Stabilization of Discrete-time Semilinear Heat Processes by Boundary Inputs

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Abstract

In this paper, we are going to study the stabilization of the semilinear heat equation with inhomogenous boundary conditions, whose solutions are not (in general) stable. Here, we use the discrete-time feedback inputs through the boundary of geometric domain to the semilinear system under some additional conditions and assumptions. It is shown that under these conditions, the stabilization can be realized by applying pole assignment argument to the principal linear part of the system and that the solutions exist globally in discrete-time t without any finite escape time.

1. Introduction

Remarkable advances in microprocessor technology have led to increasing interest in discrete-time control systems. Recently, several studies have appeared on the approach to the discrete-time formulation of systems of distributed parameter and their control.

In [1], they discussed an abstract approximation framework for linear quadratic regulator problems for systems whose state are described by a linear semigroup of operators on a Hilbert space. A design method of a digital PI-Controller was presented by [2] within the same framework. The discrete observability of the heat equation was investigated in [5].

In the present paper, we treat the stabilization problem of semilinear heat equations by a certain type of discrete boundary control, meanwhile the continous time boundary stabilization was investigated in details by [6] for the linear parabolic equations.

We provide a brief outline of the paper:

In Sec. 2 and 3, the equation of a control system is described, which is associated with an inhomogenous boundary (control) condition. By the assumption of the piecewise constant control and by changing of variables of state, we rewrite the system equation to the new one with the homogenous boundary condition in each sampling period τ, which leads us to the iterative formula of discretes state of the system. In Sec. 4, we apply the pole assignment argument to the principal linear part of the discrete system and then we prove that the semilinear heat process is stabilizable by using the contraction mapping principle. In Sec. 5, several examples are given with a different kind of nonlinearity.

2. System Description

2-1 Continous-time System

We consider the semilinear heat equation

$$(2.1)\qquad \frac{\partial w(s,x)}{\partial s} = \frac{\partial\left(a(x)\frac{\partial w(s,x)}{\partial x}\right)}{\partial x} + f(w(s,x)) \quad ,$$

where $x \in (0,1)$, $s > 0$, and $f(w(s,x))$ is nonlinear, with inhomogenous boundary conditions

(2.2a)

$$w(s,0) = 0, \quad w(s,1) = v(s) \quad \text{(boundary control)}$$

and initial condition

$$(2.2b)\qquad w(0,x) = \varphi(x) \quad .$$

The state space is considered to be $L_2(0,1)$ (denoted by H). We make the assumptions on $a(\cdot), v(\cdot)$ $\varphi(\cdot)$ and $f(\cdot)$ as follows:

$$(2.3)\begin{cases} a(\cdot) \in H^1(0,1) \quad \text{(Sobolev space of order 1)}, \\ a(x) \geq \bar{a} > 0 \quad \text{for} \quad x \in [0,1], \\ v(\cdot) \in L_2(0,\infty), \\ \varphi(\cdot) \in H, \\ f : H \to H \quad \text{is Lipschitz continuous in} \quad w \\ \text{with} \quad f(0) = 0. \end{cases}$$

We set a variable as follows:

$$(2.4)\qquad y(s,x) \equiv w(s,x) - R(x)v(s),$$

where $R(x) \equiv x$, which enables us to transform the system with the inhomogenous boundary condition, (2.1)-(2.2), into the system with homogenous boundary condition. After this transformation, it is clear that the system becomes:

(2.5)

$$\frac{\partial y(s,x)}{\partial s} = \frac{\partial\left(a(x)\frac{\partial y(s,x)}{\partial x}\right)}{\partial x} + \frac{\partial a(x)}{\partial x}v(s) + f(y(s,x) + R(x)v(s)) - R(x)\frac{\partial v(s)}{\partial s},$$

$$(2.6)\qquad y(s,0) = 0 \quad \text{and} \quad y(s,1) = 0,.$$

(homogenous boundary condition)

$$(2.7)\ y(0,x) = w(0,x) - R(x)v(0) = \varphi(x) - R(x)v(0).$$

We define an operator A as follows:

$$A \equiv \partial(a\partial(\cdot)), \quad A : D(A) \subset H \to H,$$

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where the domain $D(A)$ of the operator A is $H_0^1(0,1) \cap H^2(0,1)$ and ∂ is a differential operator on $H^1(0,1)$. The operator A is proved to be densely defined, self-adjoint and satisfies

$$< Aw, w > \leq \omega \|w\|^2, \quad w \in D(A) \tag{2.8}$$

for some negative ω, where $< \cdot, \cdot >$ denotes the inner product in H.

