Optimization for Nonlinear Systems via Block Pulse Transformation

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Abstract

This paper presents a method of suboptimal control for nonlinear systems via block pulse transformation. The adaptive optimal control scheme proposed by J.P. Matuszewski is introduced to minimize the performance index. Nonlinear systems are controlled using the obtained optimal control via block pulse transformation.

The proposed method is simple and computationally advantageous. Viablity of the this method is established with simulation results for the van der Pol equation for comparision with other methods.

1. Introduction

The orthogonal functions have been widely applied to control theory. The particular orthogonal functions used up to now are the block pulse function,the Walsh functions, shifted Legendre polynomials and etc[1-3]. The main feature of the method of using orthogonal functions is that it reduces the calculus of certain differential equations to a set linear algebraic equations through the use of the well-known operation matrix for integration via orthogonal functions.

In this paper, to obtain the optimal control of the nonlinear system, we introduce the method proposed by J.P.Matuzewski for adaptive optimal scheme[4].

The approximation method of the block pulse functions is employed to solve the optimal control problem of the nonlinear system which is linearized after every time increment Δt sec. Then the nonlinear system is controlled using the obtained optimal control via block pulse transformation. The approach used here is adaptive in nature, and viability is established with simulation result for the van der Pol quation for comparition with other approaches.

2. Block Pulse Transformations

Block pulse functions $\phi_k(t), k=1,2,\ldots,m$, defined in the interval $[t_0, t_f)$ by

$$\phi_1(t)=\begin{cases}1, & t\in[t_0,\Delta t]\\ 0, & \text{otherwise}\end{cases} \tag{2.1}$$

$$\phi_k(t)=\begin{cases}1, & t\in[(k-1)\Delta t,\ k\Delta t]\\ 0, & \text{otherwise}\end{cases} \tag{2.2}$$

where $k=2,3,\cdots\cdots\ m,\ \Delta t=\dfrac{t_f-t_0}{m}$

By applying the orthogonal property to an arbitrary function $x(t)$ which is absolutely integrable in the interval $[t_0, t_f)$, we have

$$x(t)\approx\sum_{k=1}^{m}X_k\ \phi_k(t) \tag{2.3}$$

$$\begin{aligned} X_k &= \int_{t_0}^{t_f}x(t)\ \phi_k(t)\ dt \\ &= \frac{1}{\Delta t}\int_{(k-1)\Delta t}^{k\Delta t}x(t)\ dt \\ &\approx \frac{1}{2}\Big[x(k\Delta t)+x[(k-1)\Delta t]\Big] \end{aligned} \tag{2.4}$$

where X_k is the coefficient of $\phi_k(t)$, the kth BPF. The integrals of BPFs[1] can be approximated as

$$\int_{t_0}^{t}\boldsymbol{\phi}(\tau)d\tau\approx \mathbf{P}\boldsymbol{\phi}(t) \tag{2.5}$$

$$\int_{t_0}^{t}\phi_k(\tau)d\tau\approx\frac{\Delta t}{2}\phi_k(t)+\Delta t\sum_{i=k+1}^{m}\phi_i(t) \tag{2.6}$$

using m BPFs themselves.

Similarly, the backward integrals of BPFs[1] can also be approximated as

$$\int_{t_0}^{t_f}\boldsymbol{\phi}(\tau)d\tau\approx-\mathbf{P}^T\boldsymbol{\phi}(t) \tag{2.7}$$

$$\int_{t}^{t_f}\phi_k(\tau)d\tau\approx-\frac{\Delta t}{2}\phi_k(t)-\Delta t\sum_{i=1}^{k-1}\phi_i(t) \tag{2.8}$$

where,

$$P = \begin{bmatrix} 1/2 & 1 & \cdots & 1 & 1 \\ 0 & 1/2 & \cdots & 1 & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/2 & 1 \\ 0 & 0 & \cdots & 0 & 1/2 \end{bmatrix}$$

$$\boldsymbol{\phi}^T(t)=[\phi_1(t),\phi_2(t),\ldots,\phi_m(t)]$$

Consider a linear system described by the state equation

$$\dot{\mathbf{x}}(t)= \mathbf{A}(t)\mathbf{x}(t) \tag{2.9}$$

$$\mathbf{x}(t_0)= \mathbf{x}_0$$

Suppose all elements of the vector function $\mathbf{x}(t)$ and the matrix function $\mathbf{A}(t)$ are absolutely integrable in the time interval $[t_0,t_f)$; then by using the BPF approximation we have

