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A New Approach to the Optimal Control Problem Including Trajectory Sensitivity

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ABSTRACT

We formulate optimal quadratic regulator problems with trajectory sensitivity terms as a optimization problem for a fixed controller structure. Using well-known techniques for parametric LQ problems, we give an algorithm to obtain suboptimal feedback gains by iterative solutions of two Lyapunov equations. A numerical example is given to illustrate the effectiveness of the proposed algorithm.

1. INTRODUCTION

Trajectory sensitivity method was proposed in 1960s as an approach to reduce the effect of plant parameter variation in optimal regulator problems [1-5]. This method utilizes a performance index including the trajectory sensitivity, i.e., the derivative of the state with respect to uncertain parameters. Unfortunately, the optimal solution for this problem is difficult to obtain. Several suboptimal approaches have been proposed to compute feedback gains. However, heuristic approximations are used in these algorithms.

In this paper, we propose a new approach to this problem. We consider a continuous-time time-invariant linear plant. We assume that all the state variables are measurable without error and that the system matrix and the driving matrix contain uncertain parameters with known nominal values. As usual, we consider a performance index consisting of the quadratic forms of the state, the control input and the trajectory sensitivity vector. It is reasonable to restrict our attention to a linear time-invariant controller which generates a linear combination of the measured state vector and the trajectory sensitivity vector. Imposing this restriction on the controller structure, we shall determine the feedback gains in the controller such that the performance index is minimized.

To formulate this optimization problem, we construct an extended system consisting of the plant model and the trajectory sensitivity model. The performance index is clearly quadratic with respect to the extended state vector. The optimization problem is reduced to find a state feedback gain matrix which minimizes the quadratic performance index. This problem could be regarded as a parametric LQ problem [6] which optimizes the feedback gains for a controller with a fixed structure under a quadratic performance index. However, most of parametric LQ problems deal with output feedback controllers. We must construct an algorithm efficient for this specific problem and clarify properties of the resulting control system.

Assuming that the initial state vector of the extended system is a zero mean random variable with a known covariance matrix, we can transform the problem into a matrix minimization problem. We give a descent Anderson-Moore type algorithm [6] to obtain the minimizing feedback gains by iterative solutions of two Lyapunov equations. The descent condition is satisfied if a step size parameter in the algorithm is sufficiently small. We also clarify the effect of the initial covariance matrix on the solution. To illustrate the effectiveness of the proposed method, we give a numerical example.

2. PROBLEM FORMULATION

Consider a linear time-invariant plant described by

$$\dot{x} = Ax + Bu, \qquad (2.1)$$
$$y = Cx, \qquad (2.2)$$

where $x \in R^n$, $y \in R^l$ and $u \in R^m$. We assume that all the state in (2.1) can be measured without error and that the matrices A and B contain uncertain parameters with known nominal values. Define a vector consisting of the uncertain parameters as

$$p = [\ p_1\ p_2\ \cdots\ p_h]. \qquad (2.3)$$

The performance index is given by

$$J = \int_0^\infty [y'Q_1y + y_p'Q_2y_p + u'Ru]\ dt, \qquad (2.4)$$

where $Q_1>0$, $Q_2>0$ and $R>0$ and the suffix p represents partial differentiation with respect to uncertain vector p. The second term in (2.4) is introduced to reduced the trajectory sensitivity with respect to the plant parameter variation. It is well known that the optimal control minimizing the performance index (2.4) is difficult to obtain. The following formulation to obtain a reasonable suboptimal control law can be regarded as a simplified version of the formulation given Wagie and Skelton [5].

Differentiating the both sides of (2.1) and (2.2) with respect to p, we can construct the sensitivity model as

$$\dot{x}_p = A_p x + \tilde{A} x_p + B_p u + \tilde{B} u_p, \tag{2.5}$$

$$y_p = \tilde{C} x_p, \tag{2.6}$$

where

$$\tilde{[\cdot]} = \text{block diag}\{[\cdot][\cdot] \cdots [\cdot]\}. \tag{2.7}$$

