LTI MODEL REALIZATION PROBLEM OF LINEAR PERIODIC DISCRETE-TIME SYTEMS

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ABSTRACT

In this paper, we consider linear periodic discrete-time control systems under periodic compensation. Such a closed-loop system generally represents a periodic time-varying system. We examine the problem of finding a compensator such that the closed-loop system is realized as LTI model (if possible) with the closed-loop stability being satisfied. We present a necessary and sufficient condition for solving such problem and also give the characterization of realizable LTI models.

1. INTRODUCTION

Linear periodic systems have been widely studied as an important class of linear systems. The main results on linear periodic systems, for example, structural analysis [1,2], eigenvalue assignment [3,4], optimal control [5] etc. are obtained based on state-space representation. Also, the transfer function approach for linear periodic system has been proposed in study of unified analysis [6], robust control [7], sensitivity optimization [8] and so on.

In this paper, we follow the transfer function approach to consider linear periodic discrete-time (LPDT) systems described by following linear difference equations:

$$y(k)+\sum_{L=1}^{nk} a_L(k)y(k-L)=\sum_{L=0}^{mk} b_L(k)u(k-L) \tag{1}$$

or

$$y(k)=\sum_{L=0}^{mk} b_L(k)\xi(k-L) \tag{2-a}$$

$$\xi(k)+\sum_{L=1}^{nk} a_L(k)\xi(k-L)=u(k) \tag{2-b}$$

where $u(k)\in R^M$, $y(k)\in R^P$ are the system input, output respectively and the coefficients in both (1) and (2) vary periodically with period N, i.e., $a_L(k+N)=a_L(k)$ and $b_L(k+N)=b_L(k)$.

We examine the problem of matching an LPDT system with periodic compensator to an LTI model, termed LTI model realization, i.e., finding an LPDT compensator (if possible) such that the closed-loop system is realized as desired LTI model with the closed-loop stability satisfied. When the plant is LTI, it is well-known that LTI model realization is possible if and only if the desired model contains the unstable blocking zeros information of the plant. However, it is less known about the case of periodic (or time - varying) systems. In fact, different from the case of LTI systems, LTI model realization is not always possible when the plant is periodic or time-varying.

This problem is first stated by Sakes et al [10], in which only an example is shown that the closed-loop 2-periodic LPDT scalar system is realized as LTI via 2-periodic compensation. In this paper, we focus on general multi-input multi-output (MIMO) LPDT systems.

We show that for the transfer function of an LPDT system, there exists a special doubly coprime factorization such that every factor in it is lower (block) triangular when d=0 (d is the unit delay operator). Then the parametrization of all LPDT stabilizing controllers can be obtained using the known YJB-parametrization by just taking an extra constraint that the free parameter is chosen to be lower triangular when d=0. Further, we found the LTI model realization problem of LPDT systems can be treated by solving N matrix equations which very interestingly own the same solution space. Based on the solutions of those equations, we derive the necessary and sufficient condition for realizing the closed-loop LPDT system as LTI and the characterization of realizable LTI models.

As a special case of LPDT systems, LTI discrete system is also considered. We show that, in LTI model realization i.e., the model matching under periodic compensation, the merit of using periodic compensation is only in improving the stability as pointed by Khargonekar et al [7] and so on.

In the following, we denote the polynomial matrices by $\mathbb{M}$ and the rational polynomial matrices which have no poles in Λ (Λ is the unit disk, boundary included) by $\mathbb{M}_\Lambda$.

2. LPDT STABILIZATION

For LPDT systems (1) or (2), define

$$Y(k)=[\ y^T(kN)\ \ y^T(kN+1)\ \ \dots\ y^T(kN+N-1)\ ]^T \qquad (3)$$

$$U(k)=[\ u^T(kN)\ \ u^T(kN+1)\ \ \dots\ u^T(kN+N-1)\ ]^T$$

and the d-transform ($d=z^{-1}$) as

$$Y(d)=\sum_{k=0}^{\infty} d^k Y(k). \qquad (4)$$

Then the transfer relation of LPDT systems can be represented in the LTI form as

$$Y(d)=\mathfrak{G}(d)U(d) \qquad (5)$$

using the lifting technique (see [6], [7]). We call $\mathfrak{G}(d)$ as the transfer function of LPDT systems in the view of [8,9].