$$\mathbf{x}(t) = \sum_{k=1}^{m} \mathbf{X}_k \phi_i(t) \tag{2.10}$$

$$\mathbf{A}(t) = \sum_{k=1}^{m} \mathbf{A}_k \phi_i(t) \tag{2.11}$$

$$\mathbf{A}_k = [\mathbf{A}_{1k}\ \mathbf{A}_{2k}\ \cdots\cdots\ \mathbf{A}_{nk}]$$

where $\mathbf{A}_{ik}$ indicates the kth column element of BPF for the ith column element of $\mathbf{A}(t)$.

BPFs have the following disjoint property:

$$\phi_i(t)\phi_j(t)= \begin{cases} \phi_i(t) & , \quad i = j \\ 0 & , \quad i \neq j \end{cases} \tag{2.12}$$

From the disjoint property of BPFs,

$$\mathbf{A}(t)\mathbf{x}(t) = \left[\sum_{i=1}^{m} \mathbf{A}_i \phi_i(t)\right]\left[\sum_{j=1}^{m} \mathbf{X}_j \phi_j(t)\right]$$

$$= \sum_{k=1}^{m} \mathbf{A}_k\ \mathbf{X}_k\ \phi_k(t) \tag{2.13}$$

Now, integration of eq.(2.9) from t_0 to t_f gives

$$\mathbf{x}(t)-\mathbf{x}(0) = \int_0^t \mathbf{A}(t)\mathbf{x}(\tau)d\tau \tag{2.14}$$

Substituting the BPF approximation formula eq.(2.10) and eq.(2.11), into eq.(2.14), and using eq.(2.6) and eq.(2.13), yields

$$\sum_{k=1}^{m}[\mathbf{X}\phi_k(t)-\mathbf{x}_0\phi_k(t)]=\sum_{k=1}^{m}\mathbf{A}_k\mathbf{X}_k\int_0^t \phi_k(\tau)d\tau$$

$$=\Delta t\sum_{k=1}^{m}[\mathbf{A}_k\mathbf{X}_k(\tfrac{1}{2}\phi_k(t)+\sum_{l=k+1}^{m}\phi_l(t))] \tag{2.15}$$

Equating the coefficients of $\phi_k(t),k=1,2,\ldots,m$, of both sides of the above equations gives

$$\mathbf{X}_1 - \mathbf{x}_0=\Delta t/2\ \mathbf{A}_1\mathbf{X}_1 \tag{2.16}$$

$$\mathbf{X}_{k-1}-\mathbf{x}_0=\Delta t/2\mathbf{A}_{k-1}\mathbf{X}_{k-1}+ \sum_{l=k}^{m} \mathbf{A}_l\mathbf{X}_l \tag{2.17}$$

$$\mathbf{X}_k - \mathbf{x}_0=\Delta t/2\mathbf{A}_k\mathbf{X}_k+\sum_{l=k+1}^{m} \mathbf{A}_l\mathbf{X}_l \tag{2.18}$$

Finally, we can obtain the recursive algorithm for solving $\mathbf{X}_k$ as

$$\mathbf{X}_1 = [\mathbf{I}-\Delta t/2\mathbf{A}_1]^{-1}\mathbf{x}_0$$

$$\mathbf{X}_{k+1}=[\mathbf{I}-\Delta t/2\mathbf{A}_k]^{-1}(\mathbf{I}+\Delta t/2\mathbf{A}_{k-1})\mathbf{X}_{k-1} \tag{2.19}$$

where $k=2,3,4,\cdots\cdots\cdots,m$

3. Optimal Control Scheme of Nonlinear Systems

Let us consider the physical process described by nonlinear differential equation

$$\dot{\mathbf{x}}(t)=f(\mathbf{x},\mathbf{u},\mathbf{a},t) \tag{3.1}$$

$$\mathbf{x}(t_0)=\mathbf{x}_0$$

As suggested by J.D.Pearson[5], it is assumed that nonlinear system can be represented by a time- and state-dependent model of the form

$$\dot{\mathbf{x}}(t)=\mathbf{A}(\mathbf{x},\mathbf{a},t)\mathbf{x}(t)+\mathbf{B}(\mathbf{x},\mathbf{a},t)\mathbf{u}(t) \tag{3.2}$$

