iT_1}-1)}{\bar{\lambda}_i(z-e^{\bar{\lambda}_iT_1})}\right\} \quad (2.15)$$

Substituting the relation $\bar{c}_i\bar{b}_i=(c_ib_i)^*$ to the above equation yields

$$\tilde{G}(z)=\sum_{i=1}^{p}\frac{2\{\alpha_i\mathrm{Re}(c_ib_i)+2k_{1i}\pi\mathrm{Im}(c_ib_i)\}(e^{\alpha_iT_1}-1)}{(\alpha_i^2+4k_{1i}^2\pi^2)(z-e^{\alpha_iT_1})}$$

$$+\sum_{i=p+1}^{p+q}\left\{\frac{w_it_i(e^{\lambda_iT_1}-1)}{\lambda_i(z-e^{\lambda_iT_1})}+\frac{\bar{w}_i\bar{t}_i(e^{\bar{\lambda}_iT_1}-1)}{\bar{\lambda}_i(z-e^{\bar{\lambda}_iT_1})}\right\} \quad (2.16)$$

Comparing $G(z)$ and $\tilde{G}(z)$, we obtain the following condition under which two continuous-time real-systems are equivalent in the discrete sense.

$$\frac{w_it_i}{2\alpha_i}=\frac{\{\alpha_i\mathrm{Re}(c_ib_i)+2k_{1i}\pi\mathrm{Im}(c_ib_i)\}}{(\alpha_i^2+4k_{1i}^2\pi^2)},\quad i=1,\dots,p \quad (2.17)$$

Using the vector notations (2.11), the condition is rewritten as follows:

$$\frac{\overrightarrow{c_ib_i}\cdot\vec{\bar{\lambda}}_i}{|\vec{\bar{\lambda}}_i|^2}=\frac{w_it_i}{2\alpha_i}=\text{const.},\quad i=1,\dots,p \quad (2.18)$$

where $\cdot$ designates inner product. Let $\vec{h}_i$ be the orthogonal projection vector of $\overrightarrow{c_ib_i}$ on the vector $\vec{\bar{\lambda}}_i$:

$$\vec{h}_i=\frac{\overrightarrow{c_ib_i}\cdot\vec{\bar{\lambda}}_i}{|\vec{\bar{\lambda}}_i|^2}\vec{\bar{\lambda}}_i \quad (2.19)$$

and $\vec{h}_i^{\perp}$ be the unit vector which is orthogonal to $\vec{h}_i$, then the condition (2.10) is obtained from (2.18) and (2.19). In particular, when $k=0$, $|\overrightarrow{c_ib_i}|$ is minimum and

$$\begin{cases}\mathrm{Re}(c_ib_i)=\frac{w_it_i}{2\alpha_i}\alpha_i=\frac{w_it_i}{2}\\ \mathrm{Im}(c_ib_i)=\frac{w_it_i}{2\alpha_i}2k_{1i}\pi=\frac{k_{1i}\pi w_it_i}{\alpha_i}\end{cases},\quad i=1,\dots,p \quad (2.20)$$

□

3. STATE FEEDBACK

Let's apply feedback-control laws consisting of a state feedback and a feedforward, namely

$$u=fx+v,\quad f=[f_1\ f_2\ \dots\ f_{p+1}] \quad (3.1)$$

and

$$u=\tilde{f}\tilde{x}+v,\quad \tilde{f}=[\tilde{f}_1\ \bar{\tilde{f}}_1\ \dots\ \tilde{f}_{n+p}\ \bar{\tilde{f}}_{n+p}] \quad (3.2)$$

to the n-dimensional system (2.1)–(2.2) and the $n+p$-dimensional system (2.6)–(2.7), respectively. Then the closed-loop systems become

$$\dot{x} = A_cx+bv,\quad A_c=A+bf \quad (3.3)$$
$$y = cx \quad (3.4)$$

and

$$\dot{\tilde{x}} = \tilde{A}_c\tilde{x}+\tilde{b}v,\quad \tilde{A}_c=\tilde{A}+\tilde{b}\tilde{f} \quad (3.5)$$
$$y = \tilde{c}\tilde{x} \quad (3.6)$$

Under the assumption that the number of the real-axis poles of the system (2.1)–(2.2) doesn't change through state feedback we will derive the condition that the pulse transfer function of the closed-loop system (3.5)–(3.6) is identical with that of the closed-loop system (3.3)–(3.4). In other words, we will derive the relation between the feedback-control laws of the equivalent systems in the discrete sense so that their closed-loop systems hold the equivalence in the discrete sense.

