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A Controller Design Method based on the Hessenberg Form

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Abstract

A new controller design algorithm based on the Hessenberg form for linear control systems has been proposed. The controller is composed of the dynamic compensator and the state feedback (dynamic state feedback). The algorithm gives a simple way to assign the eigenstructure (eigenvalues and eigenvectors) of the closed loop system and it also provides a method to assign the frequency shapes near the corner frequencies of the closed loop transfer function matrix. Because of this property, the algorithm is called the independent frequency shape control (IFSC) method.

1. Introduction

The Hessenberg form of a linear control system has been studied by several authors. The controllable Hessenberg form and the observable Hessenberg form have been discussed in [1], and an algorithm to get the coordination transformation has been derived. In [2], the Hessenberg form has been studied in connection with the problem of making the reduced order model. Besides these studies, in [3], we have generalized the Hessenberg form to a class of nonlinear control systems and shown that the nonlinear Hessenberg form can be successfully used to design the stabilizing controller for the nonlinear systems. There we have introduced the concept of the virtual decomposition which is almost equivalent to the decomposition of the system to the Hessenberg form. This concept has played an essential role in the derivation of the stabilizing controller for nonlinear systems. Noting that the Hessenberg form provides a classification of the states by the relative order, it has been expected that there is a design algorithm based on the Hessenberg form which can use the information of the frequency domain effectively for linear control systems. In this paper, a new design algorithm which gives a simple way to assign the eigenvalues and eigenvectors to the closed loop system by using the dynamic compensator and the state feedback based on the concept of the virtual decomposition is proposed, and it is shown that the algorithm can also assign the frequency shapes near the corner frequencies of the closed loop transfer function matrix independently. Because of this property, the algorithm will be called the independent frequency shape control (IFSC) method.

2. Hessenberg Form

Here we will give an algorithm to get a controllable Hessenberg form based on the concepts of virtual inputs and the virtual decomposition which are essential for the proposed controller design algorithm.

Let us consider a linear controllable system (C, A, B),

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where $x(t) \in R^n$ is the n dimensional state vector and $u(t) \in R^r$, $y(t) \in R^m$ are the r dimensional input vector and the m dimensional output vector respectively.

Without loss of generality, we can assume $rank(B) = r$. Hence, there exists a coordinate transformation T_1 which satisfies,

$$T_1 B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{3}$$

where $det(B_1) \neq 0$.

By this transformation, the system is decomposed as,

$$\dot{\bar{x}}_1 = \bar{A}_{11}\bar{x}_1 + \bar{A}_{12}\bar{x}_2 + \bar{B}_1 u \tag{4}$$

$$\dot{\bar{x}}_2 = \bar{A}_{21}\bar{x}_1 + \bar{A}_{22}\bar{x}_2 \tag{5}$$

where,

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = T_1 x.$$

The influence matrix B_1 of the decomposed subsystem (4) is nonsingular and hence the problem of assigning the eigenstructure (eigenvalues and eigenvectors) can be solved straightforwards. We will call such a system a full-controlled system. In general, a quantity $\phi^T x = \phi_1 x_1 + \ldots + \phi_n x_n$ is said to have relative degree k for the system (1), if k is the minimum integer satisfying,

$$\phi^T A^{k-1} B = 0 \qquad \phi^T A^k B \neq 0.$$

According to this terminology, we can say that each element of the vector $\bar{x}_1$ has the relative degree 1 for the system (1).

On the other hand, if we regard the $\bar{x}_1$ as the input vector to the subsystem (5), we will get another control system,

$$\dot{\bar{x}}_2 = \bar{A}_{22}\bar{x}_2 + \bar{A}_{21}u_2 \tag{6}$$

where $u_2 = \bar{x}_1$ will be called a virtual input. It is well known that if (A, B) is a controllable pair, then $(\bar{A}_{22}, \bar{A}_{21})$ is also a controllable pair. Hence if the system (6) is not full-controlled, we can decompose it into one full-controlled system and another controllable system by applying an appropriate coordinate transformation T_2.

For the brevity of the expression, let us omit the bars over each variables and constants ($\bar{x}_1, \bar{A}_{11}$, etc) in the sequel.

