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STABILITY ANALYSIS OF A CONTROL SYSTEM WITH AN ANTIRESET-WINDUP LIMITER BY LIAPUNOV'S SECOND METHOD

Sangsik Yang

Department of Control Engineering
Ajou University, Suwon, Korea

ABSTRACT

When a saturating control system has integral action, reset windup can cause instability as well as make the system performance unsatisfactory. An antireset-windup (ARW) limiter has been suggested to improve the stability and performance. It has been implemented with analog circuits and tested by simulations. This paper presents the stability condition of a double-integrator plant having the state feedback plus integral-action controller with the ARW limiter by using both Liapunov's second method and graphical method together.

1. INTRODUCTION

All mechanical systems have saturation nonlinearity in actuators and/or final control elements (e.g., power amplifiers). Saturation is a significant type of nonlinearity of mechanical systems. When a saturating control system has integral-action in controller, the controller output will exceed the saturation level quickly for a large reference input change. This results in a large overshoot of the system response and the phenomenon is called reset windup. Reset windup can cause instability as well as make the system performance unsatisfactory.

Saturating sampled-data systems have been analyzed by Nease[1] and Torng[2]. Kalman[3] dealt with the problem of designing an optimal nonlinear controller of saturating system. Mullin[4] suggested a digital filter which forces the systems to follow step or ramp inputs when saturation is present. To stabilize a system with a nonlinear actuator, Hsu and Meyer[5] inserted a nonlinear element in the error path of the closed-loop system. To avoid the reset windup of an integrator in a system with actuator saturation, Krikelis[6] used a nonlinear feedback element around the integrator. All of these control schemes involve tuning of the parameters that are in the filter or in the nonlinear elements. Hanus[7] proposed a technique using a proportional feedback element around the integrator, but without any stability analysis. Glattfelder and Schaufelberger[8] presented the stability analysis of systems with an antireset-windup (ARW) circuit based on the circle criterion. By using the ARW circuit, the asymptotically stable region for the system's initial state can be extended.

Another scheme to avoid reset windup is to keep the controller output to the upper (or lower) saturation level of the final control element when it tends to exceed the level. The controller resumes the integral action when the controller output intends to fall within the linear range. This scheme has been suggested by Phelan[9] for use with the 'pseudo dervative feedback' (PDF) controller, called intelligent PDF controller. It consists of PDF controller and an ARW limiter which needs no tuning of parameter and is different from the ARW circuit of Glattfelder and Schaufelberger[8]. It has been implemented successfully with analog circuits and shown effective by simulations. The stability of a second-order plant having the PDF controller with the ARW limiter has been analyzed using the describing function method in conjunction with Nyquist stability theorem by Yang[10]. However, since the describing function method is an approximate method, it does not give the necessary and sufficient condition for the asymptotic stability.

This paper presents the necessary and sufficient condition for the asymptotic stability of a double-intgrator plant having the state feedback plus integral-action controller with the

ARW limiter by using both Liapunov's second method and graphical method together.

2. A SHARP LIAPUNOV FUNCTION OF A LINEAR SYSTEM

Most mechanical positioning systems are modeled as a second-order system which consists of one integrator and one first-order dynamics with a mechanical time constant. If it is assumed that the mechanical time constant is large enough, that is, the linear damping term is negligible, the second-order system reduces to a double-integrator system. This chapter presents the stability analysis of a control system which consists of a double-integrator plant and a state feedback plus integral-action controller. The block diagram of the system is shown in Figure 1. The state equation is

$$\dot{e} = -v, \qquad \dot{v} = \frac{u}{m}, \tag{1}$$

and the controller is represented as

$$\dot{u} = k_i e - k_p v - \frac{k_d}{m} u. \tag{2}$$

The charateristic equation of the above linear system is

$$ms^3 + k_d s^2 + k_p s + k_i = 0. \tag{3}$$

By using Routh criterion, the asymptotic stability condition for the controller parameters is simply obtained as

$$k_d k_p > m k_i. \tag{4}$$

The stability proof for a nonlinear system introduced in Chapter 3 is presented by using the Liapunov's second method. In this chapter, a sharp Liapunov function for the linear system is obtained for a later use. We let a Liapunov function candidate, $V(t, \mathbf{x})$ as