3.1 State Space Approach

When the input u is sampled and fed to a zero-order hold before being applied to the system (3.3)–(3.4) and (3.5)–(3.6), the discrete-time closed-loop systems at sampling period T_2 are

$$x_{k+1} = e^{A_cT_2}x_k+\int_0^{T_2}e^{A_c\tau}b\,d\tau v_k \quad (3.7)$$
$$y_k = cx_k \quad (3.8)$$

and

$$\tilde{x}_{k+1} = e^{\tilde{A}_cT_2}\tilde{x}_k+\int_0^{T_2}e^{\tilde{A}_c\tau}\tilde{b}\,d\tau v_k \quad (3.9)$$
$$y_k = \tilde{c}\tilde{x}_k \quad (3.10)$$

Let the poles of the closed-loop system (3.3)–(3.4) be

$$\{\mu_i\mid\mu_i=\gamma_i+\theta_ij,\theta_i>0,\ i=p+1,\dots,p+q\}\cup$$
$$\{\bar{\mu}_i\mid\bar{\mu}_i=\mu_i^*,\ i=p+1,\dots,p+q\}\cup\{\mu_i\mid\mu_i=\gamma_i\in\mathbf{R},i=1,\dots,p\}$$

We consider a transformation of the state-vector of the system (3.7)–(3.8) by use of a non-singular matrix V, namely

$$x_k = V\bar{x}_k$$

where V consists of eigenvectors $\{v_i \mid i = 1, ..., n\}$ of the matrix A_c. The new state-vector $\bar{x}_k$ obtained through the above transformation satisfies

$$\begin{aligned} \bar{x}_{k+1} &= \bar{\Phi}_c \bar{x}_k + \bar{\Gamma} v_k && (3.11) \\ y_k &= \bar{c}\bar{x}_k && (3.12) \end{aligned}$$

where

$$\begin{aligned} \bar{\Phi}_c &= \text{diag}\left(\bar{\Phi}_{c1}, \bar{\Phi}_{c2}, ..., \bar{\Phi}_{c(p+q)}\right), \\ \bar{\Gamma} &= \text{diag}\left(\bar{\Gamma}_{c1}, \bar{\Gamma}_{c2}, ..., \bar{\Gamma}_{c(p+q)}\right)\bar{b}, \\ \bar{b} &= V^{-1}b = \left[\begin{array}{cccc} \hat{t}_1 & \hat{t}_2 & \ldots & \hat{t}_{p+q} \end{array}\right]^T, \\ \bar{c} &= cV = \left[\begin{array}{cccc} \hat{w}_1 & \hat{w}_1 & \ldots & \hat{w}_{p+q} \end{array}\right], \\ \bar{\Phi}_{ci} &= \begin{cases} e^{\gamma_i T_2} & , i = 1, ..., p \\ \begin{bmatrix} e^{\mu_i T_2} & \\ & e^{\bar{\mu}_i T_2} \end{bmatrix} & , i = p+1, ..., p+q \end{cases}, \\ \bar{\Gamma}_{ci} &= \begin{cases} \frac{e^{\gamma_i T_2} - 1}{\gamma_i} & , i = 1, ..., p \\ \begin{bmatrix} \frac{e^{\mu_i T_2} - 1}{\mu_i} & \\ & \frac{e^{\bar{\mu}_i T_2} - 1}{\bar{\mu}_i} \end{bmatrix} & , i = p+1, ..., p+q \end{cases} \end{aligned}$$

Since the state transformation does not change pulse transfer functions of discrete-time systems , the pulse transfer function of the system (3.7)–(3.8) becomes

$$\begin{aligned} G_c(z) &= \bar{c}\left[zI_n - \text{diag}\left(\bar{\Phi}_{c1}, ..., \bar{\Phi}_{c(p+q)}\right)\right]^{-1}\text{diag}\left(\bar{\Gamma}_{c1}, ..., \bar{\Gamma}_{c(p+q)}\right)\bar{b} \\ &= \sum_{i=1}^{p} \frac{\hat{w}_i \hat{t}_i (e^{\gamma_i T_2} - 1)}{\gamma_i (z - e^{\gamma_i T_2})} + \sum_{i=p+1}^{p+q} \left\{ \frac{\hat{w}_i \hat{t}_i (e^{\mu_i T_2} - 1)}{\mu_i (z - e^{\mu_i T_2})} + \frac{\bar{\hat{w}}_i \bar{\hat{t}}_i (e^{\bar{\mu}_i T_2} - 1)}{\bar{\mu}_i (z - e^{\bar{\mu}_i T_2})} \right\} \end{aligned}$$