By repeating these procedures, finally we will get a set of full-controlled subsystems, S_i, $i = 1, \ldots, q$, each of which has a virtual input u_i.

$$\dot{x}_i = A_{ii}x_i + A_{ii+1}x_{i+1} + \cdots + A_{iq}x_q + A_{i-1i}u_i \qquad i = 1, 2, \ldots, q \tag{7}$$

where,

$$u_i = x_{i-1}, \qquad i = 1, 2, \ldots, q \tag{8}$$

and

$$x_0 = u \qquad (real \quad input).$$

Note that from the definition, each element of the substate vector x_i of i-th subsystem S_i has relative degree i. We will call the decomposition (7) the virtual decomposition. If the relation (8) is substituted into the equations (7), the following controllable Hessenberg form is obtained.

$$\dot{x} = Ax + Bu \tag{9}$$

where,

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & \dots & A_{1q} \\ A_{21} & A_{22} & \ddots & \dots & A_{2q} \\ 0 & A_{32} & A_{33} & \dots & A_{3q} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & A_{qq-1} & A_{qq} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10}$$

Suppose that each subsystem is n_i dimensional. Then it is easily known from the above algorithm that $rank(A_{i+1i}) = n_i \le n_{i-1}$. Now let $k = \min\{l | n_l < n_{l-1}, l \ge 2\} - 1$. From the definition, for all $j \le k$, A_{j+1j} is a nonsingular matrix. If all A_{j+1j}, $j = 1, \dots, q-1$ are nonsingular, the system is called uniformly decomposable. Let us consider a dynamic compensator which is represented by,

$$\dot{z} = Dz + Ev. \tag{11}$$

$$u = Mz + Nv \tag{12}$$

where v is a new r dimensional input vector. Suppose that the compensated system can be written as,

$$\dot{X} = \hat{A}X + \hat{B}v \tag{13}$$

where $X = [z^T; x^T;]^T$ and

$$\hat{A} = \begin{bmatrix} D & 0 \\ BM & A \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} E \\ BN \end{bmatrix}.$$

Next theorem can be easily proved by induction.

Theorem 1 Suppose (1) is controllable. There exists a $r \times q - n$ dimensional compensator (11),(12) such that the compensated system (13) is uniformly decomposable.

By this theorem, we can assume that the system is always uniformly decomposable. Based on this observation, we will assume that

$$B_1 = I, \quad A_{k+1k} = I, \quad k = 1, \dots, q-1 \tag{14}$$

in the following discussions.

Note 1 The Hessenberg form is not unique, however in practice, there may be most natural form. For instance, it will be natural to consider that a high frequency mode should be excited by lower frequency modes in many mechanical systems.

3. Controller Design based on the Hessenberg Form

In this section, we will give an algorithm to assign the eigenvalues and eigenvectors based on the Hessenberg form. Let k-th subsystem can be written as,

$$\dot{x}_k = A_k x_k + \hat{A}_k x^{(k+1)} + u_k \tag{15}$$

where $A_k = A_{kk}$, $\hat{A}_k = [A_{kk+1}, A_{kk+2}, \dots, A_{kq}]$ and $x^{(k)} = [x_k, \dots, x_q]^T$. We have assumed that under this coordinate, $A_{k+1k} = I$. From equation (15), for any given matrix H_k, if the virtual input u_k satisfies,

$$u_k = (H_k - A_k)x_k - \hat{A}_k x^{(k+1)} \tag{16}$$

we will get

$$\dot{x}_k = H_k x_k. \tag{17}$$

This means that if the virtual input (16) is realizable, then the eigenstructure assignment problem for this subsystem is trivial. However, the virtual input u_k is constrained by $u_k = x_{k-1}$.

Let

$$A^{(k)} = \begin{bmatrix} A_{kk} & A_{kk+1} & \dots & \dots & A_{kq} \\ I & A_{k+1k+1} & \dots & \dots & A_{k+1q} \\ 0 & I & A_{k+3k+3} & \dots & A_{k+3q} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & I & A_{qq} \end{bmatrix} \qquad B^{(k)} = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then we can write as

$$\dot{x}^{(k)} = A^{(k)}x^{(k)} + B^{(k)}u_k \tag{18}$$

and (16) can be written as

$$u_k = F^{(k)}x^{(k)} \qquad F^{(k)} = [H_k - A_k; -\hat{A}_k].$$

Now consider the error vector $z_{k-1} = x_{k-1} - u_k$. Differentiating z_{k-1}, we will get,

$$\dot{z}_{k-1} = A_{k-1}x_{k-1} + \hat{A}_{k-1}x^{(k)} + u_{k-1} - F^{(k)}(A^{(k)}x^{(k)} + B^{(k)}x_{k-1}). \tag{19}$$