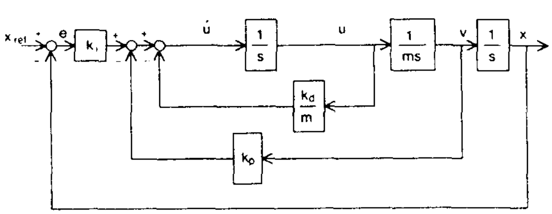

Fig. 1 The block diagram of the linear system.

$$V(t, \mathbf{x}) = \mathbf{x}^T \mathbf{Y} \mathbf{x}, \tag{5}$$

where

$$\mathbf{x} = [\, e \quad v \quad u \,]^T$$

and

$$\mathbf{Y} = \frac{1}{2}\begin{bmatrix} a & \alpha & \gamma \\ \alpha & b & \beta \\ \gamma & \beta & c \end{bmatrix}.$$

The parameters are set as

$$a = k_d k_p^2 + 2k_d^2 k_i,$$

$$b = m^2 k_i + 2mk_d k_p + m\frac{k_p^3}{k_i} + 2k_d^3,$$

$$c = \frac{k_p^2}{k_i} + 2k_d,$$

$$\alpha = -k_d^2 k_p - mk_d k_i - mk_p^2,$$

$$\beta = 2k_d^2 + mk_p,$$

$$\gamma = mk_i - k_d k_p. \tag{6}$$

If $k_d k_p > mk_i$, then,

$$V(t, \mathbf{0}) = 0, \tag{7}$$

$$V(t, \mathbf{x}) > 0 \quad \text{for all } \mathbf{x} \neq \mathbf{0}, \tag{8}$$

and

$$\dot{V}(t, \mathbf{0}) = 0, \tag{9}$$

$$\dot{V}(t, \mathbf{x}) < 0 \quad \text{for all } \mathbf{x} \neq \mathbf{0}, \tag{10}$$

which is proven in Appendix A. Since this condition is the necessary and sufficient condition for asymptotic stability of the linear system, the proposed Liapunov funtion candidate is a sharp Liapunov function.

3. STABILITY OF THE SYSTEM WITH ARW LIMITER

In general, a saturation element exists right before the double-integrator of the linear control system in Chapter 2, and the system makes unstable limit cycle for a large reference input signal change[10]. In this paper, the ARW limiter suggested by Phelan[9] is used together with the linear controller mentioned in Chapter 2 for the control of the saturating double-integrator plant. Figure 2 shows the overall block diagram of the control system with a saturation element and the ARW limiter, and Figure 3 shows the detail block diagram of the ARW limiter. The state equation of

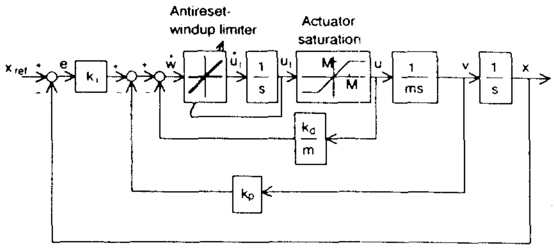

Fig. 2 The block diagram of the saturating system with the ARW limiter.

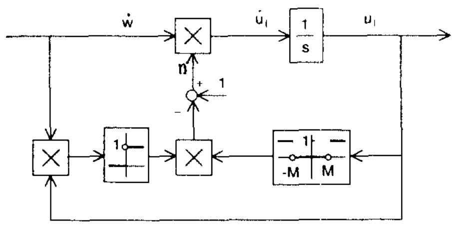

Fig. 3 The block diagram of the ARW limiter.