When the pulse transfer function of the system (3.9)–(3.10) is identical with the pulse transfer function $G_c(z)$, the system (3.9)–(3.10) must have p pairs of uncontrollable or unobservable poles. To establish this condition, p pairs of poles of the $n+p$-dimensional continuous-time system (3.5)–(3.6) must have imaginary part $\pm 2k_{2i}\pi$ $(i = 1, ..., p)$, where

$$k_{2i} = kk_{20}, \quad k_{20} > \frac{1}{2\pi} \max_{j=p+1, ..., p+q} |\text{Im}(\tilde{\mu}_j)|, \; i = 1, ..., p \tag{3.13}$$

and k is a positive integer. Let the poles of the closed-loop continuous-time system are

$$\left\{\tilde{\mu}_i \mid \tilde{\mu}_i = \tilde{\gamma}_i + \tilde{\theta}_i j, \tilde{\theta}_i \geq 0, \; i = 1, ..., p+q; \; \tilde{\theta}_i = 2k_{2i}\pi, i = 1, ..., p\right\}$$

$$\cup \{\bar{\tilde{\mu}}_i \mid \bar{\tilde{\mu}}_i = \tilde{\mu}_i^*, \; i = 1, ..., p+q\} \tag{3.14}$$

For $T_2 = \frac{1}{k_{20}}$, we consider a transformation of the state-vector of the closed-loop system (3.9)–(3.10) by use of a non-singular matrix $\tilde{V}$, namely

$$\tilde{x}_k = \tilde{V}\bar{\tilde{x}}_k$$

where $\tilde{V}$ consists of eigenvectors $\{\tilde{v}_i \mid i = 1, ..., n\}$ of the matrix $\tilde{A}_c$. The new state-vector $\bar{\tilde{x}}_k$ obtained through the above transformation satisfies

$$\begin{aligned} \bar{\tilde{x}}_{k+1} &= \bar{\tilde{\Phi}}_c \bar{\tilde{x}}_k + \bar{\tilde{\Gamma}} v_k && (3.15) \\ y_k &= \bar{\tilde{c}}\bar{\tilde{x}}_k && (3.16) \end{aligned}$$

where

$$\begin{aligned} \bar{\tilde{\Phi}}_c &= \text{diag}\left(\bar{\tilde{\Phi}}_{c1}, \bar{\tilde{\Phi}}_{c2}, ..., \bar{\tilde{\Phi}}_{c(p+q)}\right), \\ \bar{\tilde{\Gamma}} &= \text{diag}\left(\bar{\tilde{\Gamma}}_{c1}, \bar{\tilde{\Gamma}}_{c2}, ..., \bar{\tilde{\Gamma}}_{c(p+q)}\right)\bar{\tilde{b}}, \\ \bar{\tilde{b}} &= \tilde{V}^{-1}\tilde{b} = \left[\begin{array}{ccccc} \hat{b}_1 & \bar{\hat{b}}_1 & \ldots & \hat{b}_{p+q} & \bar{\hat{b}}_{p+q} \end{array}\right]^T, \\ \bar{\tilde{c}} &= \tilde{c}\tilde{V} = \left[\begin{array}{ccccc} \hat{c}_1 & \bar{\hat{c}}_1 & \ldots & \hat{c}_{p+q} & \bar{\hat{c}}_{p+q} \end{array}\right], \\ \bar{\tilde{\Phi}}_{ci} &= \begin{cases} \begin{bmatrix} e^{\tilde{\gamma}_i T_2} & \\ & e^{\tilde{\gamma}_i T_2} \end{bmatrix} & , i = 1, ..., p \\ \begin{bmatrix} e^{\tilde{\mu}_i T_2} & \\ & e^{\bar{\tilde{\mu}}_i T_2} \end{bmatrix} & , i = p+1, ..., p+q \end{cases}, \\ \bar{\tilde{\Gamma}}_{ci} &= \begin{cases} \begin{bmatrix} \frac{e^{\tilde{\gamma}_i T_2} - 1}{\tilde{\gamma}_i + 2k_{2i}\pi j} & \\ & \frac{e^{\tilde{\gamma}_i T_2} - 1}{\tilde{\gamma}_i - 2k_{2i}\pi j} \end{bmatrix} & , i = 1, ..., p \\ \begin{bmatrix} \frac{e^{\tilde{\mu}_i T_2} - 1}{\tilde{\mu}_i} & \\ & \frac{e^{\bar{\tilde{\mu}}_i T_2} - 1}{\bar{\tilde{\mu}}_i} \end{bmatrix} & , i = p+1, ..., p+q \end{cases} \end{aligned}$$

