# 두개의 ROBOT이 한물체를 다룰때의 Dynamic Model을 이용한 Stability Analysis

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# Stability Analysis using Dynamic Model of Two Industrial Robots Handling a Single Object

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**Abstract**

**Two control strategies are proposed for two arm robots; i.e. position-position control and position-force control.**

**For the proof of these control strategies, the stability analysis is conducted with robot dynamics included.**

**First, the closed form dynamic equation of the robot is derived, then it is transformed into the operational space for further analysis. Finally, Liapunov method is applied to the dynamic equation in operational space.**

## Operational Space Analysis of Dynamic Model

The closed form dynamic model of a robot can be derived using Newton-Euler formulation or Lagrangian formulation. Both are well-known equations. The dynamic equation of a robot is basically a description of the relationship between the input joint torque and the output motion, i.e., the motion of the arm. Generally, the closed form dynamic equation of an n degree of freedom manipulator can be given in the form

$$\sum_{j=1}^{n} H_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}\dot{q}_j\dot{q}_k + G_i = Q_i \tag{4.1}$$

where

$i = 1 \ldots n$ , number of joint,

$q_i$ : i-th joint variable of robot,

$h_{ijk} = \left(\frac{\partial H_{ij}}{\partial q_k} - \frac{\partial H_{jk}}{2\cdot\partial q_i}\right)$ is the Coriolis and Centrifugal coefficient,

$G_i = \sum_{j=1}^{n} m_j g^T J_{Li}^{(i)}$ is the gravity term, in which $g$ is the acceleration of gravity with respect to the base,

$Q_i = T + J^T F$ is the generalized force and torque of the manipulator, $T$ is the joint torque vector, $F$ is the external force exerted at the end-effector, and $J$ is the Jacobian,

$H_{ij}$: [i, j] component of manipulator inertia tensor matrix H,

where

$$H = \sum_{i=1}^{n} (m_i J_L^{(i)T} J_L^{(i)} + J_A^{(i)T} I_i J_A^{(i)}),$$

where

$m_i$ is the mass,

$I_i$ is the inertia tensor at the centroid of the link i with respect to the base,

$J_L^{(i)} = [J_{L1}^{(i)} \ldots J_{Li}^{(i)}\ 0 \ldots 0]$ is the 3 x n Jacobian matrix for linear velocities of link i, and

$J_A^{(i)} = [J_{A1}^{(i)} \ldots J_{Ai}^{(i)}\ 0 \ldots 0]$ is the 3 x n Jacobian matrix for angular velocities of link i.

The above equation can be represented by a simplified closed form:

$$D(q)\ddot{q} + H(q,\dot{q})\dot{q} + G(q) = T + J^T F \tag{4.2}$$

where

$D(q)$ : moment of inertia matrix,

$H(q,\dot{q})$ : Coriolis and centrifugal matrix,

$q = [q_1 \ldots q_n]^T$ and

$\dot{q}, \ddot{q}$ are the first and second time derivative vectors of q, respectively.

In order to change equation (4.2) to the operational space model, let the end-effector trajectory vector be x, then one has

$$\dot{x} = J\cdot\dot{q}, \tag{4.3}$$

$$\dot{q} = J^{-1}\cdot\dot{x} \quad \text{and} \tag{4.4}$$

$$\ddot{q} = J^{-1}\cdot\ddot{x} + \dot{J}^{-1}\cdot\dot{x}. \tag{4.5}$$

From the above three equations (4.3)-(4.5), we have

$$D(q)\ddot{q} = D(q)\cdot(J^{-1}\cdot\ddot{x} + \dot{J}^{-1}\cdot\dot{x})$$

$$= D(q)\cdot J^{-1}\cdot\ddot{x} + D(q)\cdot\dot{J}^{-1}\cdot\dot{x} \quad \text{and} \tag{4.6}$$

$$H(q,\dot{q})\dot{q} = H(q,\dot{q})\cdot J^{-1}\cdot\dot{x}. \tag{4.7}$$

Substituting $D(q)$ and $H(q,\dot{q})$ by (4.6) and (4.7), (4.2) becomes

$$D(q)\cdot J^{-1}\cdot\ddot{x} + D(q)\cdot\dot{J}^{-1}\cdot\dot{x} + H(q,\dot{q})\cdot J^{-1}\cdot\dot{x} + G(q) = T + J^{T}F. \quad (4.8)$$