the plant is the same as the one in Chapter 2. The output, u of the ARW limiter preceded by the linear controller is given by

$$\dot{u} = \dot{u}_1 = n(u,\ \dot{w})\dot{w}, \tag{11}$$

where w is the output of the linear controller and is also the input to the ARW limiter, which is represented as

$$\dot{w} = k_i e - k_p v - \frac{k_d}{m} u \tag{12}$$

and

$$n(u,\ \dot{w}) = \begin{cases} 0 & \text{if} \quad u = M \text{ and } \dot{u} > 0, \\ 0 & \text{if} \quad u = -M \text{ and } \dot{w} < 0, \\ 1 & \text{otherwise,} \end{cases} \tag{13}$$

where M is the saturation level. n is a switch to the integral-action. When the output of the ARW limiter is below the saturation level, $n = 1$, that is, the ARW limiter pass the input signal unchanged and the control system behaves the same as the linear control system in Chapter 2. If once the output of the ARW limiter reaches the saturation level and the input signal of the ARW limiter tends to exceed the saturation level, n becomes 0, that is, the integrator of the ARW limiter stops integrating and keeps the output to the saturation level until the sign of the derivative of the input signal changes. At the time of sign change n becomes 1, that is, the integrator of the ARW limiter resumes integrating to escape from saturation, and the output follows the input signal with a certain difference resulted from stoping of integration. In this way, the output of the ARW limiter is kept within the linear range of saturation element by either stoping or resuming following the input signal. The switch is controlled by the sign of the derivative of the input signal when the output of the ARW limiter is at the saturation level.

In order that the asymptotic stability of the above nonlinear control system may be illustrated, either the condition of Eq. (10) along a trajectory during the saturation period or the net decrement of the function between two points where the saturation begins and ends must be proved. In this paper, the latter proof is presented for both cases of saturation ($n = 0$), that is, when $u = M$ and $u = -M$.

When $u = M$ and $\dot{w} > 0$, the state equation becomes

$$\dot{e} = -v, \tag{14}$$

$$\dot{v} = \frac{M}{m}. \tag{15}$$

From the state equation, the relationship between e and v is obtained as

$$e = -\frac{m}{2M} v^2 + e_o, \tag{16}$$

where e_o is constant along each trajectory. Since $\dot{w} > 0$,

$$k_i e - k_p v - \frac{k_d}{m} M > 0. \tag{17}$$

As shown in Figure 4, trajectories of the system saturated at the upper level are on the left upper half plane satisfying Eq. (17). If once saturation starts at an arbitrary point, $\mathbf{x} = \mathbf{x}_1$ on the left upper plane, the system follows the trajectory and finally, saturation ends in finite time at the corresponding point, $\mathbf{x} = \mathbf{x}_2$, which is located on the boundary of the linear region. That is , the saturation end point, $\mathbf{x}_2$ is on the line given by

$$k_i e_2 - k_p v_2 - \frac{k_d}{m} u_2 = 0,\ v_2 > \frac{M k_p}{m k_i},\ u_2 = M \tag{18}$$

Obiously, the saturation end point exists and is

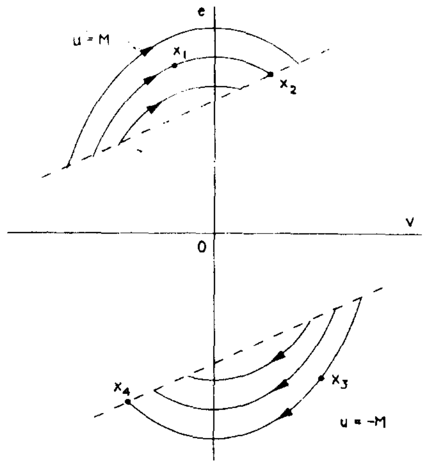

Fig. 4 Phase plane trajectories of the saturating system.

unique to every saturation start point. If $k_p k_d > mk_i$, the values of function, $V(t, \mathbf{x})$ at two points, $V(t_1, \mathbf{x}_1)$ and $V(t_2, \mathbf{x}_2)$ always satisfy that

$$V(t_1, \mathbf{x}_1) > V(t_2, \mathbf{x}_2) \ \forall \text{ set of } \mathbf{x}_1 \text{ and } \mathbf{x}_2, \tag{19}$$

of which the proof is in Appendix B.