We restrict our attention to a linear time-invariant control given by

$$u = -K_1 x - K_2 x_p. \tag{2.8}$$

Assuming that $(x_p)_p=0$, we obtain

$$u_p = -\tilde{K}_1 x_p. \tag{2.9}$$

From (2.5)-(2.9), we can construct the extended system given by

$$\dot{X} = \underline{A} X + \underline{B} u, \tag{2.10}$$

$$Y = \underline{C} X, \tag{2.11}$$

where

$$X = \begin{bmatrix} x \\ x_p \end{bmatrix}, \quad Y = \begin{bmatrix} y \\ y_p \end{bmatrix},$$

$$\underline{A} = \begin{bmatrix} A & 0 \\ A_p & \tilde{A}-\tilde{B}\tilde{K}_1 \end{bmatrix}, \tag{2.12}$$

$$\underline{B} = \begin{bmatrix} B \\ B_p \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} C & 0 \\ 0 & \tilde{C} \end{bmatrix}.$$

Then the performance index (2.4) can be rewritten as

$$J = \int_0^\infty [Y'\underline{C}'\underline{Q}\underline{C}Y + u'Ru]\, dt, \tag{2.13}$$

where

$$\underline{Q} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}. \tag{2.14}$$

In addition, we can write the linear control law (2.8) as

$$u = -KX, \tag{2.15}$$

where

$$K = [K_1 \quad K_2]. \tag{2.16}$$

Consequently, the problem is reduced to find a control gain matrix K in (2.16) minimizing the performance index (2.13) for the extended system described by (2.10)-(2.12). This optimization problem looks like a standard LQ problem. However, note that the system matrix $\underline{A}$ for the extended system contains the feedback gain matrix K_1 to be determined. Unlike the standard LQ problem, we can not reduce the optimization problem to a solution of a Riccati equation.

As a numerical method to obtain minimizing feedback gain matrices, Wagie and Skelton [5] have proposed an algorithm based on the iterative solution of a Riccati equation. In the next section, we give a new algorithm to obtain minimizing gain matrices.

3. A NEW ALGORITHM

Under quadratic performance index, several numerical algorithms have been proposed to optimize feedback gains in a linear time-invariant controller with a fixed structure. Mäkilä and Toivonen [6] have called this class of problems as a parametric LQ problem. A commonly used method is the descent Anderson-Moore algorithm [6,7] which requires iterative solutions of two Lyapunov equations.

The problem formulated in the previous section could be regarded as a parametric LQ problem. However, the problem differs from the standard parametric LQ problems in that the system matrix $\underline{A}$ for the extended system (2.10) contains the feedback gain matrix K_1 to be determined. In spite of the difference, techniques for parametric LQ problems can be applied for the problem including the trajectory sensitivity terms. We have the following result.

Proposition 1: Assume that the initial state X(0) is a zero-mean random variable with the known covariance matrix Π_0 which is positive definite. Let K denote a feedback gain matrix minimizing the averaged performance E[J], where $E[\cdot]$ denote the expectation and J is defined in (2.13). Then, it is necessary that there exist positive definite solutions P and M of the following Lyapunov equations.

$$P(\underline{A}-\underline{B}K) + (\underline{A}-\underline{B}K)'P + \underline{C}'\underline{Q}\underline{C} + K'RK = 0 \tag{3.1}$$

$$M(\underline{A}-\underline{B}K)' + (\underline{A}-\underline{B}K)M + \Pi_0 = 0 \tag{3.2}$$

In addition, K satisfies the following relation.

$$K = R^{-1}\{\underline{B}'P + [\Gamma \quad 0]M^{-1}\}, \tag{3.3}$$

The matrix Γ in (3.3) is defined as

$$\Gamma = \sum_{j=1}^{h}(\tilde{B}'P_{21}M_{12})_{jj} + \sum_{j=1}^{h}(\tilde{B}'P_{22}M_{22})_{jj}, \quad (3.4)$$

where we have defined the partition of the matrices as

$$P = \begin{array}{cc} \begin{array}{cc} n & nh \end{array} & \\ \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} & \begin{array}{c} n \\ nh \end{array} \end{array}, \quad M = \begin{array}{cc} \begin{array}{cc} n & nh \end{array} & \\ \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} & \begin{array}{c} n \\ nh \end{array} \end{array}, \quad (3.5)$$

and $(\cdot)_{jj}$ denotes the j-th nxn diagonal block matrix.

