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Self-Tuning Control with Bounded Input Constraints

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Abstract

This paper considers the design and analysis of one-step ahead optimal and adaptive controllers, under the restriction that a known constraint on the input amplitude is imposed. It is assumed that the discrete-time single-input, single-output system to be controlled is linear, except for inequality constraints on the input. The objective function to be minimized is an one-step quadratic function, where polynomial weights on the input and output are included. Both the known parameter and unknown parameter (indirect adaptive controller) cases are examined.

1 Introduction

In practice, the inputs to a system are often constrained. The outputs of a controller that are meant to drive the system cannot be faithfully applied, due to physical limitations of the actuators.

An input-constrained minimum variance control law was developed by Goodwin [1]. Unfortunately, this control law was not derived in direct implementable form. Makila [2] developed an input-constrained self-tuning regulator for MIMO system but they used simple saturation for implementation.

Some performance properties of adaptive controllers which use simple saturation to impose input constraints are known, for the deterministic case [3],[4],[5].

In this paper we consider self-tuning control problem for the case when the system is disturbed by noise and the inputs to the system is constrained. Input-constrained self-tuning controller is developed and convergence conditions for the adaptive controller with input constraints are derived.

2 One-Step Optimal Control with Input Constraints

Consider a linear discrete-time SISO system which is described by:

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + z(t) \qquad (1)$$

where q^{-1} is the backward shift operator. That is, $q^{-1}y(t) \triangleq y(t-1)$. Here d is the system dead time, $y(t)$ is the output, $u(t)$ is the input, and $z(t)$ is a general disturbance. Often special assumptions on $z(t)$ are made, such as $z(t) = C(q^{-1})\omega(t)$, where $\omega(t)$ is an uncorrelated zero mean random process. Here $A(q^{-1})$, $B(q^{-1})$ and $C(q^{-1})$ are polynomials of order of n, m and l respectively such that:

$$A(q^{-1}) = 1 + a_1q^{-1} + \ldots + a_nq^{-n} \qquad (2)$$

$$B(q^{-1}) = b_0 + b_1q^{-1} + \ldots + b_mq^{-m} \qquad (3)$$

$$C(q^{-1}) = 1 + c_1q^{-1} + \ldots + c_lq^{-l} \qquad (4)$$

where $b_0 \neq 0$. The system input is subject to the constraint:

$$M_L \leq u(t) \leq M_U \qquad (5)$$

where M_L is a finite input lower bound and M_U is a finite input upper bound.

We first consider the input-constrained one-step ahead control problem with known system parameters. A nonlinear optimal control law is derived which minimizes an one-step quadratic objective function, subject to explicit constraints on the magnitude of the control input.

We consider the following one-step quadratic objective function:

$$\begin{aligned} I &= \mathsf{E}\Big\{[P(q^{-1})y(t+d) - S(q^{-1})y^*(t+d)]^2 \\ &+ [R(q^{-1})u(t)]^2|\mathcal{F}_t\Big\} \end{aligned} \qquad (6)$$

where $\{y^*(t)\}$ is a specified reference trajectory and $P(q^{-1})$, $S(q^{-1})$, $R(q^{-1})$ are polynomials in q^{-1} of the form:

$$P(q^{-1}) = p_0 + p_1q^{-1} + \ldots + p_mq^{-m}, \quad p_0 \neq 0 \qquad (7)$$

$$S(q^{-1}) = s_0 + s_1q^{-1} + \ldots + s_mq^{-m} \qquad (8)$$

$$R(q^{-1}) = r_0 + r_1q^{-1} + \ldots + r_mq^{-m} \qquad (9)$$

Note that this is the Generalized Minimum Variance (GMV) control [6] type objective function. The minimization of (6), subject to (1), (5) is over the class of admissible functions, $\mathcal{C}$, which satisfy (1), (5) and the requirement that $u(t)$ is a function of observations $\{y(t), y(t-1), \ldots, u(t-1), u(t-2), \ldots\}$ only. That is, only causal output feedback control laws are allowed.