Furthermore it has compact resolvent, and therefore it has the point spectrum consisting of a sequence of eigenvalues λ_i satisfying

$$0 \geq \lambda_1 \geq \lambda_2 \ldots, \qquad \lambda_i \to -\infty \quad \text{as} \quad i \to \infty.$$

Moreover, it can be shown that A is the infinitesimal generator of an analytic semigroup of contractions $\{T(s);\ s \geq 0\}$ on H with

$$\|T(s)\| \leq e^{\omega s} \quad \text{for} \quad s \geq 0.$$

2-2 Discrete-time System

In continous-time control system, the feedback control is supposed to be continous in time, but when the feedback control input is generated by computer outputs, we can regard $v(s)$ as discrete (in time) signals added to the control system.

For the discrete-time feedback input

$$v(s) \equiv u(t) \quad \text{for} \quad s \in [t\tau, (t+1)\tau), \quad t = 0, 1, 2, \ldots \tag{2.9}$$

which we call a piecewise constant control, we define the discrete states as follows:

$$\theta(t\tau) \equiv \lim_{s \to t\tau - 0} w(s, \cdot) \quad \text{for} \quad t = 1, 2, \ldots \quad \text{and} \quad \theta(0) = \varphi. \tag{2.10}$$

From the above assumptions and definitions, it reveals that $y(s) = y(s, \cdot)$ satisfies

$$\begin{aligned} \dot{y}(s) =& \partial(a\partial y(s)) + \partial a u(t) \\ &+ f(y(s)) + Ru(t), \end{aligned} \tag{2.11}$$

$$y(s)|_0 = 0, \quad y(s)|_1 = 0 \quad \text{for} \quad s \in (t\tau, (t+1)\tau), \tag{2.12}$$

and

$$y(t\tau) = \theta(t\tau) - Ru(t). \tag{2.13}$$

3. Analysis of Discrete-time System

From (2.4), (2.9) and (2.10), it follows that

$$\theta((t+1)\tau) = \lim_{s \to (t+1)\tau - 0} y(s) + Ru(t). \tag{3.1}$$

On the other hand, by making use of the variation of constant formula, we have

$$\begin{aligned} y(s) =& T(s - t\tau)y(t\tau) + \int_{t\tau}^{s} T(s-p)\partial a \cdot u(t)dp \\ &+ \int_{t\tau}^{s} T(s-p)f(y(p) + Ru(t))dp \\ &\text{for} \quad s \in [t\tau, (t+1)\tau). \end{aligned} \tag{3.2}$$

Passing the limit of s to $(t+1)\tau$ in both sides of equation (3.2), we get the following equation:

$$\begin{aligned} \theta((t+1)\tau) =& T(\tau)\{\theta(t\tau) - Ru(t)\} + Ru(t) \\ &+ \int_{t\tau}^{(t+1)\tau} T((t+1)\tau - p)\partial a \cdot u(t)dp \\ &+ \int_{t\tau}^{(t+1)\tau} T((t+1)\tau - p)f(w(p))dp. \end{aligned} \tag{3.3}$$

If we can assume, in addition to (2.3), that

$$\int_{t\tau}^{(t+1)\tau} T(p)f(w(p))dp \simeq \int_{t\tau}^{(t+1)\tau} T(p)f(\theta(t\tau))dp \tag{3.4}$$

and

$$\|f(\theta(t\tau))\| \leq \alpha \|\theta(t\tau)\|^n \quad (\exists n > 1 \quad \text{and} \quad \exists \alpha > 0), \tag{3.5}$$

then we have the iterative discrete-time system of the form:

$$\begin{aligned} \theta((t+1)\tau) =& T(\tau)\theta(t\tau) + [I - T(\tau)]Ru(t) \\ &+ \int_{t\tau}^{(t+1)\tau} T((t+1)\tau - p)\partial a \cdot u(t)dp \\ &+ \int_{t\tau}^{(t+1)\tau} T((t+1)\tau - p)f(\theta(t\tau))dp. \end{aligned} \tag{3.6}$$