Proof: Using (2.10) and (2.15), the closed loop system is described by

$$\dot{X} = (\underset{\sim}{A}-\underset{\sim}{B}K)X. \quad (3.6)$$

Then the performance index (2.13) can be rewritten as

$$J = \int_0^\infty X'[\underset{\sim}{C}'Q\underset{\sim}{C}+K'RK]X\,dt. \quad (3.7)$$

Assuming that the matrix $(\underset{\sim}{A}-\underset{\sim}{B}K)$ is asymptotically stable, we can express J as

$$J = X'(0)PX(0), \quad (3.8)$$

where the matrix P is a solution of the Lyapunov equation (3.1). If the initial state X(0) is a zero-mean random variable with the covariance matrix Π_0, we can express the average performance as

$$E[J] = \mathrm{tr}\,[\Pi_0 P]. \quad (3.9)$$

Consequently, the minimization of the averaged performance index is reduced to a matrix optimization problem which minimizes the performance index (3.9) with respect to a pair of the matrices P and K satisfying (3.1). Define the Lagrangian for this matrix optimization problem as

$$L = \mathrm{tr}\,[\Pi_0 P] + \mathrm{tr}\,[M'\{P(\underset{\sim}{A}-\underset{\sim}{B}K) + (\underset{\sim}{A}-\underset{\sim}{B}K)'P+\underset{\sim}{C}'Q\underset{\sim}{C}+K'RK\}], \quad (3.10)$$

where M is a matrix Lagrange multiplier. Then $L_P=0$ and $L_K=0$, where the suffixes P and K represent partial differentiations, are clearly necessary for the minimization. It follows easily from (3.10) that $L_P=0$ is equivalent to the Lyapunov equation (3.2). We can decompose the condition $L_K=0$ into $L_{K1}=0$ and $L_{K2}=0$. Using the symmetric properties of the matrices P, M and R, we can show that the conditions $L_{K1}=0$ and $L_{K2}=0$ are equivalent to

$$B'P_{11}+B_p'P_{21}-RK_1)M_{11} + (B'P_{12}+B_p'P_{22}-RK_2)M_{21} + \Gamma = 0, \quad (3.11)$$

$$(B'P_{11}+B_p'P_{21}-RK_1)M_{12} + (B'P_{12}+B_p'P_{22}-RK_2)M_{22} = 0, \quad (3.12)$$

respectively. The above two equations can be combined into a matrix form as

$$(\underset{\sim}{B}'P-RK)M + [\Gamma \;\; 0] = 0. \quad (3.13)$$

Noting that R>0 and M>0 by the assumptions, we can obtain (3.3) from (3.13). ∎

To obtain a numerical solution for (3.1)-(3.3), we propose the following algorithm which can be regarded as a modified version of the algorithm proposed by Moerder and Calise [7] for a class of parametric LQ problems. We denote the system matrix containing the feedback gain matrix K^i by $\underset{\sim}{A}^i$.

Step 0: Set i=0. Choose K^0 such that the matrix $(\underset{\sim}{A}^0-\underset{\sim}{B}K^0)$ is asymptotically stable.

Step 1: Define P^i and M^i as the solutions of the Lyapunov equations

$$P^i(\underset{\sim}{A}^i-\underset{\sim}{B}K^i)+(\underset{\sim}{A}^i-\underset{\sim}{B}K^i)'P^i+\underset{\sim}{C}'Q\underset{\sim}{C} + (K^i)'RK^i = 0, \quad (3.14)$$

$$M^i(\underset{\sim}{A}^i-\underset{\sim}{B}K^i)' + (\underset{\sim}{A}^i-\underset{\sim}{B}K^i)M^i + \Pi_0 = 0, \quad (3.15)$$

respectively.

Step 2: Using K^i, P^i, and M^i, obtain

$$\Delta K^i = R^{-1}\{\underset{\sim}{B}'P^i+[\Gamma^i \;\; 0](M^i)^{-1}\} - K^i, \quad (3.16)$$

where

$$\Gamma^i = \sum_{j=1}^{h}(\tilde{B}'P_{21}^iM_{12}^i)_{jj} + \sum_{j=1}^{h}(\tilde{B}'P_{22}^iM_{22}^i)_{jj}. \quad (3.17)$$

Step 3: Set

$$K^{i+1} = K^i + \alpha\Delta K^i, \quad (3.18)$$

where α is chosen sufficiently small.

Step 4: Stop if $\Delta K^i \approx 0$. Otherwise set i=i+1 and go to Step 1.

For the feedback gain matrices generated by the above algorithm, we can prove the following descent property.