It is further assumed that performance index to be minimized is of the form

$$J = \frac{1}{2}\int_{t_0}^{t_f}(\mathbf{x}(t)^T\mathbf{Q}\mathbf{x}(t)+\mathbf{u}(t)^T\mathbf{R}\mathbf{u}(t))dt \tag{3.3}$$

The problem is to obtain a optimal feedback control law of the form

$$\mathbf{u}(t) = \mathbf{G}(\mathbf{x},\mathbf{a},t)\mathbf{x}(t) \tag{3.4}$$

where $\mathbf{G}$ is a gain matrix which is, in general, state- and time-independent. When the matrices $\mathbf{A}$ and $\mathbf{B}$ are not state-dependent, the optimal control law is

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^T\boldsymbol{\lambda}(t) \tag{3.5}$$

$$\boldsymbol{\lambda}(t) = \mathbf{K}(t)\mathbf{x}(t) \tag{3.6}$$

where $\mathbf{K}(t)$ satisfies the matrix Riccati equation

$$\dot{\mathbf{K}}(t)+\mathbf{K}(t)\mathbf{A}+\mathbf{A}^T\mathbf{K}(t)-\mathbf{K}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{K}(t)+\mathbf{Q}=\mathbf{0} \tag{3.7}$$

To obtain the optimal control, we introduce the method proposed by J.P.Matuzewski for adaptive optimal scheme[7].

i) The nonlinear system of eq.(3.1) is first modeled as eq.(3.2).

ii) The state in $\mathbf{A}$ and $\mathbf{B}$ matrices are considered constant at their present values $\mathbf{x}(t_i)$, with t_i being the present time. The parameter $\mathbf{a}(t)$ is also given an assumed form at this time.

iii) In the interval $[t_i,t_f]$, an optimal control is determined by the method given in the next section.

And simultaneously the nonlinear system eq.(3.1) is controlled using the control generated by eq.(3.5). The nonlinear system is controlled in this manner for a short time until some new present time t_i is reached. At this new t_i the state $\mathbf{x}(t)$ and assumed parameter variation form $\mathbf{a}(t)$ are updated, using measured information and the control $\mathbf{u}(t)$ recalculated using updated forms for the $\mathbf{A}(t)$ and $\mathbf{B}(t)$ matrices.

4. Optimization of Nonlinear system via BPF

At present time t_i nonlinear system (3.2) can be expressed as

$$\dot{\mathbf{x}}(t)=\mathbf{A}(t_i)\mathbf{x}(t)+\mathbf{B}(t_i)\mathbf{u}(t) \tag{4.1}$$

$$\mathbf{x}(t_i)=\mathbf{x}_0$$

It is well known that the optimal control variable[10] is

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^T(t_i)\boldsymbol{\lambda}(t) \tag{4.2}$$

$$\dot{\boldsymbol{\lambda}}(t)=-\mathbf{Q}\mathbf{x}(t)-\mathbf{A}^T(t_i)\boldsymbol{\lambda}(t) \ , \ \boldsymbol{\lambda}(t_f)=\mathbf{0} \tag{4.3}$$

where the adjoint variable $\boldsymbol{\lambda}(t)$ satisfies the following canonical equation:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\boldsymbol{\lambda}}(t) \end{bmatrix} = \mathbf{F} \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\lambda}(t) \end{bmatrix} \tag{4.5}$$

where

$$\mathbf{F}=\begin{bmatrix} \mathbf{A}(t_i) & -\mathbf{B}(t_i)\mathbf{R}^{-1}\mathbf{B}^T(t_i) \\ -\mathbf{Q} & -\mathbf{A}^T(t_i) \end{bmatrix}$$

Let

$$\boldsymbol{\Phi}(t_f)=\begin{bmatrix} \boldsymbol{\Phi}_{11}(t_f,t) & \boldsymbol{\Phi}_{12}(t_f,t) \\ \boldsymbol{\Phi}_{21}(t_f,t) & \boldsymbol{\Phi}_{22}(t_f,t) \end{bmatrix} \tag{4.6}$$

$$\boldsymbol{\Phi}(t_f,t_f) = \mathbf{I}$$

be the state transition matrix of eq.(4.5).
It is well known that the state transition matrix has the following property:

$$\dot{\boldsymbol{\Phi}}(t_f,t) = -\boldsymbol{\Phi}(t_f,t)\mathbf{F} \tag{4.7}$$