Proposition 1:

Given an M-input P-output causal LPDT system, one can canonically associate a PNxMN transfer function matrix $\mathfrak{G}(d)$ as in (5) which is lower (block) triangular when d=0. Conversely, given any PNxMN transfer function such that $\mathfrak{G}(0)$ is lower (block) triangular, there exists an unique N-periodic LPDT system in form of (1) or (2) which owns the same transfer relation. □

This fact motivates that the control problem of LPDT system can be analysed for $\mathfrak{G}(d)$ using the similar techniques known in LTI systems but with the transfer function of LPDT compensator satisfying Proposition 1.

Theorem 1:

Suppose $\mathfrak{G}$ is the transfer function of a causal LPDT system. Then there exists a doubly coprime factorization on $\mathfrak{G}$ such as

$$\mathfrak{G}=A_1^{-1}B_1=B_2A_2^{-1}\ ;\ A_i,\ B_i\in\mathbb{M} \qquad (6\text{-a})$$

and there exist X_i, $Y_i\in\mathbb{M}$ satisfying

$$\begin{bmatrix} X_2 & Y_2 \\ -B_1 & A_1 \end{bmatrix}=\begin{bmatrix} A_2 & -Y_1 \\ B_2 & X_1 \end{bmatrix}^{-1} \qquad (6\text{-b})$$

in which every factor is lower triangular when d=0.

Proof:

Using the known properties for matrix fraction, one can find a coprime factorization on $\mathfrak{G}$ such as

$$\mathfrak{G}=\bar{A}_1^{-1}\bar{B}_1=\bar{B}_2\bar{A}_2^{-1}\ ;\ \bar{A}_i,\ \bar{B}_i\in\mathbb{M}.$$

Let $\begin{cases} A_1=\bar{A}_1^{-1}(0)\bar{A}_1, \\ B_1=\bar{A}_1^{-1}(0)\bar{B}_1 \end{cases}$

and $\begin{cases} A_2=\bar{A}_2\bar{A}_2^{-1}(0) \\ B_2=\bar{B}_2\bar{A}_2^{-1}(0). \end{cases}$

Then we get a new coprime factorization such as $\mathfrak{G}=A_1^{-1}B_1=B_2A_2^{-1}$ where A_i, $B_i\in\mathbb{M}$ are lower triangular when d=0 since $\mathfrak{G}(0)$ is lower triangular.

So there exist x_0, $y_0\in\mathbb{M}$ which satisfy the Bezout identity

$$A_1X+B_1Y=I$$

and the general solutions of the above equation over $\mathbb{M}_\Lambda$ are derived in the form of

$$\begin{cases} X=x_0-B_2T \\ Y=y_0+A_2T \end{cases},\ T\in\mathbb{M}_\Lambda\text{: arbitrary.} \qquad (7)$$

Taking $T=-A_2^{-1}(0)\cdot y_0\in\mathbb{M}(\in\mathbb{M}_\Lambda)$ in (7), we have solution x_1, $y_1\in\mathbb{M}$ such as

$$\begin{cases} x_1=x_0+B_2\cdot A_2^{-1}(0)\cdot y_0 \\ y_1=y_0-A_2\cdot A_2^{-1}(0)\cdot y_0 \end{cases}$$

which are lower triangular when d=0 .

Similarly, there exists x_2, $y_2\in\mathbb{M}$ (lower triangular when d=0) satisfying $XA_2+YB_2=I$. So we have

$$\begin{bmatrix} x_2 & y_2 \\ -B_1 & A_1 \end{bmatrix}\begin{bmatrix} A_2 & -y_1 \\ B_2 & x_1 \end{bmatrix}=\begin{bmatrix} I & y_2x_1-x_2y_1 \\ 0 & I \end{bmatrix},$$

where $\Delta=y_2x_1-x_2y_1$ is lower triangular when d=0.

Defining $\begin{cases} A_2\Delta+y_1\to Y_1 \\ x_1-B_2\Delta\to X_1 \end{cases}$ and $\begin{cases} y_2\to Y_2 \\ x_2\to X_2, \end{cases}$

we then obtain the desired factorization.

Q.E.D.