Proposition 2: Let J^i denote the value of the performance index

$$J^i = \mathrm{tr}\,[\Pi_0 P^i], \quad (3.19)$$

where P^i satisfies (3.14). If $L_K i \neq 0$ and the step size parameter α is chosen sufficiently small, then

$$J^{i+1} < J^i, \quad i=0,1,2.... \quad (3.20)$$

Proof: First, note that the gradient of the Lagrangian with respect to K is given by

$$L_K = -2\{\underline{B}'PM - RKM + [\Gamma \quad 0]\}. \tag{3.21}$$

Define the inner product between the search direction (3.16) and the gradient (3.21) as

$$\beta(K) = \mathrm{tr}\,[L_K \Delta K']. \tag{3.22}$$

We show that $\beta(K)<0$ if $L_K \neq 0$. Using (3.16) and (3.21) in (3.22), we have

$$\beta(K) = -2\ \mathrm{tr}\,[RVMV'], \tag{3.23}$$

where

$$V = R^{-1}\underline{B}'P - K + R^{-1}[\Gamma \quad 0]M^{-1}. \tag{3.24}$$

Note that $M>0$. From (3.21) and (3.24), we have $L_K=-2VM$, which implies that $V\neq 0$ if $L_K\neq 0$. It follows from (3.23) and (3.24) that $\beta(K)<0$ if $L_K\neq 0$. Therefore, if we define the Lagrangian at the i-the iteration as

$$L^i = \mathrm{tr}\,[\Pi_0 P^i] + \mathrm{tr}\,[(M^i)'\{P^i(\underline{A}^i-\underline{B}K^i) + (\underline{A}^i-\underline{B}^iK)'P^i+\underline{C}'Q\underline{C}+(K^i)'RK^i\}], \tag{3.25}$$

we can assure that

$$L^{i+1} < L^i,\ i=0,1,2,\ldots. \tag{3.26}$$

Since P^i satisfy (3.14), it follows from (3.19) and (3.25) that $L^i=J^i$. Consequently, the descent condition (3.20) follows from (3.26). ∎

Remark 1: Moerder and Calise [7] have claimed that a sequence of feedback gain matrices generated by their algorithm converges to a stationary point of the performance index. However, Mäkilä and Toivonen [6] have pointed out that the proof given by Moerder and Calise [7] is unsatisfactory to guarantee the convergence to a stationary point. This observation also hold for our algorithm. However, as is pointed out in [6] for the algorithm of Moerder and Calise [7], the condition (3.20) is practically enough to achieve the convergence to a stationary point. To guarantee the convergence to a stationary point, we could use a modified version of the sophisticated algorithm proposed by Mäkilä [8].

Remark 2: Output feedback problems are often formulated as a parametric LQ problem. For these problems, it is not easy to find an initial stabilizing feedback gain matrix. For our problem, we can easily find an initial stabilizing gain matrix K^0 by solving a standard LQ regulator problem. The solution can be computed by use of the proposed algorithm for the performance index where we set $Q_2=0$.

In our formulation, we have randomized the initial state of the extended system. As for the choice of the covariance matrix Π_0, we have the following trivial but important result.

Proposition 3: Let ρ be a positive scalar. Assume that the initial covariance matrix of the extended state $X(0)$ is given by $\rho\Pi_0$. Then the feedback gain matrix obtained by the proposed algorithm is independent of ρ and coincides with that obtained by the initial covariance matrix Π_0.

Proof: Since $\mathrm{tr}\rho\Pi_0=\rho_0\mathrm{tr}\Pi_0$, the minimization problem with the initial matrix $\rho\Pi_0$ is clearly equivalent to that with Π_0. ∎

Remark 3: The above result suggests that the relative values of the elements of the matrix is essential. At present, we can not provide clear design guide line for the choice of the initial covariance matrix. We can treat this matrix as design parameters like the weighting matrices in the performance index (2.4). For the simplest choice, we can consider the matrix in the form

$$\Pi_0 = \mathrm{diag}[\ I_n \quad \sigma I_{nh}], \tag{3.27}$$

where we determine an appropriate value of σ by a trial and error method.

The structure of the proposed regulator is shown in Fig. 1. We can easily obtain the following result for the transfer function matrix of the regulator.