Integrating eq.(4.7) backward from t_f to t_i gives

$$\mathbf{I} - \boldsymbol{\Phi}(t_f,t) = \int_{t_f}^{t} \boldsymbol{\Phi}(t_f,\tau)\ \mathbf{F}\ d\tau \tag{4.8}$$

From eq.(4.5) and eq.(4.6)

$$\begin{bmatrix} \mathbf{x}(t_f) \\ \boldsymbol{\lambda}(t_f) \end{bmatrix}=\boldsymbol{\Phi}(t_f,t)\begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\lambda}(t) \end{bmatrix} \tag{4.9}$$

$$\boldsymbol{\Phi}_{21}(t_f,t)\mathbf{x}(t)+\boldsymbol{\Phi}_{22}(t_f,t)\boldsymbol{\lambda}(t)=\boldsymbol{\lambda}(t_f)=\mathbf{0}$$

and therefore

$$\boldsymbol{\lambda}(t) = \mathbf{K}(t)\mathbf{x}(t) \tag{4.10}$$

where $\mathbf{K}(t)=-\boldsymbol{\Phi}_{22}^{-1}(t_f,t)\boldsymbol{\Phi}_{21}(t_f,t)$. (4.11)

Sustituting eq.(4.10) into eq.(4.2), we can obtain the optimal feedback control law

$$\mathbf{u}(t) = -\mathbf{L}(t)\mathbf{x}(t) \tag{4.12}$$

with timevarying gain $\mathbf{L}(t)$:

$$\mathbf{L}(t) = -\mathbf{R}^{-1}\mathbf{B}^T(t_i)\mathbf{K}(t) \tag{4.13}$$

Here by using the BPF approximation, we obtain

$$\boldsymbol{\Phi}(t_f,t)=\sum_{k=1}^{m}\boldsymbol{\Phi}_k\phi_k(t) \tag{4.14}$$

$$\boldsymbol{\Phi}_{ij}(t_f,t)=\sum_{k=1}^{m}\boldsymbol{\Phi}_{ijk}\phi_k(t) \tag{4.15}$$

$$\mathbf{K}(t)=\sum_{k=1}^{m}\mathbf{K}_k\phi_k(t) \tag{4.16}$$

Inserting eq.(4.15) and eq.(4.16) into eq.(4.8), and applying eq.(2.8), and then equating the coefficients of $\phi_k(t)$, we can obtain the recursive algorithm for solving $\boldsymbol{\Phi}_k$ as

$$\boldsymbol{\Phi}_m = [\ \mathbf{I}-\Delta t\mathbf{F}/2\ ]^{-1}$$

$$\boldsymbol{\Phi}_k = \boldsymbol{\Phi}_{k+1}\ [\ \mathbf{I}+\Delta t\mathbf{F}/2\ ]\ [\ \mathbf{I}-\Delta t\mathbf{F}/2\ ]^{-1} \tag{4.17}$$

where $k = m-1, m-2, \cdots, 1$

From eq.(2.12),(2.13),(4.11),(4.15) and eq.(4.16) it can be shown that

$$\mathbf{K}_k = \boldsymbol{\Phi}_{22k}^{-1}\boldsymbol{\Phi}_{21k} \tag{4.18}$$

Substituting eq.(4.12) and eq.(4.13) into eq.(4.1) yields

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(t_i)-\mathbf{B}(t_i)\mathbf{R}^{-1}\mathbf{B}^T(t_i)\mathbf{K}(t)]\mathbf{x}(t) \tag{4.19}$$

$$\mathbf{x}(t_i) = \mathbf{x}_0$$

Now, integrating the above equation from t_i to t_f, and introducing the algorithm for solving the state equation described in section 2, we can obtain the recursive algorithm for solving $\mathbf{X}_k$ as

$$\mathbf{X}_1 = [\ \mathbf{I}-\tilde{\mathbf{A}}\ ]^{-1}\mathbf{x}_0$$

$$\mathbf{X}_k = [\ \mathbf{I}-\tilde{\mathbf{A}}_k\ ]^{-1}[\ \mathbf{I}+\tilde{\mathbf{A}}_{k-1}\ ]^{-1}\mathbf{X}_{k-1} \tag{4.20}$$

where $\tilde{\mathbf{A}}_k = [\mathbf{A}(t_i)-\mathbf{B}(t_i)\mathbf{R}^{-1}\mathbf{B}^T(t_i)\mathbf{K}_k]$

