# Uncertainty의 경계치 추정기법을 기초로 한 출력궤환제어

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# An output feedback control based on the adaptatation law for the estimation of the bound of the uncertainty

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**Abstract** : In deterministic design of feedback controllers for uncertain dynamical systems, the bound on the uncertainty is an important clue to guarantee the asymptotic stability or uniform ultimate boundedness of the closed-loop system. In this paper, using only the measurable output we propose an adaptation law for the estimation of the bound of the uncertainty. And based on this adaptation law an adaptive control which renders the uncertain dynamical systems uniformly ultimately bounded is constructed.

## 1. Introduction

Recently, much attention has been paid to the problem of designing feedback controllers for uncertain dynamical systems containing uncertain elements due to model-parameter uncertainty, extraneous disturbance and measurement error. To design feedback controllers of such systems, if a *priori* statistical information of the uncertainties is unavailable but bounds on the uncertainties are known, one can consider a deterministic approach.

A number of researches within the deterministic framework can categorized into two groups. One is to design feedback controls based on Lyapunov minmax approach[1-5]. The other is based on the theory of variable structure systems[6-8].

In the above deterministic design, the assumptions that the uncertainties are bounded and their bounds are available to the designer are involved. And bounds on the uncertainties are an important clue to guarantee the asymptotic stability or uniform ultimate boundedness of uncertain dynamical systems. However, sometimes bounds on the uncertainties may not be easily obtained because of the complexity of structure of uncertainties. Especially, the magnitude of extraneous disturbance can not be simply estimated. Therefore, a methodology through which the boundary values on the uncertainties can be easily obtained is required. A parameter adaptation method supplies a good tool to solve this problem. Chen[9] introduced two adaptive schemes, that is, the leakage type and the dead-zone type, for the estimation of the bound of the uncertainty, and based on these adaptive schemes Chen constructed an adaptive control which makes all the signals of overall system be uniformly bounded and uniformly ultimately bounded. In the construction of this adaptive controller, all states must be available. However, in most practical situations the state is not directly available.

In this paper, under assumptions that all uncertainties are met the matching conditions and the norm of the lumped uncertainty is cone-bounded on the state of system, using only the measurable output vector we propose an adaptation law for the estimation of the bound on the lumped uncertainty, and based on this adaptation law an output feedback controller which guarantees the uniform ultimate boundedness of every signal of overall system is constructed.

## 2. Design of a nonlinear output feedback control

Consider a class of uncertain linear dynamical systems described by

$$\dot{x}(t) = Ax(t) + (B+\Delta B(\sigma))u(t) + Wv(t) \qquad (1)$$
$$y(t) = Cx(t)$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control, $y(t) \in R^l$ is the output vector, $v(t) \in R^q$ is an extraneous disturbance, and $A, B, C$, and $W$ are constant matrices of appropriate dimensions with $B$ and $C$ of full rank. $\Delta B(\sigma)$ represents the input matrix uncertainty. The unknown function $\sigma(\cdot): R \rightarrow \sum \subset R^p$ is assumed to be Lebesque measurable with $\sum$ compact. The pair $(A, B, C)$ is controllable and observable. We assume that $\Delta B$, and $W$ satisfy the so-called *matching conditions* [1-3]. That is,

*Assumption* 1 : There exist a function $D(\cdot)$ and a constant matrix $E$ such that

$$\Delta B(\sigma) = BD(\sigma)$$
$$W = BE$$

If the following class of feedback controls is employed:

$$u(t) = Ky(t) + p(y) \tag{2}$$

where $K \in R^{m \times l}$ is a linear output feedback gain matrix such that the spectrum of $A_c = A + BKC$ is contained in the open left half plane, and $p(y)$ is a nonlinear control for suppression of the effect of uncertainties such that

$$p(y) = \begin{cases} -\dfrac{Fy\rho(y)}{\|Fy\rho(y)\|}\rho(y) & \text{if } \|Fy\rho(y)\| > \varepsilon \\ -\dfrac{Fy\rho(y)}{\varepsilon}\rho(y) & \text{if } \|Fy\rho(y)\| \le \varepsilon \end{cases} \tag{3}$$

where $F \in R^{m \times l}$ and function $\rho(\cdot): R^l \to R_+$ will be specified subsequently, the system (1) can be compactly described by

$$\dot{x}(t) = A_c x(t) + Bp(y) + Be(y, \sigma, t) \tag{4}$$
$$y(t) = Cx(t)$$

where $e(y, \sigma, t)$ is the *lumped* uncertainty as follows:

$$e(y, \sigma, t) = D(\sigma)Ky(t) + D(\sigma)p(y) + Fv(t).$$

We define function $\rho(y)$ such that

$$\|e(y, \sigma, t)\| \le \max_{\sigma \in \Sigma}\|DK\|\|y(t)\| + \max_{\sigma \in \Sigma}\|D\|\rho(y)$$
$$+ \|Fv(t)\| \triangleq \rho(y)$$