From the definition of the operator A, it is easy to find that its inverse exists and

$$\begin{aligned} A^{-1}z(\xi) = &- \int_0^{\xi} \frac{\int_0^x \frac{1}{a(\eta)}d\eta \int_\xi^1 \frac{1}{a(\alpha)}d\alpha}{\int_0^1 \frac{1}{a(\beta)}d\beta} z(x)dx \\ &- \int_\xi^1 \frac{\int_x^1 \frac{1}{a(\eta)}d\eta \int_0^\xi \frac{1}{a(\alpha)}d\alpha}{\int_0^1 \frac{1}{a(\beta)}d\beta} z(x)dx \\ &(\forall z(\cdot) \in H, 0 < \xi < 1). \end{aligned} \tag{3.7}$$

Since A^{-1}exists, and A and T are commutative,it follows, from (3.6), that

$$\begin{aligned} \theta((t+1)\tau) =& T(\tau)\theta(t\tau) + [T(\tau) - I]A^{-1}f(\theta(t\tau)) \\ &+ [I - T(\tau)](R(\cdot) - A^{-1}\partial a) \cdot u(t). \end{aligned} \tag{3.8}$$

4. Stabilization by Pole Assignment Argument

4-1 Pole Assignment Argument for Linear System

The linear operator A is, as mentioned before, a self-adjoint operator with compact resolvent. The spectrum is bounded upwards, and there exists a sequence $\{\lambda_n, n = 1, 2, \dots\}$ of eigenvalues with corresponding orthonormal eigenfunctions ϕ_n such that

$$A\phi_n = \lambda_n \phi_n \quad n = 1, 2, \dots,$$

where $0 > \lambda_1 \geq \lambda_2 \geq \dots$, $\lim_{n\to\infty} \lambda_n = -\infty$.

For every θ in H, there is a unique representation such that

$$(4.1) \qquad \theta = \sum_{n=1}^{\infty} \theta_n \phi_n, \quad \theta_n = < \theta, \phi_n > .$$

Then, the semigroup $T(s)$ is given by

$$(4.2) \qquad T(s)\theta = \sum_{n=1}^{\infty} e^{\lambda_n s} \theta_n \phi_n \quad (\forall \theta \in H).$$

Here, consider the control system

$$(4.3a) \qquad \theta(t+1) = T\theta(t) + bu(t) \quad \text{(state equation)},$$

$$(4.3b) \qquad y(t) = < g, \theta(t) > \equiv C\theta(t) \quad \text{(output equation)},$$

where b $(= [T(\tau) - I](R(\cdot) - A^{-1}\partial a)$ in (3.9)) $\in H$, T $(= T(\tau)$ in (3.9)) and $g \in H$ is a sensor influence function.

In order to stabilize the system (4.3), we will contruct a N-dimensional dynamic compensator of the form

$$(4.4) \qquad \begin{cases} u(t) = F\xi(t) \\ \xi(t+1) = E\xi(t) + Gy(t), \xi(t) \in C^N, \end{cases}$$

where E, F and G are matrices to be determined. Consider the product space $H \times \boldsymbol{C}^N$ with the inner product

$$< \zeta_1, \zeta_2 >_{H\times C^N} = < \theta_1, \theta_2 > + (\xi_1, \xi_2)_N$$
$$\text{for} \quad \zeta_i = (\theta_i, \xi_i) \in H \times C^N, \quad i = 1, 2, \dots,$$

where $(\cdot,\cdot)_N$ is the usual inner product of the euclidean space $\boldsymbol{C}^N$. Then the closed loop system constituted by (4.3)-(4.4) is represented by the evolution equation

$$(4.5) \qquad \zeta(t+1) = B\zeta(t), \quad \zeta(0) = \zeta_0,$$

where

$$B = \begin{bmatrix} T & bF \\ GC & E \end{bmatrix},$$

$$\zeta(t) = \begin{bmatrix} \theta(t) \\ \xi(t) \end{bmatrix}.$$

Our goal is to determine the matrices E, F and G, and the dimension N such that for given $0 < \sigma < 1$, the solution $\zeta(t)$ of (4.5) satisfies

$$(4.6) \qquad \|\zeta(t)\|_{H\times C^N} \leq M\sigma^t \|\zeta_0\|_{H\times C^N}, \quad t = 0, 1, \dots ,$$

for some $M \geq 1$. The procedure is in four steps.