Proposition 4: The transfer function matrix from the plant state $x(t)$ to the control input $u(t)$ is given by

$$K(s) = -\{I_m+K_2(sI_{nh}-\tilde{A}+\tilde{B}\tilde{K}_1)^{-1}B_p\}^{-1} \{K_1+K_2(sI_{nh}-\tilde{A}+\tilde{B}\tilde{K}_1)^{-1}A_p\}. \tag{3.28}$$

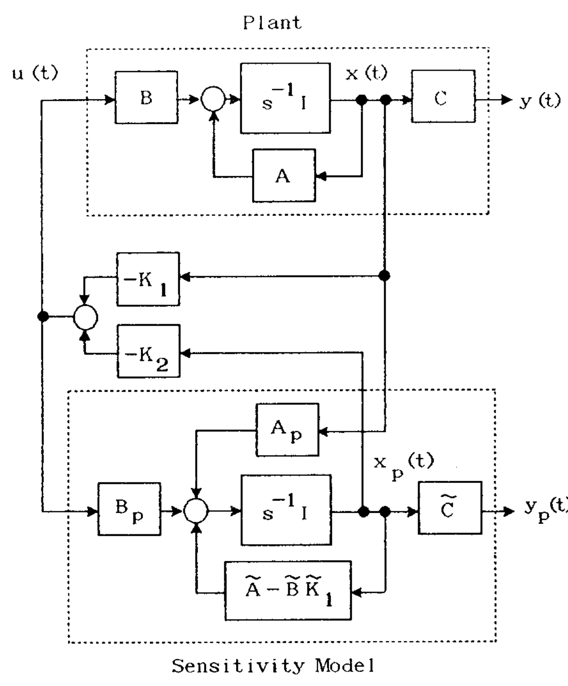

Fig. 1 Structure of the proposed controller

4. A NUMERICAL EXAMPLE

To illustrate the effectiveness of the method proposed in the previous section, we present a numerical design example for a second order plant with one uncertain parameter.

Consider a plant described by

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1+p \\ 1 \end{bmatrix} u, \qquad (4.1)$$

$$y = [\ 1 \ \ -1] x, \qquad (4.2)$$

where p is a scalar uncertain parameter. We assume that the nominal value of p is zero. Soroka and Shaked [9] have pointed out that this plant is extremely sensitive for the variation of the uncertain parameter p. They have shown that, if the standard LQ regulator is constructed for this plant, the closed system becomes unstable for infinitely small variation of p as the weighting on the control input is decreased to zero.

To apply our design method, we define the performance index as

$$J = \int_0^\infty [q_1 y^2 + q_2 y_p^2 + ru^2]\ dt. \qquad (4.3)$$

We consider the initial covariance matrix Π_0 in the form (3.27). Using the algorithm proposed in the previous section, we compute the feedback gain matrix. Note that, at each iteration, we must solve the two Lyapunov equations for 4x4 matrices. As an initial stabilizing feedback gain matrix, we choose $K^0=[K_1^0 \ \ 0]$ where K_1^0 is the optimal feedback gain matrix for the performance index (4.3) where $q_2=0$. We stop the algorithm if the maximum element of ΔK^i is less than 10^{-5}. In Table 1, we summarize the relation between the step size parameter α and the required number of the iterations for $q_1=q_2=1$, $r=10^{-3}$ and $\sigma=1$. For the step size parameter α greater than 0.9, the algorithm is divergent. As the step size is decreased, the required number of the iterations is increased. We have the same feedback gain matrix as long as the algorithm is convergent. As pointed out in Remark 1, the algorithm is practically convergent if the step size parameter is chosen sufficiently small.

Table 1. Step size parameter α versus number of iterations

α	Number of iterations
0.9	(divergent)
0.8	80
0.7	96
0.6	113
0.5	136
0.4	172
0.3	230
0.2	347
0.1	698