This means that if the virtual input u_{k-1} satisfies,

$$u_{k-1} = F^{(k-1)}x^{(k-1)} \tag{20}$$

where

$$F^{(k-1)} = [H_{k-1} - A_{k-1} + H_k - A_k; F^{(k)}A^{(k)} - H_{k-1}F^{(k)} - \hat{A}_{k-1}] \tag{21}$$

we can get two subsystems connected as,

$$\dot{z}_{k-1} = H_{k-1}z_{k-1}. \tag{22}$$

$$\dot{x}_k = H_k x_k + z_{k-1} \tag{23}$$

By repeating these processes from $k = q$, for any matrices, $H_1, \dots, H_q$, we can find the feedback control,

$$u = F^{(1)}x + v \tag{24}$$

which transform the system to the following form.

$$\dot{z}_k = H_k z_k + z_{k-1} \qquad k = 1, 2, \dots, q \tag{25}$$

where $z_0 = v$(new input) and $z_q = x_q$. The system (25) can be written in the matrix form as,

$$\dot{z} = Hz + B^{(1)}v \tag{26}$$

where,

$$H = \begin{bmatrix} H_1 & 0 & \dots & \dots & 0 \\ I & H_2 & 0 & \dots & 0 \\ 0 & I & H_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & I & H_q \end{bmatrix}.$$

From the equation (26), it will be clear that we can assign the eigenvalues and the eigenvectors by choosing appropriate matrices $H_1, \dots, H_q$.

Note 2 Note that the matrices $H_1, \dots, H_q$ may be complex matrices hence these matrices should be chosen such that if H_k is a complex matrix then for some l, $H_l = \bar{H}_k$(complex conjugate matrix) in order to guarantee the feedback control (24) to be real.

Now the controller design algorithm proposed here can be summarized as follows..

Step 1 Select $H_1, \dots, H_q$.

Step 2 Set $F^{(q)} = H_q - A_q$, and calculate $F^{(q-1)}, \dots F^{(1)}$ by (21).

Step 3 Let $u = F^{(1)}x$ and simulate the system behavior. If the result is not satisfactory, modify $H_1, \dots, H_q$ and repeat Step 1 and Step 2 for the system (26).

Example Let us consider the system

$$\dot{x} = Ax + bu \tag{27}$$

where,

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{28}$$

This system is already decomposed to the Hessenberg form. Suppose $H_1 = -2, H_2 = -3 + j, H_3 = -3 - j$. It is easy to show that,

$$\begin{aligned} u_3 &= F^{(3)}x^{(3)} = (-7 - j)x_3 \\ u_2 &= F^{(2)}x^{(2)} = -13x_2 - 51x_3 \\ u_1 &= F^{(1)}x^{(1)} = -17x_1 - 117x_2 - 319x_3 \end{aligned} \tag{29}$$

Since $u_1 = u$, we have derived the feedback control which assigns H_1, H_2, H_3. Note that the virtual input u_3 is the complex state feedback, while the u_2 is real. This observation indicates that if the complex $H_1, \dots, H_q$ are given, we should deal with the composite subsystem made up by the subsystem with H_j and it's complex conjugate subsystem with $\bar{H}_j$ to avoid the virtual input to be complex[4]. In such a case, the new subsystem may no longer be full-controlled.

4. Observer Feedback Control

To design a controller by the above method, the state feedback is required, hence the effect of the use of the observer should be considered. A state observer for the system (1) and (2) can be written as follows.

$$\dot{\tilde{x}} = A_0\tilde{x} + Ky + Bu \tag{30}$$

where

$$A_0 = A + KC. \tag{31}$$

Then the estimation error $\epsilon = \tilde{x} - x$ is given by

$$\dot{\epsilon} = A_0\epsilon. \tag{32}$$

And it is well known that the observer feedback system can be written as,

$$\begin{bmatrix} \dot{x} \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A_0 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v \tag{33}$$

where the observer feedback,

$$u = F\tilde{x} + v \tag{34}$$

has been assumed. This means that the observer feedback system is different from the pure state feedback ($u = Fx + v$) system only in the existence of the disturbance ϵ in the input channel. Hence it will be needed to design the first subsystem S_1 carefully in order to make the influence of ϵ small. Generally speaking, the eigenvalues of A_0 will be selected "faster" than the closed loop eigenvalues, eigenvalues of $H_1, \dots, H_q$. That is, the observer part will have the widest bandwidth in the closed loop system. Hence, the influence of the observation error will excites the high frequency modes unless the bandwidth of the first subsystem (eigenvalues of the matrixH_1) is selected properly.