Provided that $1 - \max_{\sigma \in \Sigma}\|D\| > 0$, the definition of $\rho(\cdot)$ is valid. Throughout this paper, vector norms are Euclidean and matrix norms are the corresponding induced ones. That is, for a real matrix $H$, $\|H\| = \sqrt{\lambda_M[H^T H]}$ where $\lambda_{M(m)}[\cdot]$ is the largest (smallest) eigenvalue of a given matrix. Thus, one obtains

$$\rho(y) = \beta_0 + \beta_1\|y\| \tag{5}$$

where

$$\beta_1 \triangleq [1 - \max_{\sigma \in \Sigma}\|D\|]^{-1}\|Fv\|,$$
$$\beta_2 \triangleq [1 - \max_{\sigma \in \Sigma}\|D\|]^{-1}\max_{\sigma \in \Sigma}\|DK\|.$$

Before stating and prooving main theorem, we introduce the following lemma.

*Lemma* 1 : For any $F \in R^{m \times l}$, if $(A_c, B, FC)$ is controllable and observable and $G_F(s) = FC(sI - A_c)^{-1}B$ is strictly positive real, then there exist symmetric positive definite matrices P and Q such that

$$PA_c + A_c^T P = -Q \tag{6}$$

and

$$FC = B^T P \tag{7}$$

Proof : see Steinberg and Corless[11].

Thus, if a matrix $F \in R^{m \times l}$ can be found such that $G_F(s)$ is strictly positive real and $(A_c, FC)$ is observable, then, $FC = B^T P$ where $P$ satisfies (6) for some symmetric positive definite matrix $Q$.

*Theorem* 1 : Consider the uncertain dynamical system described by (1). If assumption 1 is valid and there exists a matrix $F$ satisfies *Lemma* 1, then the state is uniformly bounded. That is, if $x(\cdot):[t_0, t_1] \to R^n$ is a solution to the system (1) and (2) with $\|x(t_0)\| \le r$, then $\|x(t)\| \le d(r)$ for all $t \in [t_0, t_1]$ where

$$d(r) = \begin{cases} R\sqrt{\lambda_M[P]/\lambda_m[P]} & \text{if } r \le R \\ r\sqrt{\lambda_M[P]/\lambda_m[P]} & \text{if } r > R \end{cases}$$

and $R = \sqrt{2\varepsilon/\lambda_m[Q]}$.

Furthermore, $x(t)$ is uniformly ultimately bounded. That is, if $x(\cdot):[t_0, \infty) \to R^n$ is a solution to the system (1) and (2) with $\|x(t_0)\| \le r$, then, for given $\bar{d} > R\sqrt{\lambda_M[P]/\lambda_m[P]}$

$$\|z(t)\| \le \bar{d} \qquad \text{for all } t \ge t_0 + T(\bar{d}, r)$$

where

$$T(\bar{d}, r) = \begin{cases} 0 & \text{if } r \le \bar{R} \\ \dfrac{r^2\lambda_M[P] - \bar{R}^2\lambda_m[P]}{\bar{R}^2\lambda_m[Q] - 2\varepsilon} & \text{if } r > \bar{R} \end{cases}$$

and $\bar{R} = \bar{d}\sqrt{\lambda_m[P]/\lambda_M[P]}$.

*Proof* : We consider the Lyapunov function candidate $V(\cdot): R^n \to R_+$ given by

$$V(x) = x^T P x$$

where P is defined in *Lemma* 1. The time-derivative of $V(x)$ along the system trajectory is given by

$$\dot{V}(x) = 2x^T P\dot{x}$$
$$= 2x^T P\{A_c x + Bp(y) + Be(y, \sigma, t)\}$$

In case of $\|Fy\rho(y)\| > \varepsilon$, from (3), (5), and *Lemma* 1,

$$\dot{V}(x) \le -x^T Qx - 2\|Fy\rho(y)\| + 2\|Fy\rho(y)\|$$
$$= -x^T Qx \tag{8}$$

And if $\|Fy\rho(y)\| \le \varepsilon$, by similar operations

$$\dot{V}(x) \le -x^T Qx + 2\varepsilon \tag{9}$$

Consequently, from (8) and (9)

$$\dot{V}(x) \le -\lambda_m[Q]\|x\|^2 + 2\varepsilon \tag{10}$$

for all $x \in R^n$. Then, the uniform boundedness and uniform ultimate boundedness immediately follow using the standard arguments given by Corless and Leitmann. □

In the design of this class of feedback controls, one knows that a continuous positive scalar function $\rho(\cdot)$ is an important clue to guarantee uniform ultimate boundedness of uncertain dynamical systems.