Remarks:

In fact, the existence of such factorization has been pointed by Feintuch et al [8] via time-domain analysis. Here, we provided the calculation method based on Λ-generalized matrix. □

Let $\mathfrak{G}$ and $\mathfrak{C}$ denote the transfer functions of LPDT plant and LPDT stabilizing controller respectively. Then the LPDT stabilizing problem is to parametrize the class of stabilizing controllers $\mathfrak{C}$ ($\mathfrak{C}(0)$ are lower triangular) which make the LPDT closed-loop system stable. Suppose $\mathfrak{G}$ owns the doubly coprime factorization shown in Theorem 1 and $\mathfrak{C}$ has a coprime factorization

$$\mathfrak{C}=P^{-1}Q=\tilde{Q}\tilde{P}^{-1};\ P,\ Q,\ \tilde{P},\ \tilde{Q}\in\mathbb{M}_\Lambda \qquad (8)$$

where det $P\neq0$ and det $\tilde{P}\neq0$. Then, the stabilizing problem is equivalently turned to solving the Bezout equations (see [11, 12]) :

$$PA_2+QB_2=I \quad \text{or} \quad A_1\tilde{P}+B_1\tilde{Q}=I\ . \qquad (9)$$

Based on Theorem 1, we have:

Theorem 2:

The class of all LPDT stabilizing controllers is parametrized in terms of T as

$$\tilde{C}=(Y_1-A_2T)(X_1+B_2T)^{-1} \quad (10\text{-a})$$

$$=(X_2+TB_1)^{-1}(Y_2-A_1T)\ ;T\in\mathbb{M}_A \quad (10\text{-b})$$

where $\det(Y_1-A_2T)\neq 0$, $(Y_2-A_1T)\neq 0$, but T(d) is lower triangular when d=0.

Proof:

Observing that the class (10) is YJB-parametrization, then it suffices to prove why T(0) in (10) must be lower triangular.

Suppose $\tilde{C}_o$ is an arbitrary LPDT stabilizing controller ($\tilde{C}_o(0)$ is lower triangular). Then we have that $\{I+\tilde{C}_o\tilde{G}\}^{-1}\in\mathbb{M}_A$ is lower triangular when d=0.

In case of (10-a), simple calculation shows

$$\{I+\tilde{C}_oB_2A_2^{-1}\}A_2T=Y_1-\tilde{C}_oX_1$$

for some $T\in\mathbb{M}_A$. When d=0,

$$T(0)=A_2^{-1}(0)\{I+\tilde{C}_oB_2A_2^{-1}\}^{-1}(0)\{Y_1(0)-\tilde{C}_o(0)X_1(0)\}$$

is trivially lower triangular.

The same conclusion exists also for the case of (10-b).

Q.E.D.

Using the LPDT stabilizing controllers of (10), various control problems can be treated by determining free parameter T (T(0) is lower triangular) without considering the internal stability [12].

3. LTI MODEL REALIZATION

Consider the closed-loop LPDT feedback system shown in Fig.1 ($r(k)\in R^q$, $u(k)\in R^M$, $y(k)\in R^P$). Using the LPDT stabilizing controllers given by Theorem 2, we examine the LTI model realization of LPDT systems.

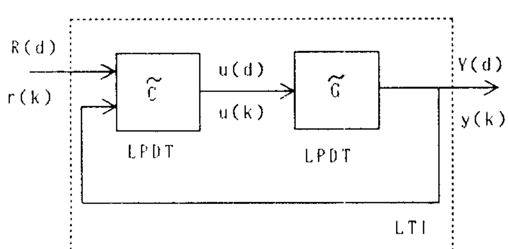

Fig.1: LPDT feedback systems

Suppose the input/output equation of LPDT compensator $\tilde{C}$ is given by

$$P\cdot U(d)=-Q\cdot Y(d)+K\cdot R(d) \quad (11)$$

where P, $[-Q\ K]\in\mathbb{M}_A$ are left coprime. Then from (10), there exists U_o such that

$$P=U_o(X_2+TB_1)$$

$$Q=U_o(Y_2-TA_1) \quad (12)$$

$$K=U_o\cdot\tilde{K}\quad ;U_o\text{: unimodular}$$

where T, $\tilde{K}\in\mathbb{M}_A$ are lower triangular when d=0 to ensure the causality of LPDT compensators.