Multiplying both sides of (4.8) by $(J^T)^{-1}$, we obtain

$$D(x)\ddot{x} + H(x,\dot{x})\dot{x} + G(x) = (J^T)^{-1}\cdot T + F \quad (4.9)$$

where

$D(x) = (J^T)^{-1}\cdot D(q)\cdot J^{-1}$,

$H(x,\dot{x}) = (J^T)^{-1}\cdot[D(q)\cdot\dot{J}^{-1} + H(q,\dot{q})\cdot J^{-1}]$ and

$G(x) = (J^T)^{-1}\cdot G(q)$.

Equation (4.9) is the robot dynamic model analyzed in the operational space. The stability of the robot system will be analyzed using this dynamic model.

## Stability Analysis of Position-Position Control

From the previous section, the operational space dynamic equations of the leader and follower become respectively,

$$D_1\cdot\ddot{x}_1 + H_1\cdot\dot{x}_1 + G_1 = (J_1^T)^{-1}\cdot T_1 + F_1 \quad \text{and} \quad (4.10)$$

$$D_2\cdot\ddot{x}_2 + H_2\cdot\dot{x}_2 + G_2 = (J_2^T)^{-1}\cdot T_2 + F_2 \quad (4.11)$$

where

subscript 1 and 2 represents leader and follower respectively.

In order to apply the Liapunov method to the above dynamic equations, an appropriate Liapunov function needs to be chosen. To do so, we first apply the two control strategies to the model, and then the dynamic equations are modified.

First, consider the position-position control strategy. By using the control strategy the applied torques of the leader and follower become respectively,

$$(J_1^T)^{-1}\cdot T_1 = k_p(x_{1d}-x_1) + k_v(-\dot{x}_1) + G_1 \quad \text{and} \quad (4.12)$$

$$(J_2^T)^{-1}\cdot T_2 = k_p(x_{2d}-x_2) + k_v(-\dot{x}_2) + G_2 \quad (4.13)$$

where

$x_{1d}$ : desired position vector of the leader,

$x_{2d}$ : desired position vector of the follower,

$k_p$: position feedback gain, a diagonal matrix and

$k_v$: velocity feedback gain, a diagonal matrix.

Note that $G_1$ and $G_2$ in (4.12) and (4.13) are the gravity compensation terms. And it is considered that the external forces $F_1$ and $F_2$ are the measured forces of the sensor. That is, we consider that the load, represented by L in Figure is a part of the leader end-effector. As a result, from Figure , one obtains

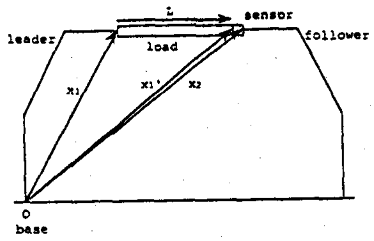


Figure Schematic Diagram of Robotic System

$$F_1 = -k_s(x_1-x_2+L) \quad \text{and} \quad (4.14)$$

$$F_2 = -k_s(x_2-x_1-L) \quad (4.15)$$

where

$k_s$ : stiffness of sensor, diagonal matrix and

L : length of load, constant.

Insert equation(4.12)-(4.15) to equations (4.10) and (4.11), one obtains

$$D_1\cdot\ddot{x}_1 + (H_1+k_v)\dot{x}_1 + k_p(x_1-x_{1d}) = -k_s(x_1-x_2+L) \quad \text{and} \quad (4.16)$$

$$D_2\cdot\ddot{x}_2 + (H_2+k_v)\dot{x}_2 + k_p(x_2-x_{2d}) = -k_s(x_2-x_1-L). \quad (4.17)$$

Define the error function vectors of the position for the leader and follower respectively as

$$e_1 = x_1-x_{1d} \quad \text{and} \quad (4.18)$$

$$e_2 = x_2-x_{2d}. \quad (4.19)$$

And the velocity and acceleration errors defined as

$$\dot{e}_1 = \dot{x}_1, \quad (4.20)$$

$$\dot{e}_2 = \dot{x}_2, \quad (4.21)$$

$$\ddot{e}_1 = \ddot{x}_1 \quad \text{and} \quad (4.22)$$

$$\ddot{e}_2 = \ddot{x}_2. \quad (4.23)$$

Using these error functions, and let $\Delta x = x_1-x_2+L$, equations (4.16) and (4.17) become

$$D_1\cdot\ddot{e}_1 + (H_1+k_v)\dot{e}_1 + k_p\cdot e_1 = -k_s\cdot\Delta x \quad \text{and} \quad (4.24)$$

$$D_2\cdot\ddot{e}_2 + (H_2+k_v)\dot{e}_2 + k_p\cdot e_2 = k_s\cdot\Delta x. \quad (4.25)$$