Step1 Fix any $\eta > 0$. Take an positive integer p such that

$$e^{\lambda_i \tau} < \sigma - \eta \quad \text{for any} \quad i > p,$$

where $e^{\lambda_i \tau}$ is the eigenvalue of T. Since

$$1 > e^{\lambda_1 \tau} \geq e^{\lambda_2 \tau} \geq \cdots, \quad \lim_{i\to\infty} e^{\lambda_1 \tau} \to 0,$$

such a number p always exists.

Step2 For each $k \in \mathbf{N}$, define the matrices

$$T_k = \begin{bmatrix} e^{\lambda_1 \tau} & & & \mathbf{0} \\ & e^{\lambda_2 \tau} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\lambda_k \tau} \end{bmatrix} \in \boldsymbol{C}^{k\times k},$$

$$b_k = \begin{bmatrix} < b, \varphi_1 > \\ \vdots \\ < b, \varphi_k > \end{bmatrix} \in \boldsymbol{C}^{k\times 1}$$

$$C_k = [< g, \varphi_1 > \dots < g, \varphi_k >] \in \boldsymbol{C}^{1\times k},$$

where φ_i is the eigenvector of A (consequently, of $T(\tau)$) corresponding to λ_i (or $e^{\lambda_i \tau}$)

Before going to the next step, we need an assumption:

Assumption 4.1. *For p taken in Step1, the system (T_p, b_p, C_p) is controllable and observable.*

Step3 Find matrices F_p and G_p such that

$$(4.7) \qquad \begin{cases} max|\sigma(T_p + b_p F_p)| \leq \sigma - \eta, \\ max|\sigma(T_p + G_p C_p)| \leq \sigma - \eta, \end{cases}$$

where $\sigma(\cdot)$ denotes the spectrum of an operator. By virtue of Assumption4.1, one can employ, for example, the method of pole assignment to contruct F_p and G_p satisfying (4.7).

Step4 Determine the coefficient matrices F, G and E of the N-dimensional compensator (4.4) as

$$(4.8) \qquad \begin{cases} F = [F_p \quad 0] \in \boldsymbol{C}^{1\times N}, \\ G = \begin{bmatrix} G_p \\ 0 \end{bmatrix} \in \boldsymbol{C}^{N\times 1}, \\ E = T_N + b_N F + GC_N \in \boldsymbol{C}^{N\times N}. \end{cases}$$

Lemma 1. *For any $0 < \sigma < 1$, if Assumption 4.1 is satisfied, there is a number N such that the state $\zeta(t)$ of closed loop system (4.5) in $H \times \boldsymbol{C}^N$ with the matrices of (4.8) is estimated by (4.6). Consequently, we have*

$$(4.9) \qquad \|\theta((t+1)\tau)\| \leq \sigma \|\theta(t\tau)\|, \quad t = 0, 1, \dots .$$

Proof: Omitted

4-2 Stabilization of Semilinear System

Lemma 2.

Let $\gamma < 1$, $1 < n$ and $ab > 0$. If a sequence $\{y_t; t = 0, 1, 2, \dots\}$ of positive real numbers satisfies

$$y_{t+1} \leq \gamma y_t + ab y_t^n$$

and

$$0 < y_0 < (\frac{1-\gamma}{\alpha b})^{\frac{1}{n-1}},$$

then for any positive η_1 *with* $0 < \gamma + \eta_1 < 1$, *it holds that*

$$y_t \le M(\gamma+\eta_1)^t y_0 \quad (\exists M \ge 1) \tag{4.10}$$

for some $M \ge 1$ *independent of* t.