The closed-loop transfer function $R(d)\rightarrow Y(d)$ in Fig.1 is

$$G_{CL}=B_2\cdot\tilde{K}\ ;\tilde{K}\in\mathbb{M}_A \quad (13)$$

where $\tilde{K}(0)$ is lower triangular. In general, G_{CL} represents a periodically time-varying system. The LTI model realization problem of finding an LPDT compensator $\tilde{C}$ such that the closed-loop system is realized as LTI, can then be interpreted as the problem of picking up the subclass of (13) which represents a set of LTI models (if it is not empty).

Let $G_m\in\mathbb{M}_A$ be a non-zero LTI model. By viewing G_m as N-periodic system, we have the transfer function of the model under N-periodic description in the form of

$$G_L(d)=\begin{bmatrix} G_1(d) & dG_N(d) & \dots & dG_2(d)\\ G_2(d) & G_1(d) & \dots & dG_3(d)\\ \dots & & & \\ G_N(d) & G_{N-1}(d) & \dots & G_1(d)\end{bmatrix} \quad (14)$$

where $G_1, G_2, \dots, G_N$ are defined from

$$G_m(\lambda)=G_1(\lambda^N)+\lambda^1G_2(\lambda^N)+\dots+\lambda^{N-1}G_N(\lambda^N)$$

($\lambda=z^{-1}$) as shown by Khargonekar et al [7]. Then the plant $\tilde{G}$ is said to be LTI model realizable if there exists $G_L\in\mathbb{M}_A$ ($\neq 0$) as in (14) such that

$$G_L=B_2\cdot\tilde{K} \quad (15)$$

being satisfied for some $\tilde{K}\in\mathbb{M}_A$ ($\tilde{K}(d)$ is lower triangular when d=0).

For any $\tilde{K}\in\mathbb{M}_A$ ($\tilde{K}(0)$ is lower triangular), we let

$$\tilde{K}(i,j)=\begin{cases} dK_{ij} & ,\ i<j\\ K_{ij} & ,\ i\geqq j\ .\end{cases}\quad ;K_{ij}\in\mathbb{M}_A \quad (16\text{-a})$$

and define

$$K_j=[K_{1j}^T\ K_{2j}^T\ \dots\ K_{Nj}^T]^T,(i,j=1\dots N). \quad (16\text{-b})$$

If LPDT plant $\tilde{G}$ is LTI model realizable, then (15) satisfies for some $G_L\in\mathbb{M}_A$ ($\neq 0$). Using definition (16), we equivalently transfer (15) into

$$b_1K_1=b_2K_2=\dots=b_NK_N=G_o \quad (17)$$

where

$$b_1=B_2,$$

$$b_j=U_2^{j-1}b_1V_j,\ j=2\dots N$$

$$b_1=U_2^Nb_1V_{N+1},$$

$$U_2=\begin{bmatrix}0 & I_{N-1}\\ d^{-1}I_1 & 0\end{bmatrix}\ \text{NxN block}, \quad (18)$$

$$V_{j+1}=\begin{bmatrix}dI_j & 0\\ 0 & I_{N-j}\end{bmatrix}\ \text{NxN block, } j=1\dots N$$

and $G_o=[G_1^T\ G_2^T\ \dots\ G_N^T]^T$.

Then solving $K_1, K_2\dots K_N\in\mathbb{M}_A$ which satisfy (17), the condition for LTI model realization and the characterization of realizable LTI models can be trivially derived. And in fact, we found that (17) can be divided into following equations:

$$U_2^1(U_2^{-1}B_2\ \ -B_2)\begin{bmatrix}\bar{K}_1\\ \bar{K}_2\end{bmatrix}=0$$
$$U_2^2(U_2^{-1}B_2\ \ -B_2)\begin{bmatrix}\bar{K}_2\\ \bar{K}_3\end{bmatrix}=0$$
$$\dots \tag{19}$$
$$U_2^{N-1}(U_2^{-1}B_2\ \ -B_2)\begin{bmatrix}\bar{K}_{N-1}\\ \bar{K}_N\end{bmatrix}=0$$
$$U_2^{N}(U_2^{-1}B_2\ \ -B_2)\begin{bmatrix}\bar{K}_N\\ d\bar{K}_1\end{bmatrix}=0$$