Now, select the Liapunov function for the leader as

$$V_1(e_1,\dot{e}_1) = \frac{1}{2}\cdot(\dot{e}_1^T\cdot D_1\cdot\dot{e}_1 + e_1^T\cdot k_p\cdot e_1) \quad (4.26)$$

which is positive. Time derivative of Liapunov function becomes

$$\dot V_1(e_1,\dot e_1) = \dot e_1^T(D_1\cdot\ddot e_1 + \frac{1}{2}\cdot\dot D_1\cdot\dot e_1 + k_p\cdot e_1). \tag{4.27}$$

It is known, from equation (4.24), that

$$D_1\cdot\ddot e_1 = -(H_1+k_v)\dot e_1 - k_p\cdot e_1 - k_s\cdot\Delta x.$$

Insert this equation into equation (4.27), we obtain

$$\dot V_1(e_1,\dot e_1) = \dot e_1^T((\frac{1}{2}\cdot\dot D_1-H_1)\dot e_1 - k_v\cdot\dot e_1 - k_s\cdot\Delta x). \tag{4.28}$$

To proceed, we need to calculate the term $\frac{1}{2}\cdot\dot D_1-H_1$ in equation (4.28). For the simplicity of formulation, eliminate the subscript 1 which means leader, so the term becomes $\frac{1}{2}\cdot\dot D-H$.

It is known that

$$\dot D = \frac{d}{dt}(D)$$

$$= \frac{d}{dt}((J^T)^{-1}\cdot D(q)\cdot J^{-1})$$

$$= (J^T)^{-1}\cdot\dot D(q)\cdot J^{-1} + 2\cdot(J^T)^{-1}\cdot D(q)\cdot\dot J^{-1}. \tag{4.29}$$

Using equation (4.29), one obtains

$$\frac{1}{2}\cdot\dot D-H$$

$$= \frac{1}{2}(J^T)^{-1}\dot D(q)J^{-1} + (J^T)^{-1}D(q)\dot J^{-1} - (J^T)^{-1}[D(q)\dot J^{-1} + H(q,\dot q)J^{-1}]$$

$$= (J^T)^{-1}(\frac{1}{2}\cdot\dot D(q)-H(q,q))J^{-1}. \tag{4.30}$$

For the calculation of $\frac{1}{2}\cdot\dot D(q)-H(q,\dot q)$ in equation (4.30), use the Lagrangian equation

$$\frac{d}{dt}(\frac{\partial L}{\partial\dot q}) - \frac{\partial L}{\partial q} = Q \tag{4.31}$$

where

$$L = K - U, \tag{4.32}$$

$K = \frac{1}{2}\cdot\dot q^T\cdot D(q)\cdot\dot q$, represents kinetic energy and

$U$ is the potential energy(depend on $q$ only).

Replacing $L$ in (4.32) by equation (4.31), one has

$$\frac{d}{dt}(\frac{\partial K}{\partial\dot q}) - \frac{\partial K}{\partial q} + \frac{\partial U}{\partial q} = Q. \tag{4.33}$$

Each element in equation (4.33) can be calculated as

$$\frac{\partial K}{\partial\dot q} = \frac{1}{2}\cdot D(q)\cdot\dot q + \frac{1}{2}\cdot\dot q^T\cdot D(q)$$

$$= D(q)\cdot\dot q, \tag{4.34}$$

$$\frac{d}{dt}(\frac{\partial K}{\partial\dot q}) = \dot D(q)\dot q + D(q)\ddot q, \tag{4.35}$$

$$\frac{\partial K}{\partial q} = \frac{1}{2}\cdot\dot q^T\frac{\partial D(q)}{\partial q}\dot q$$

