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THE RELATIONSHIP BETWEEN Z-TRANSFORM AND FOURIER TRANSFORM

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ABSTRACT

This paper presents the relationship between z-transform and Fourier transform. Using the paper's results, we can derive the Fourier transform of functions whose z-transform have poles on the unit circle.

1. INTRODUCTION

Until now, z-transform and discrete Fourier transform have been discussed separately. The relationship has not discussed. This paper presents the method of obtaining discrete Fourier transform from z-transform of functions that have poles on the unit circle.

2. FOURIER TRANSFORMS OF THE DISCRETE SIGN FUNCTION AND THE DISCRETE UNIT STEP FUNCTION

Firstly, we discuss the Fourier transform of the discrete sign function.
we define the function ϕ (t) as follows:

$$\phi(t)=\mathrm{sgn}(t)\cdot e^{-j\omega t} \quad (1)$$

The Fourier transform $\Phi(\Omega)$ of the function ϕ (t) is obtained as follows:

$$\Phi(\Omega)=2/\{j(\Omega+\omega)\} \quad (2)$$

Using Eqs. (1), (2) and the Possion sum formula, the following equation is obtained.

$$\sum_{n=1}^{\infty} e^{-j\omega n}-\sum_{n=-\infty}^{-1} e^{-j\omega n}$$

$$=\sum_{n=-\infty}^{\infty}\frac{2}{j(2\pi n+\omega)} \quad (3)$$

Generally, it is known that

$$\sum_{n=-\infty}^{\infty}\frac{1}{j\omega+2\pi nj}.$$

$$=\frac{1}{1-e^{-j\omega}}-\frac{1}{2} \quad (4)$$

On the other hand, the discrete sign function sgn(n) is defined as follows:

$$\mathrm{sgn}(n)=\begin{cases}1 & n>0\\ 0 & n=0\\ -1 & n<0\end{cases} \quad (5)$$

From Eqs. (3) and (4), the Fourier transform $S(e^{j\omega})$ of the discrete sign function (sgn(n)) is obtained.

$$S(e^{j\omega})=\sum_{n=-\infty}^{\infty}\mathrm{sgn}(n)\,e^{-jn\omega}$$

$$=\sum_{n=1}^{\infty}e^{-jn\omega}-\sum_{n=-\infty}^{-1}e^{-jn\omega}$$

$$=\frac{2}{1-e^{-j\omega}}-1 \quad (6)$$

Next, we discuss the Fourier transform $U(e^{j\omega})$ of the discrete unit step function.
The discrete unit step function u(n) is defined as follows:

$$u(n)=\begin{cases}1 & n>0\\ 1/2 & n=0\\ 0 & n<0\end{cases}$$

$$=1/2+\mathrm{sgn}(n)/2 \quad (7)$$

Comparing the relationship between the continuous unit step function and the continuous sign function , the above definition of the discrete unit function is appropriate. Using Eq.(6) and the following formula

$$\sum_{n=-\infty}^{\infty} e^{-jn\omega} = 2\pi \sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n), \quad (8)$$

the Fourier transform $U(e^{j\omega})$ of the discrete unit step function $u(n)$ is calculated as follows:

$$U(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{-jn\omega}/2 + \frac{1}{2}\sum_{n=-\infty}^{\infty} \mathrm{sgn}(n)\, e^{-jn\omega}$$

$$= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n) + \frac{1}{1 - e^{-j\omega}} - \frac{1}{2} \quad (9)$$

Next, we define the following function $u_+(n)$.

$$u_+(n) = \begin{cases} 1 & n \geqq 0 \\ 0 & n < 0 \end{cases} \quad (10)$$

Then, the Fourier transform $U_+(e^{j\omega})$ of the discrete function $u_+(n)$ is obtained as follows:

$$U_+(e^{j\omega}) = \pi \sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n) + \frac{1}{1 - e^{j\omega}} \quad (11)$$

The Fourier transform $V(e^{j\omega})$ of the discrete function $e^{jn\omega_0} u_+(n)$ are calculated as follows:

$$V(e^{j\omega}) = U_+(e^{j(\omega - \omega_0)})$$

$$= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi n) + \frac{1}{1 - e^{-j(\omega - \omega_0)}} \quad (12)$$

3. A DERIVATION OF A FOURIER TRANSFORM FROM A Z-TRANSFORM

A causal series is expressed with

$$x(n) = 0 \quad n < 0 \quad (13)$$

We assume that $X(e^{j\omega})$ and $X(z)$ are the Fourier transform and the z-transform of the discrete series $x(n)$ respectively. These are defined as follows:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n)\, e^{-jn\omega} \quad (14)$$

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} \text{ for } R < |z|, \quad (15)$$

where R is the radius of convergence of the series $x(n)$. When we derive aFourier transform from a z-transform, there are three cases for a causal series:(a) $R< 1$, (b) $R = 1$, and (c) $R> 1$.