and

$$G_0=1/N\cdot(b_1\ b_2\ \dots\ b_N)\,(K_1^T\ K_2^T\ \dots\ K_N^T)^T \tag{20}$$
$$=1/N\cdot(B_2\ U_2^1B_2\ \dots\ U_2^{N-1}B_2)(\bar{K}_1^T\ \bar{K}_2^T\dots\ \bar{K}_N^T)^T$$

where $\bar{K}_1=K_1$

$$\bar{K}_j=V_jK_j;\ j=2\dots N. \tag{21}$$

Note that, since U_2 is non-singular, N matrix equations (19) have the same solution space with the equation :

$$(U_2^{-1}B_2\ \ -B_2)X=0\ ;\ X\text{: with suitable size.} \tag{22}$$

This is a very important fact in deriving the condition for LTI model realization.

Let the general solution of (22) be denoted by

$$X=\hat{v}\cdot\omega;\ \omega\,\varepsilon\,\mathbb{M}_\Lambda\text{: arbitrary} \tag{23}$$

where $\hat{v}\,\varepsilon\,\mathbb{M}_\Lambda$ is a special solution of (22) with independent columns and there exists a matrix $\hat{v}^*\,\varepsilon\,\mathbb{M}_\Lambda$ such that $\hat{v}^*\cdot\hat{v}=I$.

From (19), we have

$$\begin{bmatrix}\bar{K}_1\\ \bar{K}_2\end{bmatrix}=\hat{v}\cdot T_1\ =\ \begin{bmatrix}\hat{v}_1T_1\\ \hat{v}_2T_1\end{bmatrix}\begin{matrix}\}\ 1/2\\ \}\ 1/2\end{matrix}$$
$$\begin{bmatrix}\bar{K}_2\\ \bar{K}_3\end{bmatrix}=\hat{v}\cdot T_2\ =\ \begin{bmatrix}\hat{v}_1T_2\\ \hat{v}_2T_2\end{bmatrix}$$
$$\dots \tag{24}$$
$$\begin{bmatrix}\bar{K}_{N-1}\\ \bar{K}_N\end{bmatrix}=\hat{v}\cdot T_{N-1}=\ \begin{bmatrix}\hat{v}_1T_{N-1}\\ \hat{v}_2T_{N-1}\end{bmatrix}$$

and

$$\begin{bmatrix}\bar{K}_N\\ d\bar{K}_1\end{bmatrix}=\hat{v}\cdot T_N\ =\ \begin{bmatrix}\hat{v}_1T_N\\ \hat{v}_2T_N\end{bmatrix}$$

where $T_1,\ T_2\dots T_N\,\varepsilon\,\mathbb{M}_\Lambda$ with suitable size and

$$\hat{v}=[\hat{v}_1^T\ \ \hat{v}_2^T]^T.$$

However the parameters $T_1,\ T_2\dots T_N$ in (24) can not be chosen freely, they must be determined in such a way that (24) stand simultaneously. Rearranging (24), we have

$$\begin{cases}\bar{K}_1=d^{-1}\hat{v}_2\cdot T_N=\hat{v}_1\cdot T_1\\ \bar{K}_2=\ \hat{v}_2\cdot T_1\ \ \ =\hat{v}_1\cdot T_2\\ \quad\dots\\ \bar{K}_N=\ \hat{v}_2\cdot T_{N-1}\ =\hat{v}_1\cdot T_N,\end{cases} \tag{25}$$

i.e., $(d\bar{v}_1-\bar{v}_2)(T_1^T\ T_2^T\ \dots\ T_N^T)^T=0$ (26-a)

and

$$(\bar{K}_1^T\ \bar{K}_2^T\ \dots\ \bar{K}_N^T)^T=\bar{v}_1\cdot(T_1^T\ T_2^T\ \dots T_N^T)^T \tag{26-b}$$

where

$$\bar{v}_1=\text{diag}(\hat{v}_1\ \hat{v}_1\ \dots\ \hat{v}_1)$$
$$\bar{v}_2=\left[\begin{array}{cccc|c}0&0&\dots&0&\hat{v}_2\\ \hline d\hat{v}_2&0&\dots&0&0\\ &\dots&&&\\ 0&0&\dots&d\hat{v}_2&0\end{array}\right]. \tag{27}$$