In this paper, our goals are to propose an adaptation law for the bound on the lumped uncertainty, $e(y, \sigma, t)$, and to design an adaptive control which guarantees the uniform ultimate boundedness of every signal of overall system.

**3. Design of an adaptive control**

Now, consider an adaptive control as follows:

1) Control law :

$$p(x) = \begin{cases} -\dfrac{Fy\bar{\rho}(y,t)}{\|Fy\bar{\rho}(y,t)\|}\bar{\rho}(y,t) & \text{if } \|Fy\bar{\rho}(y,t)\| > \varepsilon \\ -\dfrac{Fy\bar{\rho}(y,t)}{\varepsilon}\bar{\rho}(y,t) & \text{if } \|Fy\bar{\rho}(y,t)\| \le \varepsilon \end{cases} \qquad (11)$$

where $\bar{\rho}(y,t) = \bar{\beta}_0 + \bar{\beta}_1\|y\|$ and $\bar{\beta}_0$ and $\bar{\beta}_1$ are adaptive parameters on $\beta_0$ and $\beta_1$, respectively.

2) Adaptation law :

$$\dot{\bar{\beta}}_i(t) = -\phi_i\xi_i\bar{\beta}_i(t) + \phi_i\|Fy\|\|y\|^i \quad \text{for } i = 0, 1 \qquad (12)$$

where $\phi_i$ and $\xi_i$ are adaptation gains which are positive constants.

Note that if we choose the nonnegative initial values, $\bar{\beta}_i(t_0)$, a solution of (12) is always nonnegative. And $\bar{\rho}(y,t)$ is so.

Let $\beta = [\beta_0, \beta_1]^T$, $\bar{\beta}(t)=[\bar{\beta}_0(t), \bar{\beta}_1(t)]^T$, and $\tilde{\beta}(t)=\bar{\beta}(t) - \beta = [\tilde{\beta}_0(t), \tilde{\beta}_1(t)]^T$. Since we assume that $\beta_0$ and $\beta_1$ are constants, the following relationship is valid:

$$\dot{\tilde{\beta}}_i(t) = \dot{\bar{\beta}}_i(t) \quad \text{for } i = 0, 1$$

Here, we consider the following Lyapunov function, $V(\cdot) : R^n \times R^2 \to R_+$ such that

$$V(x, \tilde{\beta}) = x^T Px + \sum_{i=0}^{1}\tilde{\beta}_i^2/\phi_i \qquad (13)$$

where $\phi_i$ is defined in (12). The time-derivative of $V(x, \tilde{\beta})$ is as follows:

$$\begin{aligned} \dot{V}(x, \tilde{\beta}) &= 2x^T P\dot{x} + 2\sum_{i=0}^{1}\tilde{\beta}_i\dot{\tilde{\beta}}_i/\phi_i \\ &= 2x^T P\{A_c x + Bp(y) + Be(y,\sigma,t)\} + 2\sum_{i=0}^{1}\tilde{\beta}_i\dot{\tilde{\beta}}_i/\phi_i \\ &= -x^T Qx + 2x^T PBp(y) + 2x^T PBe(y,\sigma,t) \qquad (14) \\ &\quad + 2\sum_{i=0}^{1}\tilde{\beta}_i\dot{\tilde{\beta}}_i/\phi_i \end{aligned}$$

In case of $\|Fy\bar{\rho}(y,t)\| > \varepsilon$, substituting (5), (11), and (12) into (14), one obtains the following inequality:

$$\begin{aligned} \dot{V}(x, \tilde{\beta}) &= -x^T Qx - 2\|Fy\|\sum_{i=0}^{1}\bar{\beta}_i\|y\|^i + 2\|Fy\|\sum_{i=0}^{1}\beta_i\|y\|^i \\ &\quad + 2\|Fy\|\sum_{i=0}^{1}\tilde{\beta}_i\|y\|^i - 2\sum_{i=0}^{1}\xi_i\tilde{\beta}_i\bar{\beta}_i \\ &= -x^T Qx - 2\sum_{i=0}^{1}\xi_i\tilde{\beta}_i\bar{\beta}_i \qquad (15) \end{aligned}$$

And if $\|Fy\bar{\rho}(y,t)\| \le \varepsilon$, by similar operations

$$\dot{V}(x, \tilde{\beta}) \le -x^T Qx - 2\sum_{i=0}^{1}\xi_i\tilde{\beta}_i\bar{\beta}_i + 2\varepsilon \qquad (16)$$

Consequently, from results of (15) and (16),

$$\begin{aligned} \dot{V}(x, \tilde{\beta}) &\le -x^T Qx - 2\sum_{i=0}^{1}\xi_i\tilde{\beta}_i^2 - 2\sum_{i=0}^{1}\xi_i\tilde{\beta}_i\beta_i + 2\varepsilon \qquad (17) \\ &\le -\lambda_m[Q]\|x\|^2 - 2\xi_m\|\tilde{\beta}\|^2 + 2\xi_M\|\beta\|\|\tilde{\beta}\| + 2\varepsilon \end{aligned}$$

for all $(x, \tilde{\beta}) \in R^n \times R^2$ where $\xi_{m(M)}$ denotes the smallest (largest) component of $\xi = [\xi_0, \xi_1]^T$.