We now find the optimal I^* and $u(t)$ satisfying:

$$I^* = \min_{u(t)\in\mathcal{C}} I \qquad (10)$$

Consider the Diophantine identity:

$$T(q^{-1})P(q^{-1}) = E(q^{-1})A(q^{-1}) + q^{-d}F(q^{-1}) \qquad (11)$$

where $E(q^{-1})$ and $F(q^{-1})$ are uniquely determined from $T(q^{-1})$, $A(q^{-1})$, $P(q^{-1})$ and d. Here $T(q^{-1})$ is a stable polynomial which will be used for tailoring the disturbance effects. Using (11), the system equation (1) can be rewritten as:

$$TPy(t+d) = Fy(t) + Gu(t) + Ez(t+d) \qquad (12)$$

where $G(q^{-1}) \triangleq E(q^{-1})B(q^{-1})$. Dividing both side by T yields

$$Py(t+d) = g_0u(t) + Fy'(t) + G'u'(t-1) + \frac{E}{T}z(t+d) \quad (13)$$

where $y' \triangleq y/T$, $u' \triangleq u/T$, and $G(q^{-1}) \triangleq g_0T(q^{-1}) + q^{-1}G'(q^{-1})$ defines $G'(q^{-1})$. Substituting (13) into the objective function (6) gives

$$I = \mathsf{E}\Big\{[g_0u(t) + Fy'(t) + G'u'(t-1) - Sy^*(t+d) + \frac{E}{T}z(t+d)]^2 + [r_0u(t) + R'u(t-1)]^2|\mathcal{F}_t\Big\} \quad (14)$$

where $R(q^{-1}) \triangleq r_0 + q^{-1}R'(q^{-1})$ defines $R'(q^{-1})$.

Let us assume that $\frac{E}{T}z(t)$ is an zero mean uncorrelated sequence, independent of inputs $\{u(t), u(t-1), \ldots\}$ and outputs $\{y(t), y(t-1), \ldots\}$. This will happen when $z(t) = C(q^{-1})\omega(t)$. Then we can convert the functional minimization of (10) into an pointwise minimization with respect to $u(t)$:

$$J^* = \min_{M_L \le u(t) \le M_U} \Big\{[g_0u(t) + Fy'(t) + G'u'(t-1) - Sy^*(t+d)]^2 + [r_0u(t) + R'u(t-1)]^2\Big\} \quad (15)$$

Note that this is a deterministic optimization problem. The objective function is single variable quadratic function with respect to current control input $u(t)$. Let $v(t)$ be the unconstrained optimal solution, obtained by setting $\partial J^*/\partial u(t)$ equal to zero at each time t:

$$g_0v(t) + Fy'(t) + G'u'(t-1) - Sy^*(t+d) + \frac{r_0}{g_0}[r_0v(t) + R'u(t-1)] = 0 \quad (16)$$

Then the constrained optimal solution $u(t)$ satisfying (15) can be written as:

$$u(t) = sat[v(t), M_L, M_U] \quad (17)$$

Note that $\{u(t), u(t-1), \ldots\}$ are the past, actually applied, saturated control inputs. Multiplying (16) by $T(q^{-1})$, we get the following implementable controller form:

$$\left(g_0 + \frac{r_0^2}{g_0}\right)Tv(t) = TSy^*(t+d) - Fy(t) - \left(G' + \frac{r_0}{g_0}TR'\right)u(t-1) \quad (18)$$

$$u(t) = sat[v(t), M_L, M_U] \quad (19)$$

The one-step objective function I with assumption on the disturbance $\frac{E}{T}z(t)$ is minimized by the direct implementable nonlinear control law (18)-(19). This control law gives a closed-loop system satisfying

$$\left(BP + \frac{r_0}{g_0}RA\right)y(t+d) = BSy^*(t+d) - \left(g_0 + \frac{r_0^2}{g_0}\right)B\delta u(t) + \frac{G + \frac{r_0}{g_0}TR}{T}z(t+d) \quad (20)$$

$$\left(BP + \frac{r_0}{g_0}RA\right)v(t) = ASy^*(t+d) - \frac{F}{T}z(t) + \left(BP - (g_0 - \frac{r_0}{g_0}R'q^{-1})A\right)\delta(t) \quad (21)$$

where $\delta u(t) \triangleq v(t) - u(t)$.