Similar to (22), let the general solution of (26-a) be given as

$$(T_1^T\ T_2^T\ \dots\ T_N^T)^T=w_0\cdot\omega;\omega\,\varepsilon\,\mathbb{M}_\Lambda\text{ : arbitrary} \tag{28}$$

where $w_0\,\varepsilon\,\mathbb{M}_\Lambda$ is a special solution of (26-a) with independent column and there exists $w_0^*\,\varepsilon\,\mathbb{M}_\Lambda$ such that $w_0^*\cdot w_0=I$. So we have

$$(K_1^T\ K_2^T\ \dots\ K_N^T)^T=\text{diag}^{-1}\{\ I\ V_2\ \dots V_N\}\cdot(\bar{K}_1^T\ \bar{K}_2^T\ \dots\ \bar{K}_N^T)^T$$
$$=V\cdot\omega;\omega\,\varepsilon\,\mathbb{M}_\Lambda \tag{29}$$

in (16) and also

$$G_0=1/N\cdot(b_1\ \ b_2\ \dots\ b_N)\,(K_1^T\ \ K_2^T\ \dots\ K_N^T)^T$$
$$=\beta\cdot\omega;\ \omega\,\varepsilon\,\mathbb{M}_\Lambda \tag{30}$$

where

$$V:=\text{diag}^{-1}\{\ I\ V_2\ \dots V_N\}\cdot\bar{v}_1\cdot w_0\cdot\omega \tag{31}$$
$$\beta:=1/N\cdot(b_1\ \ b_2\ \dots\ b_N)\cdot V$$

from (20), (21).

Theorem 3:

The LTI model realization of LPDT plant is possible if and only if

$$\beta\neq 0. \tag{32}$$

And although we used the special solutions $\hat{v}$ and w_0 (not unique) , the condition (32) is independent of the choice of the special solutions.

Proof:

First part of the proof:

if: It is trivial from the deriving routine of (30).

only if : If $\beta\neq 0$, then from (31-b) there is $\bar{\beta}\,\varepsilon\,\mathbb{M}_\Lambda$ such that $\beta=d^{-1}\cdot\bar{\beta}$. Taking $\omega=d\cdot W$ ($W\,\varepsilon\,\mathbb{M}_\Lambda$), we have

$$(K_1^T\ K_2^T\ \dots\ K_N^T)^T=V\cdot d\cdot W\,\varepsilon\,\mathbb{M}_\Lambda\ (W\,\varepsilon\,\mathbb{M}_\Lambda)$$

which satisfy (15) and (17) for $G_L\,\varepsilon\,\mathbb{M}_\Lambda\neq 0$ ($G_0=\bar{\beta}\cdot W\neq 0$). i.e., The LTI model realization is possible.

Second part of the proof:

Suppose there are another pair of special solutions $\hat{p}$ and w besides $\hat{v}$ and w_0 in (23) and (28). Between the two pairs of special solutions, according to the properties of matrix, there exist two unique unimodular matrix ϕ and ψ such that

$$\hat{v}=\hat{p}\phi\ \text{ or }\ \hat{p}=\hat{v}\phi^{-1}$$

and $w_0=w\psi$ or $w=w_0\psi^{-1}$.

Defining $\hat{p}=[\hat{p}_1^T\ \hat{p}_2^T]^T$ as in (24), we have

$$\hat{p}_1=\hat{v}_1\phi^{-1}\ \text{ and }\ \hat{p}_2=\hat{v}_2\phi^{-1}.$$

Then (26-a) in form of $\hat{p}_1$ and $\hat{p}_2$ becomes

$$(d\bar{p}_1-\bar{p}_2)(T_1{}' \ T_2{}' \ldots T_N{}')=0 . \quad (33)$$

Since w is the special solution of (26-a) with independent columns besides w_0.

