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# Robust Model Reference Direct Adaptive Pole Placement Control

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**Abstract**

Robustness of a model reference direct adaptive pole placement control for not necessarily minimum phase systems is studied subject to unmodeled dynamics and bounded disturbances. The adaptive control scheme involves two estimators for the system and the controller parameter estimation, respectively. The robustness is obtained under some weak assumptions and by using both a normalized least-squares algorithm with dead zone and an appropriate nonlinear feedback.

## 1. Introduction

One of the recent issues in adaptive control systems is the robust stability. The robustness problem has been investigated in several approaches by a number of researchers (see References in [1]). Most of the proofs of robust stability in the above approaches were established based on the *a priori* boundedness of the external disturbances. In the potential application, however, boundedness of the disturbances cannot be assumed *a priori*. To solve this problem, Praly [2] suggested a normalization in the adaptation algorithm, and Kreisselmeier and Anderson [1] introduced the relative dead zone which acts on a suitably normalized, relative identification error.

In a recent paper, Giri *et al.* [3] presented a robust pole placement direct adaptive control algorithm which covers both bounded disturbances and unmodeled dynamics. Their work may be the first complete one on robust stability of a direct adaptive control scheme for nonminimum phase systems.

This paper presents an adaptive control algorithm which guarantees robust stability of the resulting closed-loop adaptive control system with respect to bounded disturbances and unmodeled dynamics. As an adaptive control algorithm, a model reference direct adaptive pole placement control algorithm using specially structured nonminimal models in [4] is employed. Since this control algorithm is derived from both pole placement and zero placement equations, it can be applied to not necessarily minimum phase systems. As described in [4] and [5], this algorithm has exponential data weighting properties for past measurement data by the method of selecting the characteristic polynomials of the sensitivity function filters. The robustness of this algorithm is achieved subject to following assumptions : upper bounds on the system and the controller parameters are known, including basic assumptions of the adaptive control. This adaptive control scheme involves two estimators for the system model parameter estimation and the controller parameter estimation, respectively. The control law is designed from both a general causal feedback and an additional nonlinear feedback [6] which is used to ensure that some identification mismatch error is sufficiently small.

As in [3], a normalized least-squares algorithm with dead zone is used, the nonlinear feedback is shown to be switched off within a finite time, and the tracking case is considered subject to unmodeled dynamics and bounded disturbances. And compared to [3], the nonlinear feedback term is slightly modified, and a more accurate proof is presented for the self-excitation capability.

In this paper, Section 2 presents the robustness problem, Section 3 describes the proposed control algorithm, and Section4 is devoted to the robust stability analysis of the resulting closed-loop adaptive control system. Finally some concluding remarks follow in Section 5.

## 2. Statement of the Problem

Consider a discrete-time system with measurable input $u(t)$ and output $y(t)$. This system to be controlled is assumed to be modeled as a linear, time-invariant, nth order system with modeling error $\eta(t)$ as follows:

$$\begin{aligned} A(q^{-1})z(t) &= u(t) \\ y(t) &= B(q^{-1})z(t)+\eta(t) \end{aligned} \tag{2.1}$$

where $A(q^{-1}) = 1 + a_1q^{-1} + \cdots + a_nq^{-n}$
$B(q^{-1}) = b_1q^{-1} + \cdots + b_nq^{-n}$

and $z(t)$ is the partial state, $q^{-1}$ denotes the backward shift operator, and the order n is chosen by the designer. The system (2.1) can also be written as

$$y(t) = \phi_1(t)^T\theta_1 + \eta_1(t) \tag{2.2}$$

where $\theta_1 = [a_1 \cdots a_n \; b_1 \cdots b_n]^T$ (2.2a)

$\phi_1(t) = [-y(t-1) \cdots - y(t-n) \; u(t-1) \cdots u(t-n)]^T$ (2.2b)

$\eta_1(t) = A(q^{-1})\eta(t).$ (2.2c)

It is assumed that :

A1 : $A(q^{-1})$ and $B(q^{-1})$ are coprime.

A2 : $\|\theta_1\| \le \rho_1$ where $\rho_1$ is a known positive scalar.