The closed-loop response to the general disturbance $z(t)$ is tailored by the polynomial $T(q^{-1})$. Therefore $T(q^{-1})$ can be used as a design polynomial, to improve disturbance rejection of the closed-loop system response. This is discussed for unconstrained systems by Clarke [7]. For most practical unconstrained applications, $T(q^{-1})$ can be taken as a fixed first-order polynomial where $\frac{1}{T}$ is a low-pass filter (Clarke *et al.* [8]). We have extended this to the constrained input case. The closed-loop equations have different output regulation errors due to the input bound constraint, for each choice of T. The output regulation error is filtered through $\frac{1}{T}$. Thus we can use T to filter the output regulation error due to the input saturation. This is true even in the noise free case. The point here is that by choice of T we handle the effects of the disturbance when it interacts with the constraints, and alter the regulation error as desired. Therefore T can be used as design polynomials which are determined to improve the input-constrained controller performance. Of course, a bad choice of T could make things worse. A design method of $T(q^{-1})$ based on the stability property of the input-constrained control system, is given in [9]. For the colored noise case (*i.e.* $z(t) = C(q^{-1})\omega(t)$), $T = C$ yields the minimum output variance of the $\frac{E}{T}z(t)$ in the objective function.

3 Adaptive Control with Input Constraints

We next consider the control of input-constrained systems whose parameters are unknown. Specially, we investigate the convergence and stability attributes of input-constrained adaptive control for the stochastic SISO case with a MV (Minimum Variance) objective function. Without the input constraints, the asymptotic properties of this algorithm are known. In Goodwin *et al.* [10], global convergence is proved. Here we extend these results to the input-constrained case.

We assume that the disturbance term $z(t)$ is modeled as $C(q^{-1})\omega(t)$ and make the following assumptions about the system (1):

A.1 System delay d is known.

A.2 Upper bounds for n, m and l are known.

A.3 $C(q^{-1})$ is stable polynomial.

Note that we do not assume stability of $B(q^{-1})$ here as in Goodwin *et al.* [10], because the constraint on the input $u(t)$ makes this unnecessary. The following independence and variance assumptions are made on the process $\omega(t)$:

B.1 $\mathsf{E}\{\omega(t)|\mathcal{F}_{t-1}\} = 0$ a.s.

B.2 $\mathsf{E}\{\omega(t)^2|\mathcal{F}_{t-1}\} = \sigma^2$ a.s.

B.3 $\limsup_{N\to\infty} \sum_{t=1}^{N} \omega(t)^2 < \infty$ a.s.

The indirect adaptive control version of the one-step optimal control law (18)-(19) with MV objective function is given by:

$$\hat{C}(q^{-1},t)y^*(t+d) = \hat{F}(q^{-1},t)y(t) + \hat{g}_0(t)\hat{C}(q^{-1},t)v(t) + \hat{G}'(q^{-1},t)u(t-1) \quad (22)$$

$$u(t) = sat[v(t), M_L, M_U] \quad (23)$$

where

$$\hat{G}(q^{-1},t) = \hat{g}_0(t)\hat{C}(q^{-1},t) + q^{-1}\hat{G}'(q^{-1},t) \quad (24)$$

Here $\hat{C}(q^{-1},t)$, $\hat{F}(q^{-1},t)$, $\hat{G}(q^{-1},t)$ are estimates of $C(q^{-1})$, $F(q^{-1})$, $G(q^{-1})$ respectively. Next we define $\bar{y}(t)$ as:

$$\hat{C}(q^{-1},t)\bar{y}(t+d) \triangleq \hat{F}(q^{-1},t)y(t) + \hat{G}(q^{-1},t)u(t) \quad (25)$$

This can be written in the form:

$$\bar{y}(t+d) = \bar{\phi}(t)^T\hat{\theta}(t) \quad (26)$$

where

$$\bar{\phi}(t) = [y(t), \ldots, y(t-n+1), u(t), \ldots, u(t-m+1), -\bar{y}(t+d-1), \ldots, -\bar{y}(t+d-l)]^T$$

$$\hat{\theta}(t) = [\hat{f}_0(t), \ldots, \hat{f}_{n-1}(t), \hat{g}_0(t), \ldots, \hat{g}_{m-1}(t), \hat{c}_1(t), \ldots, \hat{c}_l(t)]^T \quad (27)$$