Since the state transformation does not change pulse transfer functions, the pulse transfer function of the system (3.9)–(3.10) becomes

$$\begin{aligned} \tilde{G}_c(z) &= \bar{\tilde{c}}\left[zI_{n+p} - \text{diag}\left(\bar{\tilde{\Phi}}_{c1}, ..., \bar{\tilde{\Phi}}_{c(p+q)}\right)\right]^{-1}\text{diag}\left(\bar{\tilde{\Gamma}}_{c1}, ..., \bar{\tilde{\Gamma}}_{c(p+q)}\right)\bar{\tilde{b}} \\ &= \sum_{i=1}^{p} \frac{\{\tilde{\gamma}_i(\hat{c}_i\hat{b}_i + \bar{\hat{c}}_i\bar{\hat{b}}_i) - 2k_{2i}\pi j(\hat{c}_i\hat{b}_i - \bar{\hat{c}}_i\bar{\hat{b}}_i)\}(e^{\tilde{\gamma}_i T_2} - 1)}{(\tilde{\gamma}_i^2 + 4k_{2i}^2\pi^2)(z - e^{\tilde{\gamma}_i T_2})} \\ &\quad + \sum_{i=p+1}^{p+q} \left\{ \frac{\hat{c}_i\hat{b}_i(e^{\tilde{\mu}_i T_2} - 1)}{\tilde{\mu}_i(z - e^{\tilde{\mu}_i T_2})} + \frac{\bar{\hat{c}}_i\bar{\hat{b}}_i(e^{\bar{\tilde{\mu}}_i T_2} - 1)}{\bar{\tilde{\mu}}_i(z - e^{\bar{\tilde{\mu}}_i T_2})} \right\} \\ &= \sum_{i=1}^{p} \frac{2\{\tilde{\gamma}_i\text{Re}(\hat{c}_i\hat{b}_i) + 2k_{2i}\pi\text{Im}(\hat{c}_i\hat{b}_i)\}(e^{\tilde{\gamma}_i T_2} - 1)}{(\tilde{\gamma}_i^2 + 4k_{2i}^2\pi^2)(z - e^{\tilde{\gamma}_i T_2})} \\ &\quad + \sum_{i=p+1}^{p+q} \left\{ \frac{\hat{c}_i\hat{b}_i(e^{\tilde{\mu}_i T_2} - 1)}{\tilde{\mu}_i(z - e^{\tilde{\mu}_i T_2})} + \frac{\bar{\hat{c}}_i\bar{\hat{b}}_i(e^{\bar{\tilde{\mu}}_i T_2} - 1)}{\bar{\tilde{\mu}}_i(z - e^{\bar{\tilde{\mu}}_i T_2})} \right\} \end{aligned}$$

Comparing $G_c(z)$ and $\tilde{G}_c(z)$ leads to the following theorem.

Theorem 2: *The closed-loop system (3.5)–(3.6) is equivalent to the system (3.3)–(3.4) in the discrete sense if*

$$\begin{aligned} \tilde{\gamma}_i &= \gamma_i, \quad i = 1, ..., p \\ \tilde{\mu}_i &= \mu_i, \quad i = p+1, ..., p+q \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &\frac{2\{\gamma_i\text{Re}(\hat{c}_i\hat{b}_i) + 2k_{2i}\pi\text{Im}(\hat{c}_i\hat{b}_i)\}}{(\gamma_i^2 + 4k_{2i}^2\pi^2)} = \frac{\hat{w}_i\hat{t}_i}{\gamma_i}, \quad i = 1, ..., p \\ &\begin{cases} \hat{w}_i\hat{t}_i = \hat{c}_i\hat{b}_i \\ \bar{\hat{w}}_i\bar{\hat{t}}_i = \bar{\hat{c}}_i\bar{\hat{b}}_i \end{cases}, \qquad i = p+1, ..., p+q \end{aligned} \tag{3.18}$$