Let $z(t) = [x(t)^T, \tilde{\beta}(t)^T]^T$, $\gamma_3 = \min\{\lambda_m[Q], 2\xi_m\}$, and $\gamma_4 = 2\xi_M\|\beta\|$. Then, (17) can be finally rewritten as follows:

$$\dot{V}(z) \le -\gamma_3\|z\|^2 + \gamma_4\|z\| + 2\varepsilon \quad \text{for all } z \in R^2.$$

Now, we are ready to state the following theorem.

*Theorem* 2 : Consider the uncertain dynamical system described by (1). If Assumption 1 is valid and there exists a matrix $F$ satisfies *Lemma* 1, then $z(t)$ is uniformly bounded. Furthermore, $z(t)$ is uniformly ultimately bounded.

*Proof* ; By procedure presented in [10], we can prove that every signal, that is, the adaptation error vector $\tilde{\beta}(t)$ and the state $x(t)$ are uniformly ultimately bounded to an arbitrary neighborhood of the zero state. □

*Remark* : By choosing appropriate $\phi$, one can adjust the rate of parameter adaptation. In theory, as $\phi$ is getting larger, the rate of parameter adaptation is getting higher. In practice, $\phi$ is limited by the bound of control input and other practical considerations. And $\xi$ prevents the fast adaptation. Thus, the system response is improved as $\xi$ is getting smaller.

**4. Illustrative Example**

In order to illustrate the effectiveness of the proposed method, we consider a following uncertain linear system by

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & -3 \\ 0 & 0 & -3 \end{bmatrix}, \quad B+\Delta B(\sigma) = \begin{bmatrix} 1+\sigma_1(t) & 0 \\ 0 & 1+\sigma_2(t) \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let a set of desired eigenvalues of the closed-loop system be {-1, -2, -3}. Then the matrices $K$, $F$, $P$, and $Q$ are

$$K = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & -5 \\ 0 & -5 & 34 \end{bmatrix}.$$

Now, $\sigma_1(t) = \sin x_1$, $\sigma_2(t) = \cos 3t$, $v(t) = \sin 3t$, $\hat{\beta}(0) = 0$, and $\varepsilon = 0.01$. For $x(0) = [\pi/2,\ 0]^T$, observe the system response, while changing the value of $\phi$. In Fig. 1, fixed $\xi = [0.001\ 0.001]^T$, dynamic responses ($x_1$) are shown for $\phi = [1\ 1\ 1]^T$ and $[3\ 3\ 3]^T$, respectively. From this figure, as $\phi$ is getting larger, we know that the system response is improved. This is the reason that the adaptation rate is getting faster as $\phi$ is getting larger. This fact can be confirmed by the progress of the parameter adaptation shown in Fig. 2.

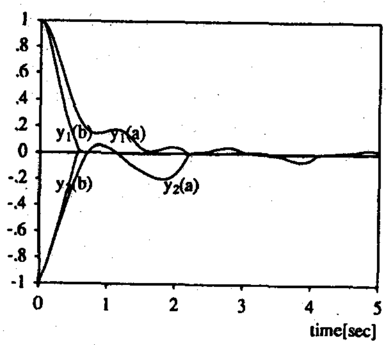


Fig. 1. Output trajectories.
( a: $\phi = [1\ 1]^T$, b: $\phi = [3\ 3]^T$)

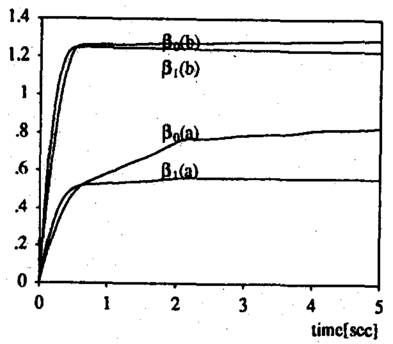


Fig. 2. The progress of parameter adaptation.
( a: $\phi = [1\ 1]^T$, b: $\phi = [3\ 3]^T$)

## 5. Conclusions

In order to control the uncertain dynamical system, the bound of the uncertainty is an important factor but may not be easily obtained by several reasons. Based on the adaptation law for the estimation of the bound of the uncertainty, the adaptive control is constructed using only the measurable output vector. The proposed method guarantees the augmented overall system combined the original uncertain dynamical system plus the adaptive algorithm uniformly ultimately bounded. The simulation results show that the proposed method effectively controls the uncertain dynamical system.

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