Note that $\bar{y}(t)$ is not needed by the controller (22). It is a quantity which corresponds to the prediction of output $y(t)$ based on the estimated parameters. To estimate the system parameters $\hat{\theta}(t)$, a stochastic gradient method [10] is used. Since $\bar{y}(t)$ is filtered through $\hat{C}(q^{-1},t)$, divergence problems can occur if $\hat{C}(q^{-1},t)$ is not kept within its necessary stability region. Because $\hat{C}(q^{-1},t)$ is an estimated quantity, it can become unstable as a result of noisy or bad data, or a lack of adequate excitation. To prevent this, an orthogonal projection of the parameter estimates onto a set such that the roots of the estimated $\hat{C}(q^{-1},t)$ polynomial lie strictly inside the unit circle will be done whenever $\hat{C}(q^{-1},t)$ becomes unstable.

We summarize the above as follows:

Input-Constrained Adaptive Control Algorithm

$$\hat{\theta}'(t) = \hat{\theta}(t-d) + \frac{\bar{a}}{r(t-d)}\bar{\phi}(t-d)[y(t) - \bar{\phi}(t-d)^T\hat{\theta}(t-d)] \quad (28)$$

$$\hat{\theta}(t) = Pj[\hat{\theta}'(t)] \quad (29)$$

$$r(t-d) = r(t-d-1) + \bar{\phi}(t-d)^T\bar{\phi}(t-d), \qquad r(0) = 1 \quad (30)$$

$$\hat{C}(q^{-1},t)y^*(t+d) = \hat{F}(q^{-1},t)y(t) + \hat{g}_0(t)\hat{C}(q^{-1},t)v(t) + \hat{G}'(q^{-1},t)u(t-1) \quad (31)$$

$$u(t) = sat[v(t), M_L, M_U] \quad (32)$$

$$\bar{y}(t+d) = \bar{\phi}(t)^T\hat{\theta}(t) \quad (33)$$

where $\bar{a} > 0$ and $Pj[\cdot]$ denotes an operator which will project $\hat{\theta}'(t)$ orthogonally onto the surface of a set such that $\hat{\theta}(t)$ remains within the stability region of $\hat{C}(t,q^{-1})$.

The approach used by Goodwin *et al.* [10] will be utilized here to prove convergence of the above adaptive control algorithm. First, we define the following variables:

$$\bar{e}(t) \triangleq y(t) - \bar{y}(t) = y(t) - \bar{\phi}(t-d)^T\hat{\theta}(t-d) \quad (34)$$

$$\bar{z}(t) \triangleq \bar{e}(t+d) - \xi(t+d) \quad (35)$$

Note that $\bar{e}(t)$ corresponding to prediction error and $\bar{z}(t)$ corresponds to deterministic part of the prediction error. The following lemma is key to proving convergence of the adaptive control algorithm.

Lemma 1 *Subject to the system assumptions set (A) and the noise assumptions set (B), and provided that*

$$C(q^{-1}) - \frac{\bar{a}}{2} \quad (36)$$

is strictly positive real, then the algorithm (28)-(33) ensures that with probability 1,

(i)

$$\limsup_{t\to\infty} \|\hat{\theta}(t) - \theta_0\|^2 < \infty \quad (37)$$

(ii)

$$\lim_{N\to\infty} \frac{1}{r(N)} \sum_{t=0}^{N} \bar{z}(t)^2 = 0 \qquad \text{a.s.} \quad (38)$$

Proof: See [9].

That is, as long as the constant $\bar{a}$ in the algorithm is chosen, relative to noise coloring polynomial $C(q^{-1})$, such that the stability requirement (36) is met (as $t \to \infty$), the norm of parameter estimation error will be bounded and the normalized deterministic part of prediction error will converge to zero. The main result is the following:

Theorem 1 *Let the assumptions sets (A) and (B) hold for the system (1)-(5). Further, assume that* $\left[C(q^{-1}) - \frac{\bar{a}}{2}\right]$ *is strictly positive real and that algorithm (28)-(33) is used. If the open-loop system is asymptotically stable, then with probability* 1*:*

(a) *The input-constrained adaptive control system (1), (28)-(33) is globally convergent in the following sense:*

(i)

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} y(t)^2 < \infty \quad (39)$$

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} u(t)^2 < \infty \quad (40)$$

(ii)

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathsf{E}\{(y(t+d) - \bar{y}(t+d))^2|\mathcal{F}_t\} = \gamma^2 \quad (41)$$

(b) *If there exists an integer $\tau > 0$ such that $M_L \le v(t) \le M_U$ for all $t \ge \tau$, then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathsf{E}\{(y(t+d) - y^*(t+d))^2|\mathcal{F}_t\} = \gamma^2 \quad (42)$$

where γ^2 is the minimum possible mean square control error achievable with any causal linear feedback control (this includes the controller using the true system parameters). That is, if asymptotically the constraint is inactive, then the controller converges to the unconstrained MV controller.