3.2 Transfer Function Approach

The pulse transfer function $G(z)$ can be obtained from the transfer function $G(s)$ using z-transform, through

$$G(z) = (1 - z^{-1})\mathfrak{Z}\{G(s)/s\} \tag{3.19}$$

where $\mathfrak{Z}\{\}$ designates z-transform[4]. Comparing $\mathfrak{Z}\{G_c(s)/s\}$ and $\mathfrak{Z}\{\tilde{G}_c(s)/s\}$, we derive the condition that two closed-loop continuous-time systems are equivalent in the discrete sense. For simplicity, we will deal with a scalar system. The transfer functions associated with the closed-loop systems (3.3)–(3.4) and (3.5)–(3.6) are given by

$$G_c(s) = \frac{w_1 t_1}{s - \gamma_1}$$

$$\tilde{G}_c(s) = \frac{2\mathrm{Re}(c_1 b_1)s - \frac{w_1 t_1}{\alpha_1}(\alpha_1^2 + 4k_1^2\pi^2)}{(s - \alpha_1 - 2k_1\pi j - b_1\bar{f}_1)(s - \alpha_1 + 2k_1\pi j - \bar{b}_1\bar{\bar{f}}_1) - b_1\bar{b}_1 \bar{f}_1\bar{\bar{f}}_1}$$

At sampling period T_2, the z-transform of $G_c(s)/s$ is given by

$$\mathfrak{Z}\left\{\frac{G_c(s)}{s}\right\} = \mathfrak{Z}\left\{\frac{1}{s}\frac{w_1 t_1}{s - \gamma_1}\right\} = \frac{w_1 t_1}{\gamma_1}\frac{z(e^{\gamma_1 T_2} - 1)}{(z-1)(z - e^{\gamma_1 T_2})} \tag{3.20}$$

Since $\tilde{G}_c(s)$ is partially expanded as follows:

$$\tilde{G}_c(s) = \frac{\tilde{\eta}_1}{s - \tilde{\mu}_1} + \frac{\bar{\tilde{\eta}}_1}{s - \bar{\tilde{\mu}}_1}$$

where

$$\begin{aligned} \tilde{\eta}_1 + \bar{\tilde{\eta}}_1 &= 2\mathrm{Re}(c_1 b_1) \\ \tilde{\eta}_1\bar{\tilde{\mu}}_1 + \bar{\tilde{\eta}}_1\tilde{\mu}_1 &= \frac{w_1 t_1}{\alpha_1}(\alpha_1^2 + 4k_1^2\pi^2) \end{aligned}$$

The z-transform of $\tilde{G}_c(s)/s$ is given by

$$\begin{aligned} \mathfrak{Z}\left\{\frac{\tilde{G}_c(s)}{s}\right\} &= \mathfrak{Z}\left\{\frac{1}{s}\frac{\tilde{\eta}}{s - \tilde{\mu}_1} + \frac{1}{s}\frac{\bar{\tilde{\eta}}_1}{s - \bar{\tilde{\mu}}_1}\right\} \\ &= \frac{\tilde{\eta}_1}{\tilde{\mu}_1}\frac{z(e^{\tilde{\mu}_1 T_2} - 1)}{(z-1)(z - e^{\tilde{\mu}_1 T_2})} + \frac{\bar{\tilde{\eta}}_1}{\bar{\tilde{\mu}}_1}\frac{z(e^{\bar{\tilde{\mu}}_1 T_2} - 1)}{(z-1)(z - e^{\bar{\tilde{\mu}}_1 T_2})} \end{aligned}$$

And $\mathfrak{Z}\{G_c(s)/s\}$ and $\mathfrak{Z}\{\tilde{G}_c(s)/s\}$ have the same poles, if and only if

$$e^{\tilde{\mu}_1 T_2} = e^{\bar{\tilde{\mu}}_1 T_2} = e^{\gamma_1 T_2} \tag{3.21}$$

, namely

$$\begin{aligned} \mathrm{Re}(\tilde{\mu}_1) &= \mathrm{Re}(\bar{\tilde{\mu}}_1) = \gamma_1 && (3.22) \\ \mathrm{Im}(\tilde{\mu}_1) &= -\mathrm{Im}(\bar{\tilde{\mu}}_1) = 2k_2\pi, \quad k_2 = 1/T_2 && (3.23) \end{aligned}$$

From this condition, $\mathfrak{Z}\{\tilde{G}_c(s)/s\}$ becomes

$$\mathfrak{Z}\left\{\frac{\tilde{G}_c(s)}{s}\right\} = \frac{\tilde{\eta}_1\bar{\tilde{\mu}}_1 + \bar{\tilde{\eta}}_1\tilde{\mu}_1}{\tilde{\mu}_1\bar{\tilde{\mu}}_1}\frac{z(e^{\gamma_1 T_2} - 1)}{(z-1)(z - e^{\gamma_1 T_2})}$$