Substituting eq.(4.18) and eq.(4.20) into eq.(4.12) gives

$$\mathbf{U}_k = -\mathbf{R}^{-1}\mathbf{B}^T(t_i)\mathbf{K}_k\mathbf{X}_k \tag{4.21}$$

where $k=1,2,\ldots,m$

Since at new present time the state $\mathbf{x}(t)$ and assumed parameter variation form $\mathbf{a}(t)$ are updated using measured information, the nonlinear system is controlled using the control generated by eq.(4.21) for a short time $[t_i, t_i+\Delta t]$ until some new present time t_i is reached. By using $\underline{m}$ BPFs in the time interval $[t_0, t_f)$, the initial state vector is updated at new present time as

$$x(t_i) = 2X_i - x(t_i - \Delta t) \qquad (4.22)$$

where the size of the time increment is $\Delta t = (t_f - t_0)/m$. In the next time interval $[t_i, t_f)$, the control is generated by eq.(4.21) using m-i BPFs.

5. Illustrative Example

Consider a example of Van der Pol equation[4-9]. This example is used because published results of other suboptimal techniques use the same example. This system is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1-x_1^2(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad (5.1)$$

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (5.2)$$

with the following performance index to be minimized

$$J = \frac{1}{2} \int_0^{t_f} x_1^2(t) + x_2^2(t) + u(t)\,dt$$

Table 1 contains a summary of the values of the performance index by various approaches.

Control u(t) and state x(t) trajectories are shown in Fig.1 and Fig.2. Results are given for the present method using $t_f=10$, $m=50$ and $\Delta t=0.2$.

Table 1. The Values of the Performance Index Compared with Other Method

Method	Performance Index
Matuszewski[4]	2.6375
Pearson[5]	2.6188
Garrard[6]	2.6335
Burghart[7]	2.6101
Mahamoud[8]	2.5941
Permar[9]	2.5887
Present method	2.6130
Optimal	2.5635

6. Conclusions

To obtain the optimal control of the nonlinear system, we introduce the method proposed by Matuzewski for adaptive optimal scheme.

The nonlinear system is first represented by a linearized time- and state-dependent model. The state in system matrices are considered constant at their present values, with t_i being the present time.

The approximation method of the block pulse functions is employed to solve the optimal control problem of the nonlinear system which is linearized after every time increment Δt sec. And simultaneously the nonlinear system is controlled, using the obtained optimal control via block pulse transformation, for a short time until some new present time t_i is reached.

The proposed method is simple and computationally advantageous.

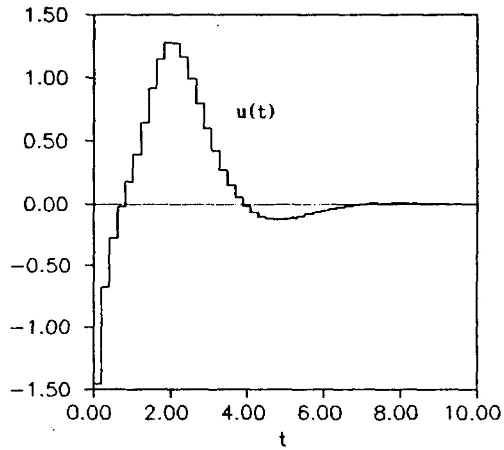

Fig.1 Control u(t) via the present method

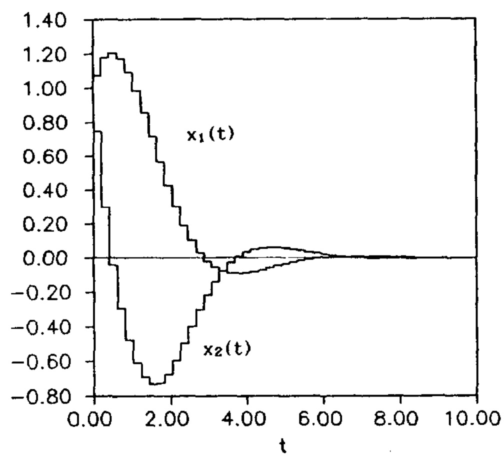

Fig.2 States x(t) via the present method

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