In the case (a)($R< 1$), the unit circle is included in the analytic region. Then, the Fourier transform is derived from the z-transform as follows:

$$X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}} \quad (16)$$

In the case (c)($R> 1$), the discrete unit circle is not included in the analytic region. In this case, we can not obtain the Fourier transform by inserting $e^{j\omega}$ in z of the z-transform $X(z)$. Therefore, the z-transform exists but the Fourier transform does not exist.

In the case (b) ($R = 1$), $X(z)$ is analytic in $1 < |z|$ but has at least a pole on the unit circle.

Firstly, we discuss the Fourier transform of the discrete unit step function. The z-transform $U(z)$ of the discrete unit step function is obtained as follows:

$$U(z) = \sum_{n=-\infty}^{\infty} u(n)\, z^{-n}$$

$$= \frac{1}{1 - z^{-1}} - \frac{1}{2} \text{ for } 1 < |z| \quad (17)$$

The relationship between the z-transform and the Fourier transform of the discrete unit step function can be obtained from Eqs. (9) and (17).

$$U(e^{j\omega}) = U(z)\Big|_{z=e^{j\omega}} + \pi \sum_{n=-\infty} \delta(\omega - 2\pi n) \quad (18)$$

The Fourier transform is obtained by adding the above infinite δ-function series to the insertion of $e^{j\omega}$ in the z-transform. Therefore, the Fourier transform can not be obtained by the insertion of $e^{j\omega}$ directly. Next, we obtain Fourier transforms of $u_+(n)$ and

$e^{jn\omega_0} \cdot u_+(n)$. z-transform of these functions are expressed as follows:

$$U_+(z) = \frac{1}{1 - z^{-1}} \quad \text{for } 1 < |z| \tag{19}$$

$$V(z) = \frac{1}{1 - e^{j\omega_0} z^{-1}} \quad \text{for } 1 < |z| \tag{20}$$

These functions $U_+(z)$ and $V(z)$ have poles at the unit circle. Then, Fourier transform can be obtained by inserting $e^{j\omega}$ to z-transform and adding the infinite δ -function series. These results are already calculated in Eqs. (11) and (12).

Next, we discuss the case that X(z) has a pole of order 1 at $\omega = \omega_0$. Then, X(z) is decomposed as follows:

$$X(z) = X_1(z) + \frac{1}{1 - e^{j\omega_0} z^{-1}} \tag{21}$$

We assume that $X_1(z)$ has no poles on the unit circle. From Eqs. (12), (16) and (20), the Fourier transform $X(e^{j\omega})$ can be obtained.

$$X(e^{j\omega}) = X_1(z)\Big|_{z=e^{j\omega}} + \pi \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi n) + \frac{1}{1 - e^{-j(\omega - \omega_0)}} \tag{22}$$

If X(z) has a pole of order k at $\omega = \omega_0$, X(z) is decomposed as follows:

$$X(z) = X_1(z) + \frac{1}{(1 - e^{j\omega_0} z^{-1})^k} \tag{23}$$

We assume that $X_1(z)$ has no poles on the unit circle. The Fourier transform $X(e^{j\omega})$ can be obtained similarly.

$$X(e^{j\omega}) = X_1(z) + \frac{1}{(k-1)!} j^{(k-1)} \pi e^{-j(k-1)\omega_0} * \sum_{n=-\infty}^{\infty} \delta^{(k-1)}(\omega - \omega_0 - 2\pi n) + \frac{1}{(1 - e^{j\omega_0} e^{-j\omega})^k} \tag{24}$$

Using Eqs. (22) and (24), we can obtain the Fourier transform $X(e^{j\omega})$ whose z-transform has poles on the unit circle. Then, if X(z) is a rational function, the Fourier transform of X(z) can be derived by using the above results.

Inversely, z-transform can be obtained from the Fourier transform using the above discussions.

Example: The z-transform of a causal series is expressed as follows:

$$X(z) = \frac{1}{(1 - e^{2j} z^{-1})(1 - e^{-2j} z^{-1})} = \frac{1}{1 - 2z^{-1}\cos(2) + z^{-2}}$$

The fourier transform of the causal series is obtained from above discussions.

$$X(e^{j\omega}) = \frac{1}{1 - 2e^{-j\omega}\cos(2) + e^{-2j\omega}} + \frac{\pi}{2j\sin(2)} \sum_{n=-\infty}^{\infty} [\, e^{2j}\delta(\omega - 2 - 2\pi n) - e^{-2j}\delta(\omega + 2 - 2\pi n)]$$

4. CONCLUSIONS

This paper presents the relationship between the Fourier transform and the z-transform of discrete series. Using this paper's discussions, we can obtain the Fourier transform from the z-transform and the z-transform from the Fourier transform easily.

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