Proof:

Consider the iterative system:

$$z_{t+1} = \gamma z_t + \alpha b z_t^n \tag{4.11}$$

where $n > 1$, $\gamma < 1$, $\alpha b > 0$ and $z_t > 0$ for $t = 0, 1, 2, \ldots$, and the initial value is

$$0 < z_0 < (\frac{1-\gamma}{\alpha b})^{\frac{1}{n-1}}.$$

If we set

$$f(z_t) = \gamma z_t + \alpha b z_t^n,$$

then we get

$$f(z_t) < z_t \quad \text{for} \quad 0 < z_t < (\frac{1-\gamma}{\alpha b})^{\frac{1}{n-1}}.$$

From the initial value condition, then z_t is a monotonously decreasing sequence. It is also bounded downwards. Hence it is a convergent sequence and if we put z as its limit , then $z = f(z)$ .Thus $z = 0$, and from the above we get the result that

$$0 < z_0 < (\frac{1-\gamma}{\alpha b})^{\frac{1}{n-1}} \Rightarrow z_t \to 0 \quad \text{as} \quad t \to \infty. \tag{4.12}$$

Here, we set $x_t = \frac{z_t}{(\gamma+\eta_1)^t z_0}$ and z_0 satisfies the condition $z_0 < (\frac{1-\gamma}{\alpha b})^{\frac{1}{n-1}}$. From (4.11),we have

$$\begin{aligned} x_{t+1} &= \frac{z_{t+1}}{(\gamma+\eta_1)^{t+1} z_0} \\ &= \frac{\gamma x_t}{\gamma+\eta_1} + \gamma_t x_t^n \\ &\quad (\text{where} \quad \gamma_t = \alpha b(\gamma+\eta_1)^{tn-t-1} z_0^{n-1}) \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} x_t - x_{t+1} &= x_t(\frac{\eta_1}{\gamma+\eta_1} + \gamma_t x_t^{n-1}). \\ &= \frac{x_t}{\gamma+\eta_1}(\eta_1 - b\alpha z_t^{n-1}) \end{aligned} \tag{4.14}$$

Since $z_t^{n-1} \to 0$ as $t \to \infty$, there is an N' large enough such that for any $t \ge N'$,

$$x_{t+1} \le x_t. \tag{4.15}$$

Therefore, x_t decreases monotonously. Moreover, since x_t is a positive value, for arbitrary t, the sequence is bounded downwards. Hence, it is a convergent sequence and we can set

$$\lim_{t\to\infty} x_t = x (< \infty). \tag{4.16}$$

Passing the limit of t to ∞ in both sides of equation (4.14),we get

$$x = \frac{\gamma}{\gamma+\eta_1} x \quad \Rightarrow \eta_1 x = 0, \tag{4.17}$$

and thus $x = 0$, which implies that there exists an N'' large enough such that for $t > N''$

$$x_t \le 1, \tag{4.18}$$

and then from (4.13), we get

$$z_{t+1} \le (\gamma+\eta_1)^{t+1} z_0. \tag{4.19}$$

Therefore,

$$\exists M \ge 1 \quad s.t. \quad z_t \le M(\gamma+\eta_1)^{t+1} z_0 \quad \text{for all} \quad t.$$

For the system

$$y_{t+1} \le \gamma y_t + \alpha b y_t^n \quad (n > 1, \quad \gamma < 1 \quad \text{and} \quad \alpha b > 0),$$

we set that $z_0 = y_0$ and get the inequality

$$\begin{aligned} y_t &\le z_t \\ &\le M(\gamma+\eta_1)^t y_0 \\ &\quad \text{for} \quad t = 1, 2, \ldots \end{aligned}$$

Proposition 3.