Proof: (i) First, using the fact that the system is asymptotically stable and using assumptions (A.3) and (B.3), it follows that there exists an N_1 such that

$$\frac{1}{N}\sum_{t=0}^{N} y(t+d)^2 \leq \frac{K_1}{N}\sum_{t=0}^{N} u(t)^2 + K_2, \quad N > N_1 \quad \text{a.s.} \tag{43}$$

Thus, since $M_L \leq |u(t)| \leq M_U < \infty$, it follows that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} u(t)^2 < \infty \tag{44}$$

and

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} y(t)^2 < \infty \tag{45}$$

(ii) From Lemma 1 we know that $\hat{F}(q^{-1},t), \hat{G}(q^{-1},t)$ and $\hat{C}(q^{-1},t)$ are bounded and $\hat{C}(q^{-1},t)$ is kept within the stability region by the orthogonal projection. Therefore from (25) it follows that there exists an N' such that

$$\frac{1}{N}\sum_{t=0}^{N} \bar{y}(t+d)^2 \leq \frac{K_1}{N}\sum_{t=0}^{N} y(t)^2 + \frac{K_2}{N}\sum_{t=0}^{N} u(t)^2 + \frac{K_3}{N}, \quad N > N' \quad \text{a.s.} \tag{46}$$

Hence using the definition of $r(N)$ and $\bar{\phi}(t)$, it follows that

$$\frac{r(N)}{N} \leq \frac{K_4}{N}\sum_{t=0}^{N} u(t)^2 + K_5 \quad \text{for} \quad N > N' \quad \text{a.s.} \tag{47}$$

So

$$\limsup_{N\to\infty} \frac{r(N)}{N} < \infty, \tag{48}$$

and hence

$$\liminf_{N\to\infty} \frac{N}{r(N)} > \frac{1}{K} > 0 \tag{49}$$

Then from Lemma 1 we have that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N} \bar{z}(t)^2 = 0 \quad \text{a.s.} \tag{50}$$

Now, from (34) and (35)

$$\bar{z}(t) = y(t+d) - \bar{y}(t+d) - \xi(t+d) \tag{51}$$

Hence

$$\mathsf{E}\{(y(t+d) - \bar{y}(t+d))^2 | \mathcal{F}_t\} = \bar{z}(t)^2 + \gamma^2 \quad \text{a.s.} \tag{52}$$

and from (50)

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N} \mathsf{E}\{(y(t+d) - \bar{y}(t+d))^2 | \mathcal{F}_t\} = \gamma^2 \quad \text{a.s.} \tag{53}$$

which completes the proof.

(b) The fact that $M_L \leq v(t) \leq M_U$ implies that

$$v(t) = u(t) = sat[v(t), M_L, M_U], \quad \text{for all} \quad t \geq T \tag{54}$$

which implies $\bar{y}(t+d) = y^*(t+d)$. Thus

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N} \mathsf{E}\{(y(t+d) - y^*(t+d))^2 | \mathcal{F}_t\} = \gamma^2 \quad \text{a.s.} \tag{55}$$

This completes the proof. ∎

From part (b) of Theorem 1 we can see that the input-constrained control law converges to the unconstrained control law if it remains inside of bound after certain amount of time τ.

Note that we can establish Theorem 1 for the case of known system parameters. This implies that input-constrained adaptive control algorithm (28)-(33) has a self-tuning property.

4 Conclusion

This paper considers the design and analysis of one-step optimal and adaptive controllers, under the restriction that a known constraint on the input is imposed. An input-constrained control law which minimizes one-step quadratic objective function with additional filter, subject to the input constraints, is derived. The convergence of output prediction error, model parameters and tracking error and a self-tuning property of the constrained adaptive control algorithm (using the Minimum Variance cost function and a stochastic gradient parameter estimation method) are also established.

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