When $u = -M$ and $\dot{w} < 0$, the state equation becomes

$$\dot{e} = -v, \tag{20}$$

$$\dot{v} = -\frac{M}{m}. \tag{21}$$

From the state equation, the relationship between e and v is obtained as

$$e = \frac{m}{2M} v^2 - e_o, \tag{22}$$

where e_o is constant along each trajectory. Since $\dot{w} < 0$,

$$k_i e - k_p v + \frac{k_d}{m} M < 0. \tag{23}$$

Trajectories of the system saturated at the lower level are on the right lower half plane satisfying Eq. (23) as shown in Figure 4. All the statements for the points, $\mathbf{x}_1$ and $\mathbf{x}_2$, are also true of two points, $\mathbf{x}_3$ and $\mathbf{x}_4$. In the same way as the case of saturation at the upper level, if $k_p k_d > mk_i$, the values of function, $V(t, \mathbf{x})$ at two points, $V(t_3, \mathbf{x}_3)$ and $V(t_4, \mathbf{x}_4)$ always satisfy that

$$V(t_3, \mathbf{x}_3) > V(t_4, \mathbf{x}_4) \ \forall \text{ set of } \mathbf{x}_3 \text{ and } \mathbf{x}_4, \tag{24}$$

of which the proof is the same to the one in Appendix B except for the signs.

Therfore, from the Eqs. (19) and (24), it is concluded that the function, $V(t, \mathbf{x})$ may increase partially during the system is saturated, but the net change of $V(t, \mathbf{x})$ between the start point and the end point of any saturation period is negative.

Summarizing all the results derived above, if $k_p k_d > mk_i$, it is satisfied that

(A) $V(t, \mathbf{0}) = 0,$ (25)

(B) $V(t, \mathbf{x}) > 0 \ \ \forall \mathbf{x} \neq \mathbf{0}$ independent of n, (26)

(C) $\dot{V}(t, \mathbf{0}) = 0,$ (27)

(D) $\dot{V}(t, \mathbf{x}) < 0 \ \ \forall \mathbf{x} \neq \mathbf{0}$ along all trajectories when $n = 1$ (system not saturated), (28)

(E) $V(t_1, \mathbf{x}_1) > V(t_2, \mathbf{x}_2)$ when $n = 0$ and $u = M$, (29)

(F) $V(t_3, \mathbf{x}_3) > V(t_4, \mathbf{x}_4)$ when $n = 0$ and $u = -M$, (30)

where $\mathbf{x}_1$ and $\mathbf{x}_2$, are any saturation start points and $\mathbf{x}_3$ and $\mathbf{x}_4$ are the corresponding saturation end points. When $n = 0$, using Eq. (29) the infimum of V, V_i is

$$V_i = \inf_{n=0} V(t, \mathbf{x}) = \inf_{u=M} V(t, \mathbf{x}_2) \tag{31}$$

Using Eqs. (5) and (18), it can be easily seen that there exists the infimum of V. If $V(0, \mathbf{x}) \le V_i$, the system never saturate and it behaves as the linear system of Chapter 2. So, the system trajectory approaches asymptotically the equilibrium point, $\mathbf{x} = \mathbf{0}$. If $V(0, \mathbf{x}) > V_i$, the system may saturate at finite time or may not saturate. If the system does not saturate, it will follow the trajectory in the linear region. If the system saturate, it will escape from the saturation in finite time with the value of V reduced, and follow the trajectory in the linear region. If V is not reduced enough in the linear trajectory, the system may saturate again. This phenomenon will be repeated until $V(t, \mathbf{x}) \le V_i$. If once $V(t, \mathbf{x}) \le V_i$, the system never saturate and the stability analysis ends up with that of the linear system in Chapter 2. Therefore, the nonlinear system is globally asymptotically stable if and only if the stability condition of the linear system is satisfied.

The time response of the saturating system with the ARW limiter to a step reference input in the case of stability limit is shown in Figure 5. The figure illustrates that in the beginning the

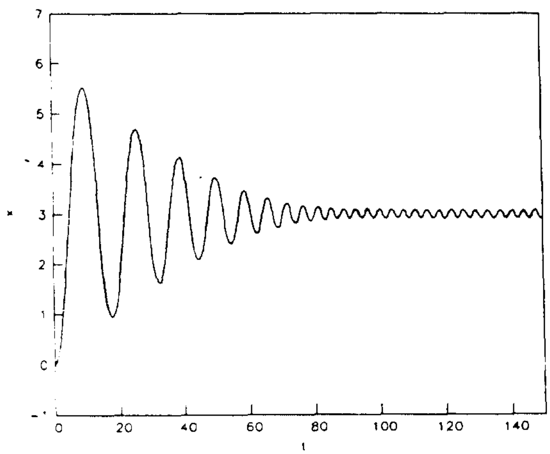

Fig. 5 Time response of the saturating system with the ARW limiter in the case of stability limit.

amplitude of oscillation decreases until V is reduced enough below V_i. After the final saturation, its behaviour is the same as the one of the linear system, and it keeps oscillating.