Therefore, two pulse transfer functions $G_c(z)$ and $\tilde{G}_c(z)$ are identical if the following condition is satisfied.

$$\begin{aligned} \frac{\tilde{\eta}_1\bar{\tilde{\mu}}_1 + \bar{\tilde{\eta}}_1\tilde{\mu}_1}{\tilde{\mu}_1\bar{\tilde{\mu}}_1} &= \frac{w_1 t_1}{\alpha_1}(\alpha_1^2 + 4k_1^2\pi^2)\frac{1}{(\gamma_1 + 2k_2\pi j)(\gamma_1 - 2k_2\pi j)} \\ &= \frac{w_1 t_1(\alpha_1^2 + 4k_1^2\pi^2)}{\alpha_1(\gamma_1^2 + 4k_2^2\pi^2)} = \frac{w_1 t_1}{\gamma_1} && (3.24) \end{aligned}$$

Rearranging the last equation leads to the following theorem.

Theorem 3: *For a scalar system, the system (3.5)–(3.6) is equivalent to the system (3.3)–(3.4) in the discrete sense if and only if*

$$\tilde{\gamma}_1 = \gamma_1, \quad \frac{\alpha_1}{\alpha_1^2 + 4k_1^2\pi^2} = \frac{\gamma_1}{\gamma_1^2 + 4k_2^2\pi^2} \tag{3.25}$$

that is,

$$\tilde{\gamma}_1 = \gamma_1, \quad \frac{k_1^2}{\left(\dfrac{4\alpha_1\pi^2}{\gamma_1 - \alpha_1}\right)} - \frac{k_2^2}{\left(\dfrac{4\gamma_1\pi^2}{\gamma_1 - \alpha_1}\right)} = 1 \tag{3.26}$$

Remark 1: *From equation (3.25), α_1 and γ_1 have the same sign, i.e., unstable systems can not be stabilized. This is due to the fact that the discrete-time system (2.13)–(2.14) is uncontrollable. From equation (3.26), for*

$$2\pi\sqrt{\frac{\alpha_1}{\gamma_1 - \alpha_1}} < k_1 < \infty$$

k_2 moves on a hyperbola and takes values between 0 and ∞ as shown in Fig.3.1.

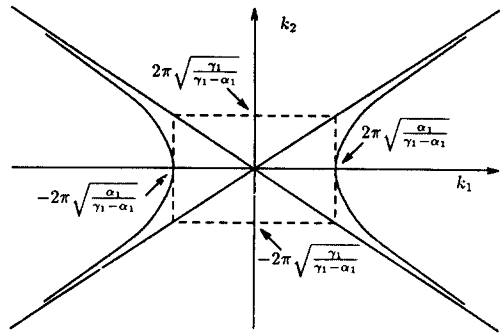

Fig.3.1 Relation between k_1 and k_2

3.3 Example 1

Consider a continuous-time scalar system

$$\begin{aligned} \frac{d}{dt}x &= -2x + 3u \\ y &= 4x \end{aligned}$$

where $\alpha_1 = -2, t_1 = 3$. The specified closed-loop pole is $\mu_1 = \gamma_1 = -4$. Determine the feedback-control law of the equivalent system so that its closed-loop system is also equivalent in the discrete sense to the closed-loop system of the given system. Let $b_1 = 1$ and $k_1 = 1$, then the equivalent system in the discrete sense is described by

$$\begin{aligned} \frac{d}{dt}\tilde{x} &= \begin{bmatrix} -2 + 2\pi j & 0 \\ 0 & -2 - 2\pi j \end{bmatrix}\tilde{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u \\ y &= \begin{bmatrix} 6 - 18.8495j & 6 + 18.8495j \end{bmatrix}\tilde{x} \end{aligned}$$

Two discrete-time systems for $T_1 = 1/k_1 = 1$ become

$$\begin{aligned} x_{k+1} &= 0.1353x_k + 1.2969u_k \\ y_k &= 4x_k \\ \tilde{x}_{k+1} &= \begin{bmatrix} 0.1353 & 0 \\ 0 & 0.1353 \end{bmatrix}\tilde{x}_k + \begin{bmatrix} 0.0397 + 0.1249j \\ 0.0397 - 0.1249j \end{bmatrix}u_k \\ y_k &= \begin{bmatrix} 6 - 18.8495j & 6 + 18.8495j \end{bmatrix}\tilde{x}_k \end{aligned}$$

and the pulse transfer functions are

$$\begin{aligned} G(z) &= \frac{5.1879}{z - 0.1353} \\ \tilde{G}(z) &= \frac{2.5939}{z - 0.1353} + \frac{2.5939}{z - 0.1353} = \frac{5.1879}{z - 0.1353} \end{aligned}$$