If the initial data φ *in (2.2b) is restricted to the ball of radius* δ *in* H *such that*

$$\begin{aligned} \|\varphi\| &\le \delta \\ &\equiv \{\frac{(1-\sigma+\eta_1)\bar{a}}{\alpha(1+e^{\lambda_1 \tau})}\}^{\frac{1}{n-1}}, \end{aligned} \tag{4.20}$$

then there is a constant $M \ge 1$ *with the following property. The norm of the states in equation (3.8) is estimated as*

$$\|\theta((t+1)\tau)\| \le M\sigma^{t+1}\|\varphi\|$$

for $t = 0, 1, 2, \ldots$, *where* $\sigma - \eta_1 < 1$, $\bar{a}$ *is a constant in (2.3),* λ_1 *is the first eigenvalue of* A *,*τ *is the sampling periods, and* n *and* α *stand for the characteristics of nonlinearity in (3.5).*

Proof: We can get the result directly by applying Lemma 1 and Lemma 2, and the proof is omitted here.

5. Examples

5-1 The case of $f(w(s,x)) = w(s,x) \cdot sin(w(s,x))$

The system is governed by

$$\frac{\partial w(s,x)}{\partial s} = \frac{\partial(a(x)\frac{\partial w(s,x)}{\partial x})}{\partial x} + w(s,x) \cdot sin(w(s,x)) \tag{5.11}$$

where $x \in (0,1)$ and $s > 0$ .and the boundary and initial conditions are same as (2.2a) and (2.2b). By applying the arguments in Sec.2 and Sec.3, we get the following equation:

(5.12)

$$\theta((t+1)\tau) = T(\tau)\theta(t\tau) + [T(\tau) - I]A^{-1}\theta(t\tau)\cdot sin(\theta(t\tau)) + [I - T(\tau)](R - A^{-1}\partial a)\cdot u(t).$$

Using the pole assignment argument and the estimate of $\|\theta(t\tau)\cdot sin(\theta(t\tau))\| \leq \|\theta(t\tau)\|^2$,we obtain the following inequality:

(5.13)

$$\begin{aligned}\|\theta((t+1)\tau)\| &\leq (\sigma - \eta_1)\|\theta(t\tau)\| + (1+e^{\lambda_1\tau})\|A^{-1}\|\cdot\|\theta(t\tau)\|^2 \\ &\leq (\sigma-\eta_1)\|\theta(t\tau)\| + (1+e^{\lambda_1\tau})\frac{1}{\bar{a}}\|\theta(t\tau)\|^2.\end{aligned}$$

Here, if the condition of Proposition 3 is satisfied, we can see that the system is stablized asymptotically by the boundary input.

5-2 The case of $f(w(s,x)) = w^2(s,x)$

Although this system doesn't statisfy the condition of assumption (3.5), the asymptotic stability is still assured by the initial condition $w(0)$ subjected to the restriction given by (4.20) .

The following equation is obtained by going through the Sec.2 and Sec.3.

(5.21)

$$\begin{aligned}\theta((t+1)\tau) &= T(\tau)\{\theta(t\tau) - Ru(t)\} + Ru(t) \\ &\quad + \int_{t\tau}^{(t+1)\tau} T((t+1)\tau - p)\partial a\cdot u(t)dp \\ &\quad + \int_{t\tau}^{(t+1)\tau} T((t+1)\tau - p)\theta^2(p)dp \\ &= T(\tau)\theta(t\tau) + [T(\tau) - I]A^{-1}\theta^2(t\tau) \\ &\quad + [I - T(\tau)](R - A^{-1}\partial a)\cdot u(t).\end{aligned}$$

By applying the pole assignment argument and taking norms in both sides, we get

$$\|\theta((t+1)\tau)\| \leq (\sigma - \eta_1)\|\theta(t\tau)\| + \|T(\tau) - I\| \cdot\|A^{-1}\theta^2(t\tau)\|. \tag{5.22}$$