A3 : $|\eta(t)| \le \mu m(t)$ where $\mu$ is some positive scalar, and for arbitrary $0 < \sigma < 1$, m(t) is defined by

$$m(t) = \sigma m(t-1) + \max\{|u(t-1)| + |y(t-1)|, 1\}. \tag{2.3}$$

Then it follows from assumptions A2 and A3 that

$$|\eta_1(t)| \le \nu_1 \mu m(t) \tag{2.4}$$

where $\nu_1 = 1 + \sqrt{n}\rho_1\sigma^{-n}.$ (2.4a)

The problem to be considered here is to design a robust model reference direct adaptive pole placement controller in the sense that, for all $0 \le \mu \le \mu_0$, where $\mu_0$ is some positive scalar : i) all the signals in the closed-loop adaptive control system remain bounded ; ii) the poles of the closed-loop system are arbitrarily assigned to the desired locations.

## 3. Model Reference Direct Adaptive Control

The adaptive controller consists of the identification of the system parameters and the controller parameters, and the control law including the identification mismatch error. A normalized least-squares algorithm will be employed as an identification algorithm as in [3], and a model reference direct adaptive pole placement in [4,5] will be used as a controller structure.

### 3.1 Identification of the System

The system parameters $\theta_1$ will be estimated using the following normalized least-squares algorithm with dead zone, wher $i = 1$ [3] :

$$\theta_i(t) = \theta_i(t-1) + \frac{\alpha_i P_i(t)\bar{\phi}_i(t) D(\bar{e}_i(t), d_i(\mu_0))}{1 + \bar{\phi}_i(t)^T P_i(t-1)\bar{\phi}_i(t)} \tag{3.1}$$

$$P_i(t) = P_i(t-1) - \frac{P_i(t-1)\bar{\phi}_i(t)\bar{\phi}_i(t)^T P_i(t-1)\lambda_i(t)}{1 + \bar{\phi}_i(t)^T P_i(t-1)\bar{\phi}_i(t)\lambda_i(t)} \tag{3.2}$$

$$\lambda_i(t) = \begin{cases} \frac{\alpha_i D(\bar{e}_i(t), d_i(\mu_0))}{\bar{e}_i(t)(1+\bar{\phi}_i(t)^T P_i(t-1)\bar{\phi}_i(t)} & \text{if } |\bar{e}_i(t)| > d_i(\mu_0) \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

$$D(\bar{e}_i(t), d_i(\mu_0)) = \begin{cases} \bar{e}_i(t) - d_i(\mu_0) & \text{if } \bar{e}_i(t) > d_i(\mu_0) \\ 0 & \text{if } \bar{e}_i(t) \le d_i(\mu_0) \\ \bar{e}_i(t) + d_i(\mu_0) & \text{if } \bar{e}_i(t) < -d_i(\mu_0) \end{cases} \tag{3.4}$$

$$d_i(\mu_0) = \mu_0 \nu_i \sqrt{1+\alpha_i} \tag{3.5}$$

where $\bar{s}(t) = s(t)/m(t)$ for all scalar or vector sequence $\{s(t)\}$, $0 < \alpha_i \le 1$ and $\mu_0 > 0$ are arbitrarily chosen and

$$e_1(t) = y(t) - \phi_1(t)^T \theta_1(t-1). \tag{3.6}$$

### 3.2 Identification of the Controller

The reference model and the characteristic polynomials $Q_1(q^{-1})$ and $Q_2(q^{-1})$ of the sensitivity function filters are used in the controller. They are selected for exponential data weighting in the measurement data vectors as follows [4] :

$$A^*(q^{-1}) = \sum_{i=0}^{n} a_i^* q^{-i}, \quad a_i^* = (\frac{1}{a})^i, \quad a > 1 \tag{3.7}$$

$$B^*(q^{-1}) = \sum_{i=1}^{n} b_i^* q^{-i}, \quad b_i^* = (\frac{1}{b})^{i-1} b_1^*, \quad b > 1 \tag{3.8}$$

and

$$Q_1(q^{-1}) = \sum_{j=0}^{n} q_{1j} q^{-j}, \quad q_{10} = 1 \tag{3.9}$$

$$Q_2(q^{-1}) = \sum_{j=0}^{n} q_{2j} q^{-j}, \quad q_{20} = 1 \tag{3.10}$$

where

$$q_{1j} = \sum_{i=0}^{j-1} q_{1i}(\kappa a_{j-i-1}^* - a_{j-i}^*)$$

$$q_{2j} = [\sum_{i=0}^{j-1} q_{2i}(b_1^* b_{n-i}^* - b_{n-i+1}^*)] / b_1^*, \quad b_{n+1}^* = 0.$$