$$\cdot \quad \text{diag}\{\phi \ \phi \ \ldots \ \phi\}\cdot w$$

is the similar solution of (33). Then under the new pair of special solutions,

$$\begin{aligned}\beta_{new} &= 1/N\cdot(b_1 \ \ b_2 \ldots b_N)\cdot V_{new}\\ &= 1/N\cdot(b_1 \ \ b_2 \ldots b_N)\cdot \text{diag}^{-1}\{ I \ V_2 \ldots V_N\}\cdot \bar{p}_1\\ &\qquad\qquad \cdot \text{diag}\{\phi \ \phi \ \ldots \ \phi\}\cdot w\\ &= 1/N\cdot(b_1 \ \ b_2 \ldots b_N)\cdot \text{diag}^{-1}\{ I \ V_2 \ldots V_N\}\cdot \bar{v}_1\cdot\\ &\quad \text{diag}^{-1}\{\phi\phi\ldots\phi\}\cdot \text{diag}\{\phi\phi\ldots\phi\}\cdot w_0\cdot\psi\\ &= \beta\cdot\psi;\ \psi\text{: unimodular.}\end{aligned}$$

Clearly, $\beta\neq 0$ is independent of the choice of the special solution.

Q.E.D.

Theorem 3 shows the necessary and sufficient condition for LTI model realization via solving matrix equation. We can also trivially parametrize the class of realizable LTI models for LPDT plants.

Observe that in (29) $\text{diag}^{-1}\{I \ V_2 \ \ldots \ V_N\}$ has unstable pole at d=0. Generally V will have the same unstable pole, it results in that ω in (29) or (30) can not be freely chosen to ensure $K_1 \ K_2 \ldots K_N \in \mathbb{M}_\Lambda$ and $G_0 \in \mathbb{M}_\Lambda$ (i.e., in (15) $\tilde{k}$, $G_L \in \mathbb{M}_\Lambda$).

Define the set Q_L as follows.

$$Q_L:=\{Q \mid Q\in\mathbb{M}_\Lambda \text{ and } V\cdot Q\in\mathbb{M}_\Lambda\} \quad (34)$$

Q: with suitable size.

Obviously this set has non-zero elements if we choose $Q=d\cdot W$, $W\in\mathbb{M}_\Lambda$ ($\neq 0$). Then, ω in (29) or (30) should be chosen from the elements of Q_L.

Theorem 4:

The class of realizable LTI models is given by

$$G_0=\beta\cdot\omega;\omega\in Q_L\text{: arbitrary} \quad (35)$$

where

$$G_0=[G_1{}^T \ G_2{}^T \ldots G_N{}^T]^T$$

and the LTI model is

$$G_m(\lambda)=G_1(\lambda^N)+\lambda^1 G_2(\lambda^N)+\ldots+\lambda^{N-1}G_N(\lambda^N),$$

$(\lambda=z^{-1})$.

Proof: It is trivial from Theorem 3 and above statements.

Q.E.D.

4. LTI MODEL REALIZATION OF LTI PLANTS

For LTI discrete systems, LTI model realization problem is the model matching problem under periodic compensation. Let the transfer function of LTI discrete plant be denoted by G which owns doubly coprime factorization:

$$G=A_1{}^{-1}B_1=B_2A_2{}^{-1}\ ,\ A_i,\ B_i\in\mathbb{M} \quad (36\text{-a})$$

and there exists X_i, $Y_i\in\mathbb{M}$ such that

$$\begin{bmatrix} X_2 & Y_2\\ -B_1 & A_1\end{bmatrix}=\begin{bmatrix} A_2 & -Y_1\\ B_2 & X_1\end{bmatrix}^{-1}. \quad (36\text{-b})$$

When LTI compensation is used, it is known that the class of matching models is given by

$$G_M=B_2\cdot K;\ K\in\mathbb{M}_\Lambda . \quad (37)$$

However in this case, the model matching with stable compensator is possible if and only if the P.I.P. condition on the plant is satisfied [12].