4. CONCLUSIONS

The antireset-windup limiter was used with the state feedback plus integral-action controller for control of a double-integrator plant with saturation nonlinearity. The necessary and sufficient condition for the asymptotic stability of the nonlinear control system was obtained by using both Liapunov's second method and graphical method together. The nonlinear control system is globally asymptotically stable if and only if the stability condition of the corresponding linear system is satisfied.

REFERENCES

[1] R. F. Nease, "Analysis and Design of Nonlinear Sampled-Data Control Systems," *WADC Technical Note*, Servomechanisms Laboratory, Massachusetts Institute of Technology, pp. 57-162, 1957.

[2] H. C. Torng, "The Analysis of Non-linear Sampled-Data Control Systems Containing a Saturating Element," *NEREM Record*, Boston, pp. 88-89, 1959.

[3] R. E. Kalman, "Optimal Control of Saturating Systems by Intermittent Action," *IRE WESCON Convention Record*, Part 4, pp. 130-135, 1957.

[4] F. J. Mullin, "The Stability and Compensation of Saturating Sampled-Data Systems," *AIEE Trans. Communications and Electronics*, pp.270-278, 1959.

[5] J. C. Hsu and A. U. Meyer, *Modern Control Principles and Applications*, McGraw-Hill Book Co., New York, 1986, pp. 225-232.

[6] N. J. Krikelis, "State Feedback Integral Control with 'Intelligent' Integrators," *International Journal of Control*, Vol. 32, No. 3, pp. 465-473, 1980.

[7] R. Hanus, "A New Technique for Preventing Control Windup," *Journal A*, Vol. 21, No. 1, pp. 15-20, 1980.

[8] A. H. Glattfelder and W. Schaufelberger, "Stability Analysis of Single Loop Control Systems with Saturation and Antireset-Windup Circuits," *IEEE Trans.*, Vol. AC-28, No. 12, pp. 1074-1081, 1983.

[9] R. M. Phelan, *Automatic Control Systems*, Cornell University Press Ltd., 1977, pp. 116-120.

[10] S. Yang, "Analysis on a Saturating System with an Intelligent Limiter," *Proc. of 1989 KACC*, Vol. 2, pp. 1091-1096, 1989.

APPENDIX A - A SHARP LIAPUNOV FUNCTION

If all principal minors of $\mathbf{Y}$ are positive, the quadratic function, V is positive definite. The three principal minors are as follows:

(i) $$|\,a\,| = a = \frac{k_d k_p^2 + 2k_d^2 k_i}{2} > 0.$$

(ii) $$\begin{vmatrix} a & \alpha \\ \alpha & b \end{vmatrix} = ab - \alpha^2$$

$$= \frac{k_d^2}{4}[(mk_i + k_d k_p)^2 + mk_p^3 + 4k_d^3 k_i]$$

$$+ \frac{mk_p^2}{4k_i}(k_p^2 + k_d k_i)(k_d k_p - mk_i).$$

(iii) $$\begin{vmatrix} a & \alpha & \gamma \\ \alpha & b & \beta \\ \gamma & \beta & c \end{vmatrix} = abc + 2\alpha\beta\gamma - b\gamma^2 - c\alpha^2 - a\beta^2$$

$$= S(k_d k_p - mk_i).$$

where S is a positive constant. So, if $k_p k_d > mk_i$, V is positive definite.

For the time derivative of V,

$$\dot{V}(t,\ \mathbf{x}) = -\mathbf{x}^{\mathrm{T}}W\mathbf{x},$$

where

$$W = -(F^{\mathrm{T}}Y + YF),$$

and from Eqs. (1) and (2)

$$F = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1/m \\ k_i & k_p & -k_d/m \end{bmatrix}.$$

And, W is obtained as

$$W = (k_d k_p - m k_i)\begin{bmatrix} k_i & 0 & 0 \\ 0 & k_d & 0 \\ 0 & 0 & k_p/mk_i \end{bmatrix}.$$

So, if $k_p k_d > m k_i$, $\dot{V}$ is negative definite.