Since $Q_1(q^{-1})$ and $Q_2(q^{-1})$ should be stable, $b_1^*$ and $\kappa$ should be chosen in the following ranges, respectively :

$$\frac{1}{a} < \kappa < 1 \tag{3.11}$$

$$\frac{1}{b} < b_1^* < 1 \quad \text{for sufficiently large } n. \tag{3.12}$$

From assumption A1, there exist unique polynomials $H(q^{-1})$, $K(q^{-1})$, $R(q^{-1})$, and $S(q^{-1})$ all of degree $n$ where $H(0) = K(0) = R(0) = S(0) = 0$, such that

$$A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1}) = Q_1(q^{-1})(A(q^{-1}) - A(q^{-1})*) \tag{3.13}$$

$$A(q^{-1})K(q^{-1}) + B(q^{-1})H(q^{-1}) = Q_2(q^{-1})(B^*(q^{-1}) - B(q^{-1})). \tag{3.14}$$

Equation (3.13) is a pole placement equation, and (3.14) is a zero placement equation. From these equations, we get the following specially structured nonminimal model :

$$F(q^{-1})y(t) = G(q^{-1})u(t) + F(q^{-1})\eta(t) \tag{3.15}$$

where
$$\begin{aligned} F(q^{-1}) &= Q_1(q^{-1})Q_2(q^{-1})A^*(q^{-1}) \\ &\quad + Q_2(q^{-1})B^*(q^{-1})R(q^{-1}) + Q_1(q^{-1})A^*(q^{-1})H(q^{-1}) \\ G(q^{-1}) &= Q_1(q^{-1})Q_2(q^{-1})B^*(q^{-1}) \\ &\quad - Q_2(q^{-1})B^*(q^{-1})S(q^{-1}) - Q_1(q^{-1})A^*(q^{-1})K(q^{-1}) \end{aligned}$$

whose common factor $L(q^{-1})$ is obtained as

$$\begin{aligned} L(q^{-1}) &= Q_1(q^{-1})Q_2(q^{-1}) + Q_1(q^{-1})H(q^{-1}) - Q_2(q^{-1})S(q^{-1}) \\ &\quad - S(q^{-1})H(q^{-1}) + R(q^{-1})K(q^{-1}). \end{aligned}$$

The above nonminimal system model (3.15) can be written as

$$y^*(t) = \phi_2(t)^T \theta_2 + \eta_2(t) \tag{3.16}$$

where
$$\begin{aligned} \theta_2 &= [r_1 \cdots r_n \; s_1 \cdots s_n \; h_1 \cdots h_n \; k_1 \cdots k_n]^T \\ \phi_2(t) &= [Q_2(q^{-1})B^*(q^{-1})y(t-1) \cdots Q_2(q^{-1})B^*(q^{-1})y(t-n) \\ &\quad Q_2(q^{-1})B^*(q^{-1})u(t-1) \cdots Q_2(q^{-1})B^*(q^{-1})u(t-n) \\ &\quad Q_1(q^{-1})A^*(q^{-1})y(t-1) \cdots Q_1(q^{-1})A^*(q^{-1})y(t-n) \\ &\quad Q_1(q^{-1})A^*(q^{-1})u(t-1) \cdots Q_1(q^{-1})A^*(q^{-1})u(t-n)]^T \\ y^*(t) &= Q_1(q^{-1})Q_2(q^{-1})(B^*(q^{-1})u(t) - A^*(q^{-1})y(t)) \\ \eta_2(t) &= F(q^{-1})\eta(t). \end{aligned}$$

From assumption A3, it follows :

$$|\eta_2(t)| \le \nu_2 \mu m(t) \tag{3.17}$$

where

$$\begin{aligned} \nu_2 = [&(1 + \frac{n(a-1)}{a})(1 + \frac{n(b-1)}{b})(\frac{a}{a-1}) + (1 + \frac{n(a-1)}{a}) \\ &\times (\frac{a}{a-1})\rho_2\sqrt{n} + (1 + \frac{n(b-1)}{b})(\frac{b}{b-1}) b_1^* \rho_2 \sqrt{n}]\,\sigma^{-3n} \end{aligned}$$

and $\rho_2$ is an arbitrary positive scalar such that $\rho_2 \ge \|\theta_2\|$. As in [3], the following assumption is used.

A4 : An upper bound $\rho_2$ on $\|\theta_2\|$ is known.

The controller parameters $\theta_2$ are estimated using the algorithm (3.1)-(3.5) where $i = 2$, and

$$e_2(t) = y^*(t) - \phi_2(t)^T \theta_2(t-1). \tag{3.18}$$

Since the measurement data vector $\phi_2(t)$ is composed of the past data, this controller is more easily implementable than that of [3].