Here, from (3.7) , we know that,

$$\begin{aligned}|A^{-1}\theta^2(t\tau)| &= |\int_0^\xi C_1(x)\theta^2(t\tau)dx| + |\int_\xi^1 C_2(x)\theta^2(t\tau)dx| \\ &\leq (\int_0^\xi {C_1}^2(x)\theta^2(t\tau)dx)^{\frac{1}{2}}(\int_0^\xi \theta^2(t\tau)dx)^{\frac{1}{2}} \\ &\quad + (\int_\xi^1 {C_2}^2(x)\theta^2(t\tau)dx)^{\frac{1}{2}}(\int_\xi^1 \theta^2(t\tau)dx)^{\frac{1}{2}} \\ &\leq \{(\int_0^\xi {C_1}^2(x)\theta^2(t\tau)dx)^{\frac{1}{2}} \\ &\quad + (\int_\xi^1 {C_2}^2(x)\theta^2(t\tau)dx)^{\frac{1}{2}}\}\cdot\|\theta(t\tau)\| \\ &\leq \{(\int_0^\xi (\int_\xi^1 \frac{1}{a(\alpha)}d\alpha)^2\theta^2(t\tau)dx)^{\frac{1}{2}} \\ &\quad + (\int_\xi^1 (\int_0^\xi \frac{1}{a(\alpha)}d\alpha)^2\theta^2(t\tau)dx)^{\frac{1}{2}}\}\cdot\|\theta(t\tau)\| \\ &\leq \{\int_\xi^1 \frac{1}{a(\alpha)}d\alpha + \int_0^\xi \frac{1}{a(\alpha)}d\alpha\}\|\theta(t\tau)\|^2 \\ &\leq \frac{1}{\bar{a}}\|\theta(t\tau)\|^2,\end{aligned}$$

where $C_1(x) = \frac{\int_0^x \frac{1}{a(\eta)}d\eta\int_\xi^1\frac{1}{a(\alpha)}d\alpha}{\int_0^1\frac{1}{a(\beta)}d\beta}$ and $C_2(x) = \frac{\int_x^1 \frac{1}{a(\eta)}d\eta\int_0^\xi\frac{1}{a(\alpha)}d\alpha}{\int_0^1\frac{1}{a(\beta)}d\beta}$.

Therefore the inequality becomes, finally,

$$\|\theta((t+1)\tau)\| \leq (\sigma-\eta_1)\|\theta(t\tau)\| + \|T(\tau) - I\| \cdot\frac{1}{\bar{a}}\|\theta(t\tau)\|^2.$$

This, together with Proposition 3,shows that the system is obviously asymptotically stable.

6. Conclusions

In this paper, we have studied the application of the boundary discrete-time input to stabilize the semilinear heat processes. First, we formulated the system with homogenous boundary condition from the system with inhomogenous boundary condition by changing variables of state, and then we derived the system of iterative discrete-time states. Next, from the iterative discrete-time system, we replaced a sequence of eigenvalues of the operator A with new points spectra by using pole assignment method. Proposition 3 reveals that under the condition that initial state (data) satisfies the inequality (4.20), then the semilinear heat processes can be stabilized asymptotically. Last, we showed 2 examples to clarify the main points of the whole discussions there.

We have not given any numerical results, in this paper, to check the approximation given by (3.4). Moreover, it seems to be reasonable that Proposition 3 cannot be held for the case $n = 1$ as far as we insists relying on the pole assignment argument for linear principal parts. Different approaches will be available to overcome these difficulties, such as nonlinear semigroup method and the second method of Liapunov.

References

1. J.S.Gibson & I.G.Rosen, *Numerical Approximation for the Infinite-Dimensional Discrete-Time Optimal Linear-Quadratic Regulator Problem*, SIAM J. Control and Optimization. **26**,**no 2** (March 1988).
2. Toshihiro Kobayashi, *A Digital PI-Controller for Distributed Parameter Systems*, SIAM J. Control and Optimization **26**, **no 6** (Nov. 1988).
3. A. Pazy, *Semigroup of Linear Operator and Application to Partial Differential Equations*, Springer-Verlag (1983).
4. J. A. Walker, *Dynamical Systems and Evolution Equations*, Prenum Press (1980).
5. David Gilliam, Zhu Li & Clyde Martin, *Discrete Observability of the Heat Equation on Bounded Domains*, Int. J. Control **48**,**no 2** (1988).
6. Takao Nambu, *On Stabilization of Partial Differential Equations of Parabolic Type: Boundary Observation and Feedback*, FUNKCIALAJ EKVACIOJ (SERIO INTERNACIA) **28**,**no 3** (Dec. 1985).
7. A. V. Balakrishnan, *Applied Functional Analysis*, Springer-Verlag (1976).