The feedback-control law to assign the pole of the given system at -4 is given by

$$u = -0.6667x + v$$

and the pulse transfer function of the discrete-time closed-loop system is

$$G_c(z) = \frac{2.8481}{z - 0.0506}$$

According to theorem 3,

$$\begin{aligned} \tilde{\gamma}_1 &= \gamma_1 = -4 \\ k_{21} &= 1.3406 \end{aligned}$$

The poles of the equivalent closed-loop system must be assigned at $\{-\tilde{\gamma}_1 \pm 2k_{21}\pi j\} = \{-4 \pm 8.4235j\}$. Such a feedback-control law is given by

$$u = \begin{bmatrix} -2 + 2.8232j & -2 - 2.8232j \end{bmatrix} \tilde{x} + v$$

and the closed-loop system is

$$\begin{aligned} \frac{d}{dt}\tilde{x} &= \begin{bmatrix} -4 + 9.1064j & -2 - 2.8232j \\ -2 + 2.8232j & -4 - 9.1064j \end{bmatrix} \tilde{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v \\ y &= \begin{bmatrix} 6 - 18.8495j & 6 + 18.8495j \end{bmatrix} \tilde{x} \end{aligned}$$

The discrete-time closed-loop system becomes

$$\begin{aligned} \tilde{x}_{k+1} &= \begin{bmatrix} 0.0506 & 0 \\ 0 & 0.0506 \end{bmatrix} \tilde{x}_k + \begin{bmatrix} 0.0218 + 0.0685j \\ 0.0218 - 0.0685j \end{bmatrix} v_k \\ y_k &= \begin{bmatrix} 6 - 18.8495j & 6 + 18.8495j \end{bmatrix} \tilde{x}_k \end{aligned}$$

and we find the pulse transfer function

$$\tilde{G}_c(z) = \frac{1.4240}{z - 0.0506} + \frac{1.4240}{z - 0.0506} = \frac{2.8481}{z - 0.0506}$$

is identical with $G_c(z)$.

4. COMPLEX-SYSTEM

4.1 Complex-System

Since the equivalent system in the discrete sense has distinct ($n' = p + q$) pairs of complex conjugate poles and no real-axis poles, its corresponding complex-system can be obtained[3]. The equivalent system is transformed into a suitable form to define a complex-system using the following lemma.

Lemma 1: *By a state transformation $\tilde{x} = U\bar{x}$, the $2n'$-dimensional real system (2.6)–(2.7) is transformed to the system*

$$\begin{aligned} \frac{d}{dt}\bar{x} &= \bar{A}\bar{x} + \bar{b}u \qquad (4.1) \\ y &= \bar{c}\bar{x} \qquad (4.2) \end{aligned}$$

where

$$U = \frac{1-j}{2} \operatorname{diag}\left(\begin{bmatrix} 1 & j \\ j & 1 \end{bmatrix}, \begin{bmatrix} 1 & j \\ j & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & j \\ j & 1 \end{bmatrix} \right) \in \mathbf{C}^{2n' \times 2n'},$$

$$\bar{A} = \operatorname{diag}\left(\begin{bmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_{n'} & -\beta_{n'} \\ \beta_{n'} & \alpha_{n'} \end{bmatrix} \right) \in \mathbf{C}^{2n' \times 2n'},$$

$$\bar{b} = \left[\begin{bmatrix} b_{1r} & b_{1i} \end{bmatrix}, \ldots, \begin{bmatrix} b_{n'r} & b_{n'i} \end{bmatrix} \right]^T \in \mathbf{C}^{2n' \times 1},$$

$$\begin{aligned} \bar{c} &= \frac{1}{2} \left[\begin{bmatrix} c_{1r} & c_{1i} \end{bmatrix}, \ldots, \begin{bmatrix} c_{n'r} & c_{n'i} \end{bmatrix} \right] \in \mathbf{C}^{1 \times 2n'} \\ &= 2 \left[\begin{bmatrix} \mathrm{Re}(c_1 b_1) \\ \mathrm{Im}(c_1 b_1) \end{bmatrix}^T, \ldots, \begin{bmatrix} \mathrm{Re}(c_{n'} b_{n'}) \\ \mathrm{Im}(c_{n'} b_{n'}) \end{bmatrix}^T \right] \begin{bmatrix} \bar{B}_1 & & \\ & \ddots & \\ & & \bar{B}_{n'} \end{bmatrix}, \qquad (4.3) \end{aligned}$$

$$\begin{aligned} \bar{B}_i &= \frac{1}{b_{ir}^2 + b_{ii}^2} \begin{bmatrix} b_{ir} & b_{ii} \\ b_{ii} & -b_{ir} \end{bmatrix}, i = 1, \ldots, n'; \\ \beta_i &= 2k_{1i}\pi, \quad i = 1, \ldots, p \end{aligned}$$