### 3.3 The Control Law

Let us first introduce the following polynomial which is the difference between the desired and the estimated closed-loop characteristic polynomials :

$$\begin{aligned} Q(t, q^{-1}) &= Q_1(q^{-1})(A^*(q^{-1}) - A(t, q^{-1})) \\ &\quad + A(t, q^{-1})S(t, q^{-1}) + B(t, q^{-1})R(t, q^{-1}) \\ &= \sum_{i=1}^{2n} q_i(t) q^{-i}. \end{aligned} \tag{3.19}$$

The identification mismatch error is defined as follows :

$$e(t) = \sum_{i=1}^{2n} q_i(t) \tag{3.20}$$

and the following time interval is defined :

$$I_\epsilon = \{t/t = 4nk, k = 1, 2, \cdots \text{and there exists } \tau : \\ t - 6n + 1 \le \tau \le t - 2n \text{ such that } e(\tau) > \epsilon.\} \quad (3.21)$$

where $\epsilon$ is a given positive scalar which is chosen small enough to ensure the robust stability of the closed-loop control system.

Given the adaptive schemes producing $R(t, q^{-1}), S(t, q^{-1})$, and $e(t)$, the feedback control is generated by the control law

$$\begin{aligned} Q_1(q^{-1})u(t) &= R(t, q^{-1})y(t) + S(t, q^{-1})u(t) \\ &\quad + Q_1(q^{-1})y^*(t) + f(t) \end{aligned} \quad (3.22)$$

where $$f(t) = \begin{cases} \alpha + \beta m(t) & \text{if } t \in I_\epsilon \\ 0 & \text{otherwise} \end{cases} \quad (3.22a)$$

$$\alpha = 4n(1 + \tfrac{n(a-1)}{a})y^*/k_v \quad (3.22b)$$

and $\beta$ is a positive constant which will be defined later, and $y^*(t)$ is a bounded reference sequence, i.e.,

$$|y^*(t)| \le y^* \quad \text{for all } t. \quad (3.22c)$$

## 4. Robust Stability of the Adaptive Control System

In this section, similar result as [3] will be abtained for the robust stability of the closed-loop adaptive control system.

### 4.1 Properties of the Control Algorithm

The first property to be stated is the boundedness and asymptotic time-invariance of the parameter estimates.

**Proposition 1** *Given an arbitary positive scalar $\mu_0$, for all $0 \le \mu \le \mu_0$, the estimation algorithm (3.1)-(3.5) has the following properties :*

$$a) \|\theta_i(t)\| \le \rho_i', \quad i = 1, 2$$
$$b) \lim_{t\to\infty} D(\bar{e}_i(t), d_i(\mu_0)) = 0, \quad i = 1, 2$$
$$c) \lim_{t\to\infty} \|\theta_i(t) - \theta_i(t-1)\| = 0, \quad i = 1, 2$$
$$d) e(t) \le k_e(\|\tilde{\theta}_1(t)\| + \|\tilde{\theta}_2(t)\|), \quad \text{for all } t$$

*where $\tilde{\theta}_i(t) = \theta_i(t) - \theta_i$ and $\rho_1', \rho_2', k_e$ are constants depending on $\rho_1, \rho_2, P_1(0)$, and $P_2(0)$.*

**Proof :** This proposition is proved in [3].

The following exponential boundedness property for $m(t)$ can be obtained.

**Proposition 2** *There exists a positive constant $\nu_0$ such that for $t \ge 3n$, one has*

$$\sigma \le \frac{m(t+1)}{m(t)} \le \begin{cases} (\nu_0 + \nu_1\mu + \beta) & \text{if } t \in I_\epsilon \\ (\nu_0 + \nu_1\mu) & \text{otherwise} \end{cases} \quad (4.1)$$

**Proof :** Let us first prove the upper bound. From (3.22) it follows

$$\begin{aligned} |u(t)| &\le \{\sum_{i=1}^{n} [|r_i(t)| + |s_i(t) - q_{1i}|]\}\sigma^{1-n} m(t) \\ &\quad + |Q_1(q^{-1})y^*(t)| + f(t) \\ &\le c_1 m(t) + f(t) \end{aligned} \quad (4.2)$$

where $c_1$ is a positive constant. The second inequality follows from proposition 1 and the fact that $m(t) \ge 1$. From (2.2), there exists a positive constant $c_2$ such that

$$\begin{aligned} |y(t)| &\le \sum_{i=1}^{n}(|a_i| + |b_i|)\sigma^{1-n} m(t) + |\eta_1(t)| \\ &\le (c_2 + \nu_1\mu)m(t) \end{aligned} \quad (4.3)$$