$$= \frac{1}{2}\cdot\dot D(q)\dot q \quad \text{and} \tag{4.36}$$

$$\frac{\partial U}{\partial q} = G(q). \tag{4.37}$$

Using equations (4.34)-(4.37), the left side of the Lagrangian equation (4.33) becomes

$$\dot D(q)\dot q + D(q)\ddot q - \frac{1}{2}\cdot\dot D(q)\dot q + G(q).$$

Using the dynamic equation (4.2), this equation becomes

$$\frac{1}{2}\cdot\dot D(q)\dot q - H(q,\dot q)\dot q + Q. \tag{4.38}$$

From equation (4.33), equation (4.38) should be equal to Q. As a result,

$$\frac{1}{2}\cdot\dot D(q)\dot q - H(q,\dot q)\dot q = 0, \tag{4.39}$$

consequently, one has

$$\frac{1}{2}\cdot\dot D-H = 0. \tag{4.40}$$

This means that $\frac{1}{2}\cdot\dot D_1-H_1 = 0$, in equation (4.28).

From the equation (4.28), one has

$$\dot V_1(e_1,\dot e_1) = \dot e_1^T(-k_v\cdot\dot e_1-k_s\cdot\Delta x). \tag{4.41}$$

In the follower case, select the Liapunov function as

$$V_2(e_2,\dot e_2) = \frac{1}{2}(\dot e_2^T\cdot D_2\cdot\dot e_2 + e_2^T\cdot k_p\cdot e_2). \tag{4.42}$$

From (4-42) one may obtain

$$\dot V_2(e_2,\dot e_2) = -\dot e_2^T\cdot k_v\cdot\dot e_2 + \dot e_2^T\cdot k_s\cdot\Delta x. \tag{4.43}$$

Define the overall Liapunov function V as

$$V = V_1 + V_2 + \frac{1}{2}(x_1-x_2+L)^T\cdot k_s(x_1-x_2+L). \tag{4.44}$$

The time derivative of Liapunov function V is

$$\dot V = \dot V_1 + \dot V_2 + (\dot x_1-\dot x_2)^T\cdot k_s(x_1-x_2+L). \tag{4.45}$$

Replacing $\dot V_1$ and $\dot V_2$ in (4.45) by (4.41) and (4.43), and using $\Delta x = x_1-x_2+L$, (4.20), and (4.21), equation (4.45) becomes

$$\dot V = -\dot e_1^T\cdot k_v\cdot\dot e_1 - \dot e_2^T\cdot k_v\cdot\dot e_2 \le 0. \tag{4.46}$$

Thus the stability of the position-position control strategy is proved. Equality holds when the velocity errors of the leader and follower end-effectors become zero ($\dot e_1$=0 and $\dot e_2$=0), and the position errors become constant, which means that the system is always asymptotically stable.

### Stability Analysis of Position-Force Control

From the previous section, it is known that dynamic equations of the leader and follower are described by equations (4.10) and (4.11). In the position-force control scheme, position control of the leader uses PD control, while force control of the follower uses Proportional control. Gravitational term is also compensated for simplicity of calculation. So the position controller for the leader is the same as equation (4.12), but the force controller for the follower becomes

$$(J_2^T)^{-1}T_2 = k_f(F_d - F_c) + G_2 \tag{4.47}$$

where

$F_d = ks(x_{2d} - x_{1d} - L)$, desired force,

$F_c = ks(x_2 - x_1 - L)$, actual reaction force, and

$k_f$ is the force feedback gain, which is adiagonal matrix.

The dynamic equations with the position-force control strategy become

$$D_1 \cdot \ddot{x}_1 + (H_1 + k_v)\dot{x}_1 + k_p(x_1 - x_{1d}) = F_1 \quad \text{and} \tag{4.48}$$

$$D_2 \cdot \ddot{x}_2 + H_2 \cdot \dot{x}_2 - k_f(F_d - F_c) = F_2. \tag{4.49}$$

$F_1$ and $F_2$ are the external forces defined as in the previous section. But, because we are now applying the position-force control strategy, $F_1$ and $F_2$ should be defined as

$$F_1 = -(F_d - F_c) \quad \text{and} \tag{4.50}$$

$$F_2 = F_d - F_c \tag{4.51}$$

which means if the reaction force is the same as the desired force, the external force to the robotic system is zero. Otherwise, the external force should be regulated.