APPENDIX B

When the system is saturated ($n = 0$),

$$\dot{u} = 0,$$

and

$$\dot{V}(t,\ \mathbf{x}) = -\mathbf{x}^{\mathrm{T}}(F_n{}^{\mathrm{T}}Y + YF_n)\mathbf{x},$$

where

$$F_n = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1/m \\ k_i & k_p & -k_d/m \end{bmatrix}.$$

Then,

$$\dot{V}(t,\ \mathbf{x}) = -\alpha ev - \alpha v^2 + \frac{\alpha}{m}eu + \left(\frac{b}{m} - \gamma\right)uv + \frac{\beta}{m}u^2.$$

When $u = M$ and $\dot{w} > 0$, by using Eq. (16),

$$\dot{V}(t,\ \mathbf{x}) = \frac{\alpha m}{2M}v^3 - \frac{3}{2}\alpha v^2 + \left(\frac{b}{m}M - \gamma M - \alpha e_o\right)v$$

$$+ \frac{\beta}{m}M^2 + \frac{\alpha}{m}e_o M$$

along the trajectory specified with a constant e_o. The net change of $V(t,\ \mathbf{x})$ from an arbitrary saturation start point, $\mathbf{x}_1$ to a corresponding saturation end point, $\mathbf{x}_2$ is represented as

$$\Delta V = V(t_2,\ \mathbf{x}_2) - V(t_1,\ \mathbf{x}_1) = \int_{t_1}^{t_2} \dot{V}(t,\ \mathbf{x})dt.$$

Since $\dot{v} = M/m$,

$$\Delta V = \frac{m}{M}\int_{v_1}^{v_2} \dot{V}(t,\ \mathbf{x})dv.$$

If saturation starts at a point, $\mathbf{x}_1$, then e_o is specified and v_2 is determined. We let v_2 as

$$v_2 = -\sigma + \omega, \qquad 0 < \omega < \infty$$

where

$$\sigma = \frac{Mk_p}{mk_i}, \quad \omega = \frac{Mk_p}{mk_i}\left(1 - \frac{2k_d k_i}{k_p{}^2} + \frac{2mk_i{}^2}{Mk_p{}^2}e_o\right)^{1/2}$$

As shown in Figure 4, v_1 can be of any value represented as

$$v_1 = -\sigma + \rho\omega, \qquad -1 < \rho < 1$$

Performing the integration, ΔV can be represented as a polynomial of ω.

$$\Delta V = A\omega^4 + B\omega^3 + C\omega^2 + D\omega,$$

where

$$A = -\frac{m^2}{8M^2}(k_d{}^2 k_i + k_d k_p{}^2)(1 - \rho^2)^2 < 0,$$

$$B = \frac{1}{2M}(3k_d{}^2 k_p + \frac{k_d}{k_i}k_p{}^3 + mk_d k_i + mk_p{}^2)(1 - \rho^2)\rho,$$

$$C = \frac{k_p{}^3}{k_i{}^2}(k_d k_p - mk_i)(1 - \rho^2) > 0,$$

$$D = \frac{M}{m}\Big[(k_d k_i - k_p{}^2)^2$$
$$+ k_i k_p(k_d k_p + mk_i)\Big](mk_i - k_d k_p)(1 - \rho) < 0,$$

for all $\rho \in (-1,\ 1)$ if $k_p k_d > mk_i$. Throughout a long complicated derivation, it is proved that

$$\frac{d(\Delta V)}{d\omega} < 0, \qquad \forall\ \omega > 0.$$

Since

$$\Delta V\big|_{\omega=0} = 0,$$

$$\Delta V < 0, \qquad \forall\ \omega > 0,\ \forall\ \rho \in (-1,\ 1),$$

that is,

$$V(t_1,\ \mathbf{x}_1) > V(t_2,\ \mathbf{x}_2) \quad \forall \text{ set of } \mathbf{x}_1 \text{ and } \mathbf{x}_2.$$