Then one may ask what is the difference when periodic compensation is used. In this case, by viewing the plant as N-periodic LPDT plant (N:arbitrary chosen), we have the doubly coprime factorization of the plant under N-periodic representation:

$$\tilde{G}=\tilde{A}_1{}^{-1}\tilde{B}_1=\tilde{B}_2\tilde{A}_2{}^{-1},\ \tilde{B}_i,\ \tilde{A}_i\in\mathbb{M} \quad (38\text{-a})$$

and there exists $\tilde{X}_i$, $\tilde{Y}_i\in\mathbb{M}$ such that

$$\begin{bmatrix} \tilde{X}_1 & \tilde{Y}_2\\ -\tilde{B}_1 & \tilde{A}_1\end{bmatrix}=\begin{bmatrix} \tilde{A}_2 & -\tilde{Y}_1\\ \tilde{B}_2 & \tilde{X}_1\end{bmatrix}^{-1} \quad (38\text{-b})$$

where $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{X}_i$ and $\tilde{Y}_i$ are transformations of A_i, B_i, X_i and Y_i defined as getting G_L from G_m in (14). The doubly coprimeness in (38) can be easily checked via matrix calculation based on (14).

The LTI model realization in this case is to find $\tilde{k}\in\mathbb{M}_\Lambda$ which is lower triangular when d=0 such that

$$G_L=\tilde{B}_2\tilde{k} \quad (39)$$

stands where G_L represents some LTI model. (39) can be then equivalently transferred into

$$b_1K_1=b_2K_2=\ldots=b_NK_N=G_0 \quad (40)$$

where $b_1=\tilde{B}_2$,

$$b_j=b_1V_j\ ,\ j=2..N,$$

$$V_j=\begin{bmatrix} 0 & I_{j-1}\\ I_{N-j+1} & 0\end{bmatrix} \text{ NxN block.}$$

Clearly (39) is solvable for any plant if we take $K_j=V_jK_1$ (K_1: arbitrary), i.e.,

$$\tilde{k}=\begin{bmatrix} a_1 & da_N & \ldots & da_2\\ a_2 & a_1 & \ldots & da_3\\ \ldots & & & \\ a_N & a_{N-1} & \ldots & a_1\end{bmatrix},\ a_1,\ a_2\ldots a_N\in\mathbb{M}_\Lambda\text{ :arbitrary,} \quad (41)$$

that means LTI model realization is possible for any LTI discrete plant.

On the class of realizable LTI models, we have:

Theorem 5:

The class of realizable LTI models for LTI plants is the same with (37) which represents the class of matching models when LTI compensation is used.

Proof:

For any model $G_m\in\mathbb{M}_\Lambda$ in (37), there exists $K_m\in$

M_Λ such that

$$G_m = B_2 \cdot K_m.$$

Then translating G_m, K_m into $\tilde{G}_m$, $\tilde{K}_m$ as in (14), there is

$$\tilde{G}_m = \tilde{B}_2 \tilde{K}_m$$

i.e., any models in (37) is realizable under periodic compensation.

Inversely, suppose there is a model which is realizable under periodic compensation but can not be matched under LTI compensation i.e., there is a model $G_L \in M_\Lambda$ (N-periodic representation of model G_M) and $\tilde{K} \in M_\Lambda$ ($\tilde{K}(0)$ is lower triangular) such that

$$G_L = \tilde{B}_2 \tilde{K} \quad \text{or} \quad b_1 K_1 = b_2 K_2 = \ldots = b_N K_N = G_0$$

as in (39) or (40).

Note that if we let $K_1 = [a_1^T \; a_2^T \; \ldots \; a_N^T]^T$ and construct $\tilde{K}_m$ in form of (41), then $G_L = \tilde{B}_2 \tilde{K}_m$ also stands, which is equivalent to

$$G_M = B_2 \cdot K_m \tag{42}$$

where

$$K_m(\lambda) = a_1(\lambda^N) + \lambda^1 a_2(\lambda^N) + \ldots + \lambda^{N-1} a_N(\lambda^N)$$

and $K_m \in M_\Lambda$ ($\lambda = z^{-1}$).

Obviously, (42) contradicts the assumption.

Q.E.D.

Remarks:

This theorem means no better models can be realized by using periodic compensation than using LTI compenation for LTI discrete plant. □

However, by suitably choosing the period number N, it is pointed out that the unstable blocking zeros of the original plant can be made to disappear under the N-periodic representation. And consequently, the strong stabilization is possible for any LTI discrete plant via periodic compensation (see [7] and so on).

5. CONCLUSION

In this paper, we have proposed and solved the LTI model realization problem of LPDT systems. A necessary and sufficient condition for realizing the LPDT closed-loop systems as LTI model is given. We have also shown the difference between using periodic compensation and using LTI compensation in model matching problem for LTI discrete systems.

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