Defining the position error functions as (4.18) and (4.19), the force error function can be defined as

$$e_f = F_c - F_d. \tag{4.52}$$

Using equations (4.18),(4.19) and (4.50)~(4.52) to replace the related terms in dynamic equations (4.48) and (4.49), one can obtain

$$D_1 \cdot \ddot{e}_1 + (H_1 + k_v)\dot{e}_1 + k_p \cdot e_1 - e_f = 0 \quad \text{and} \tag{4.53}$$

$$(k_f + 1)^{-1}(D_2 \cdot \ddot{e}_2 + H_2 \cdot \dot{e}_2) + e_f = 0. \tag{4.54}$$

There are three independent variables in (4.53) and (4.54), namely, $e_1$, $e_2$, and $e_f$. The Liapunov function must include all these three variables. Select the Liapunov function as

$$V = \frac{1}{2}(\dot{e}_1^T \cdot D_1 \cdot \dot{e}_1 + e_1^T \cdot k_p \cdot e_1) + \frac{1}{2}\dot{e}_2^T \cdot (k_f + 1)^{-1} \cdot D_2 \cdot \dot{e}_2 + \frac{1}{2}e_f^T \cdot k_s^{-1} \cdot e_f, \tag{4.55}$$

which is clearly positive. Take the time derivative of equation (4.55), and rearrange the result using the dynamic equations (4.53) and (4.54), one obtains

$$\dot{V} = -\dot{e}_1^T \cdot k_v \cdot \dot{e}_1 + (\dot{e}_1^T - \dot{e}_2^T)e_f + \dot{e}_f^T \cdot k_s^{-1} \cdot e_f. \tag{4.56}$$

By using the error equations ((4.18),(4.19), and (4.52)), and reaction force relations ((2.4) and (2.8)), the relation between the position and force errors can be calculated as

$$e_f = F_c - F_d$$

$$= k_s(x_2 - x_1 - L) - k_s(x_{2d} - x_{1d} - L)$$

$$= -k_s(e_1 - e_2) \quad \text{and} \tag{4.57}$$

$$\dot{e}_f^T = -k_s(\dot{e}_1^T - \dot{e}_2^T). \tag{4.58}$$

Using equation (4.58), equation (4.56) becomes

$$\dot{V} = -\dot{e}_1^T \cdot k_v \cdot \dot{e}_1 \le 0. \tag{4.59}$$

Consequently, the given dynamic equation of position-force control is always stable. Equality holds when velocity error of the leader end-effector becomes zero ($\dot{e}_1=0$), which means that the position error becomes constant. In this case, the system is always asymptotically stable.

### Conclusions

Stability analysis is conducted with robot dynamics included in the operational space using the Liapunov second method. Two control mechanisms are studied in the analysis, namely, position-position and position-force control. By the analysis, both control mechanisms are found to be always stable.

In the analysis, the external force is considered as the reaction force. This is true when the two robots use the position controller. But in the position-force control strategy, if the reaction force is equal to the desired force, the external force becomes zero. Otherwise, the external force should be regulated.

### REFERENCES


- Zheng, Y.F. and Luh, J.Y.S., "Joint Torques for Control of Two Coordinated Moving Robots," Proc. 1986 IEEE Intern. Conference on Robotics and Automation, April 7-10, 1986, San Francisco, CA, pp. 1375-1380.
- Mason, M.T., "Compliance and Force Control for Computer Controlled Manipulators," IEEE Trans. on System, Men and Cybernatics, Vol. SMC-11, No. 6, June 6, 1981, pp.418-432.
- Krasovskii, N.N., Stability of Motion, Stanford Univ. Press, 1963.
- Paul, R.P., Robot Manipulators: Mathematics, Programming and Control, MIT Press, 1981.
- Asada, H. and Slotine, J.J.E., Robot Analysis and Control, A Wiley-Interscience Publication, 1986.

• Kab Il Kim and Y.F. Zheng, "Two Strategies of Position and Force Control for Two Industrial Robots Handling a Single Object," Journal of Robotics and Autonomous Systems, Vol. 5, No. 4, North-Holland, 1989. PP. 395-403.

• Kab Il Kim and Y.F. Zheng, "Stability Analysis of Position and Force Control for Two Industrial Robots Handling a Single Object," Proceeding, 2nd International Symposium on Robotics and Manufacturing, New Mexico, November 1988.