5. Transfer Function Matrix of the Closed Loop System

In this section, we will consider the transfer function matrix of the closed system,

$$\dot{z} = Hz + B^1 v \tag{35}$$

$$y = \hat{C}z \tag{36}$$

Let

$$G_c(s) = \hat{C}(sI - H)^{-1}B^{(1)} \tag{37}$$

be a transfer function matrix of the closed system (35), (36). From the structure of H matrix, we know

$$(sI - H)^{-1}B^{(1)} = \begin{bmatrix} (sI - H_1)^{-1} \\ (sI - H_2)^{-1}(sI - H_1)^{-1} \\ \vdots \\ (sI - H_q)^{-1} \cdots (sI - H_1)^{-1} \end{bmatrix} \tag{38}$$

Hence the transfer function matrix $G(s)$ is given by the linear combination of the elements $(sI - H_1)^{-1}, \dots (sI - H_q)^{-1}$. In the frequency domain, each element $(j\omega I - H_k)^{-1}$ can be regarded to represent the characteristics of the closed loop system in the neighborhood of frequencies determined by the eigenvalues of H_k (so called corner frequencies). We have shown that we can assign H_k independently by using the above algorithm. In other words, our algorithm gives the method of assigning frequency shapes near corner frequencies independently to the closed system. Let us call the algorithm the independent frequency shape control (IFSC) method. The coefficient matrix $\hat{C}$ is determined by the coordinate transformation T which has transformed the system (10) into (35), and T is determined uniquely by $H_1, \dots, H_q$. Indeed, T can be written as,

$$T = \begin{bmatrix} I & -F_2^{(2)} & \dots & \dots & -F_q^{(2)} \\ 0 & I & -F_3^{(3)} & \dots & -F_q^{(3)} \\ 0 & 0 & I & \dots & -F_q^{(4)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & I \end{bmatrix} \tag{39}$$

and

$$\hat{C} = CT^{-1}. \tag{40}$$

Hear $F_k^{(i)}$ is the element of the feedback vector defined by (21), that is,$F^{(i)} = [F_i^{(i)}, F_{i+1}^{(i)}, \dots, F_q^{(i)}]$.

On the other hand, from (38), we know that the transfer function matrix $G_c(s)$ can be written as,

$$G_c(s) = G_0(s)G_1(s) \tag{41}$$

where,

$$G_0(s) = \hat{C} \begin{bmatrix} (sI - H_2) \cdots (sI - H_q) \\ (sI - H_3) \cdots (sI - H_q) \\ \vdots \\ I \end{bmatrix} \tag{42}$$

$$G_1(s) = (sI - H_q)^{-1}(sI - H_{q-1})^{-1} \cdots (sI - H_1)^{-1} \tag{43}$$

As is known from (42), $G_0(s)$ is the feedback invariant part of $G_c(s)$, since the matrices $H_2, \dots, H_q$ are invariant with respect to the state feedback,

$$v = Fz + Lw. \tag{44}$$

It is easy to show that the transfer function matrix after the state feedback (44) is written as,

$$\hat{G}_c(s) = G_0(s)G_{F,L}(s). \tag{45}$$

Let us write the i-th block of $G_0(s)$ as $\rho_i(s)$. Then $G_{F,L}(s)$ is given by the next theorem.

Theorem 2 Suppose $det(L) \neq 0$. Then,

$$G_{F,L}(s)^{-1} = L^{-1}((s - H_1)\rho_1(s) - \sum_{j=1}^{q} F_j\rho_j(s)). \tag{46}$$

If the closed loop system is again transformed to the triangular form as in (26), $G_{F,L}$ should be written as,

$$G_{F,L}(s) = (sI - \hat{H}_q)^{-1} \cdots (sI - \hat{H}_1)^{-1}.$$

Hence, $G_{F,L}(s)$ is the transfer function matrix from the input to the substate z_q of the relative degree q. This means that the transfer function matrix of the closed loop system is the product of the feedback invariant part $G_0(s)$ and the transfer function matrix $G_{F,L}(s)$ from the input to the substate with the largest relative degree q. IFSC method gives a simple way to design the $G_{F,L}(s)$ by assigning the frequency shape to each mode independently. Note $G_{F,L}$ can also be designed by using the relationship,

$$G_d^{-1}(s) = G_{F,L}^{-1}(s) \tag{47}$$

where $G_d(s)$ is the desired polynomial matrix. Some design methods ,which design the denominator matrix of the transfer function matrix directly, have been proposed [5]. However, it will not be so easy to design the specific frequency shape by such approaches, since the physical meanings of $G_d(s)$ are not always clear and it will be determined by trial and error in general.

6. Conclusion

We have proposed a new controller design method (IFSC method) based on the Hessenberg form of linear control systems. IFSC method gives a simple way to assign the eigenstructure to the closed loop system. Further, we can use the information about the frequency characteristics (information about the phase delay for instance) through the design process. To analyze the robustness properties will be a very important and interesting subject in the future.

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