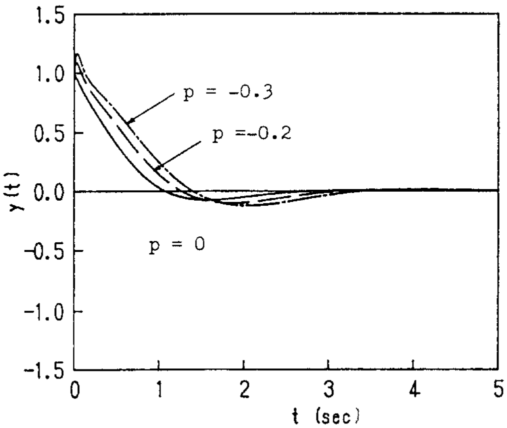

Fig. 2 Response of the proposed regulator

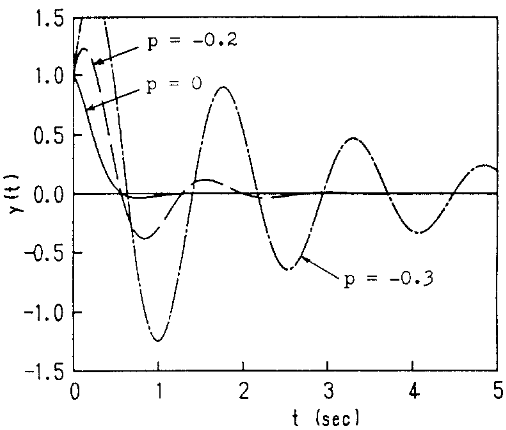

Fig. 3 Response of the LQ regulator

We compare the performance of the proposed regulator obtained by setting $q_1=q_2=1$, $r=10^{-3}$ and $\sigma=1$ with the conventional LQ regulator with $q_1=1$ and $r=10^{-3}$. First, we compare the response for the initial condition $x(0)=[1 \ \ 0]'$. For three values of the uncertain parameter, we show the response obtained by the proposed regulator in Fig. 2. The corresponding results for the conventional regulator are shown in Fig. 3. Apparently, the response obtained by the proposed regulator is robust against the parameter variation while the conventional LQ regulator provides the extremely sensitive result.

Next, we compare the frequency domain properties of the both regulators. Comparisons of the sensitivity functions and the complementary sensitivity functions at the input of the plant are shown in Fig. 4 and 5, respectively. It is interesting to note that the gain of the

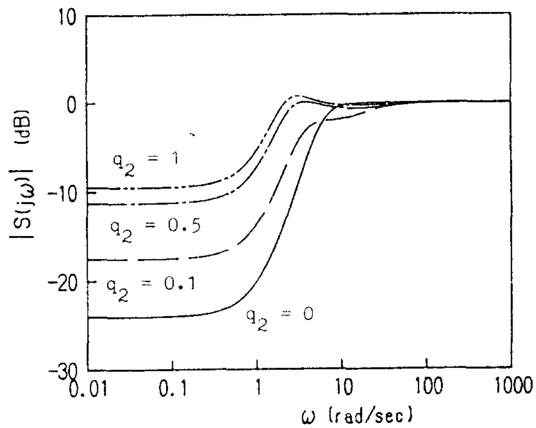

Fig. 4 Sensitivity functions

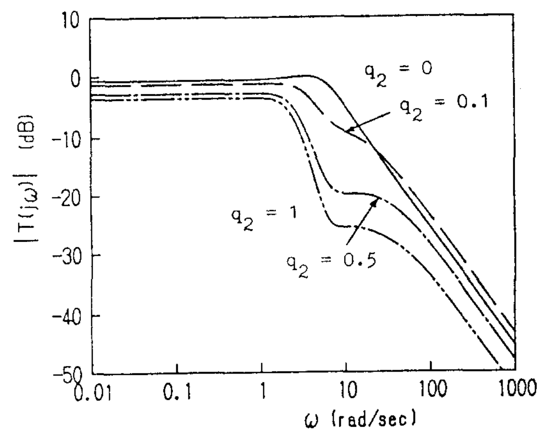

Fig. 5 Complementary sensitivity functions

sensitivity function at the low frequency region increases as the weighting coefficient q_2 for the trajectory sensitivity is increased. This means that, for the variation of the parameter p, the reduction of the trajectory sensitivity does not imply that of the sensitivity function. As shown in Fig. 5, the complementary sensitivity function for the conventional regulator decreases 20dB per decade as is suggested by the well-known Kalman inequality. It has been pointed out that, due to the 20dB per decade property, the conventional LQ design often provides poor robustness against unstructured perturbations. For the proposed regulator, the frequency region where the gain decreases at the rate of 40dB decade appears as the weighting coefficient q_2 is increased. Although the gain decreases at the rate of 20dB per decade in sufficiently high frequency region, the proposed regulator provides improved robustness against unstructured perturbations in the frequency region where the steep decrease is achieved.

5. CONCLUSIONS

We have proposed a new algorithm for the quadratic optimal control problem including the trajectory sensitivity. In comparison with the existing algorithms, the proposed algorithm has clear meaning of approximation and the computational advantage. As is shown by the numerical example, the frequency domain properties achieved by introducing the trajectory sensitivity term can not be obtained by the conventional LQ design method. A controller that is robust against both structured and unstructured perturbations could be designed by a systematic use of the proposed algorithm, which is under current investigation.

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