Proof: $\bar{A}$, $\bar{b}$, and $\bar{c}$ are calculated as follows:

$$\begin{aligned} \bar{A} &= U^{-1}AU = \operatorname{diag}\left(\begin{bmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_{n'} & -\beta_{n'} \\ \beta_{n'} & \alpha_{n'} \end{bmatrix} \right), \\ \bar{b} &= U^{-1}b = \frac{1}{1-j} \left[\begin{bmatrix} b_1 - \bar{b}_1 j \\ -jb_1 + \bar{b}_1 \end{bmatrix}^T \cdots \begin{bmatrix} b_{n'} - \bar{b}_{n'} j \\ -jb_{n'} + \bar{b}_{n'} \end{bmatrix}^T \right]^T, \\ \bar{c} &= cU = \frac{1-j}{2} \left[\begin{bmatrix} c_1 + \bar{c}_1 j \\ jc_1 + \bar{c}_1 \end{bmatrix}^T \cdots \begin{bmatrix} c_{n'} + \bar{c}_{n'} j \\ jc_{n'} + \bar{c}_{n'} \end{bmatrix}^T \right] \end{aligned}$$

In order to verify that $\bar{b}$ and $\bar{c}$ satisfy the relation (4.3) we calculate $2[\mathrm{Re}(c_i b_i)\ \mathrm{Im}(c_i b_i)]\bar{B}_i$. Letting $c_i b_i = \mathrm{Re}(c_i b_i) + \mathrm{Im}(c_i b_i)j$, it follows that

$$\begin{aligned} & 2 \begin{bmatrix} \mathrm{Re}(c_i b_i) & \mathrm{Im}(c_i b_i) \end{bmatrix} \bar{B}_i \\ &= \frac{2}{b_{ir}^2 + b_{ii}^2} \begin{bmatrix} \mathrm{Re}(c_i b_i) & \mathrm{Im}(c_i b_i) \end{bmatrix} \begin{bmatrix} b_{ir} & b_{ii} \\ b_{ii} & -b_{ir} \end{bmatrix} \\ &= \frac{1-j}{2} \begin{bmatrix} c_i + \bar{c}_i j & jc_i + \bar{c}_i \end{bmatrix} \end{aligned}$$

This completes the proof of the Lemma. □

Using the above lemma, we now define a complex-system corresponding to the real-system (2.1)–(2.2) in the following definition.

Definition 2: *The complex system corresponding to the real-system (2.1)–(2.2) is*

$$\frac{d}{dt}\begin{bmatrix} x_{c1} \\ \vdots \\ x_{cn} \end{bmatrix} = \begin{bmatrix} \alpha_1 + j\beta_1 & & \\ & \ddots & \\ & & \alpha_n + j\beta_n \end{bmatrix} \begin{bmatrix} x_{c1} \\ \vdots \\ x_{cn} \end{bmatrix} + \begin{bmatrix} b_{1r} + jb_{1i} \\ \vdots \\ b_{nr} + jb_{ni} \end{bmatrix} (u_r + ju_i)$$

$$\begin{aligned} (y_r + jy_i) &= \begin{bmatrix} \frac{c_{1r} - jc_{1i}}{2} & \cdots & \frac{c_{nr} - jc_{ni}}{2} \end{bmatrix} \begin{bmatrix} x_{c1} \\ \vdots \\ x_{cn} \end{bmatrix} \\ x_{ci} &= x_{ir} + jx_{ii} \end{aligned}$$

5. CONCLUSIONS

We introduced the relation called equivalence in the discrete sense and defined a complex-system corresponding to a real-system with real-axis poles as well as complex conjugate poles which is a generalization of a complex-system considered in [3]. The introduction of such complex system makes it more suitable to treat not only the robust control but also pole assignment in the separate regions. It is possible to treat root-clustering problem in three separate regions: in two symmetric regions with respect to the real-axis and on the real-axis segment.

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