If $t \in I_\epsilon$, from (4.2), (4.3), and (2.3), we obtain

$$\begin{aligned} m(t+1) &\le \sigma m(t) + (c_1 + c_2 + \nu_1\mu)m(t) + 1 + f(t) \\ &\le \sigma m(t) + (c_1 + c_2 + \nu_1\mu + 1)m(t) + 1 + f(t) \\ &\le (\nu_0 + \nu_1\mu + \beta)m(t) \end{aligned}$$

where $\nu_0 = \sigma + c_1 + c_2 + \alpha + 1$. The second and the last inequalities follow from the fact that $m(t) \ge 1$. And if $t \notin I_\epsilon$,

$$m(t+1) \le (\nu_0 + \nu_1\mu)m(t)$$

The lower bound can be derived directly from (2.3). ∎

In the following proposition a self-excitation capability of the adaptive control system is established through the nonlinear feedback $f(t)$. The proof is adapted from [3].

**Proposition 3** *Given an arbitrary positive scalar $\beta$, there exist positive scalars $\mu(\beta)$ and $\delta(\beta)$ and a finite time $t_0(\beta)$ such that if $0 \le \mu \le \mu(\beta), t \ge t_0(\beta)$, and $e(t) > \epsilon$, then for arbitrary unit vector $\omega_1$ and $\omega_2$ of dimensions $2n$ and $4n$, respectively,*

$$|\omega_i^T \bar{\phi}_i(t + \tau)| \ge \delta(\beta), \quad i = 1, 2$$

*for at least one $\tau : 1 \le \tau \le 10n - 2$.*

**Proof :** Substituing (2.1) in (3.22) yields for $t \ge n$

$$\begin{aligned} &(Q_1(q^{-1})A(q^{-1}) - S(t, q^{-1})A(q^{-1}) - R(t, q^{-1})B(q^{-1}))z(t) \\ &\quad = f(t) + Q_1(q^{-1})y^*(t) + R(t, q^{-1})\eta(t). \end{aligned} \quad (4.4)$$

Defining the state vector $x(t) = [1\ q^{-1} \cdots q^{-4n+1}]z(t-1)$, (4.4) can be written

$$x(t+1) = G(t)x(t) + g(f(t) + Q_1(q^{-1})y^*(t) + R(t, q^{-1})\eta(t)) \quad (4.5)$$

where

$$G(t) = \begin{bmatrix} -\gamma_1(t) & \cdots & -\gamma_{2n}(t) & 0 & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & \ddots & \ddots & & & \vdots \\ & & \ddots & \ddots & & \vdots \\ & \mathbf{0} & & \ddots & \ddots & \vdots \\ & & & & 1 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

and $\gamma_i(t)$ is the coefficient of $q^{-i}$ on the left-hand side of (4.4). From (3.16) and (2.1), we get

$$\begin{aligned} \phi_2(t)^T = &[q^{-1}Q_2B^*B \cdots q^{-n}Q_2B^*B\ q^{-1}Q_2B^*A \cdots q^{-n}Q_2B^*A \\ &q^{-1}Q_1A^*B \cdots q^{-n}Q_1A^*B\ q^{-1}Q_1A^*A \cdots q^{-n}Q_1A^*A]\, z(t) \\ &+ [q^{-1}Q_2B^* \cdots q^{-n}Q_2B^*\ 0 \cdots 0 \\ &\quad q^{-1}Q_1A^* \cdots q^{-n}Q_1A^*\ 0 \cdots 0]\, \eta(t) \end{aligned} \quad (4.6)$$

for $t \ge n$. The $4n$ polynomial in the first term of the right-hand side of (4.6) form a basis for the space of polynomials with degree less than or equal to $4n$. Then there exists a nonsingular $4m \times 4n$ matrix $H_2$ such that

$$\phi_2(t) = H_2 x(t) + N_2(t) \quad (4.7)$$

where $$N_2(t)^T = [q^{-1}Q_2B^* \cdots q^{-n}Q_2B^*\ 0 \cdots 0 \\ q^{-1}Q_1A^* \cdots q^{-n}Q_1A^*\ 0 \cdots 0]\, \eta(t)$$

Similarly,

$$\begin{aligned} \phi_1(t)^T &= [-q^{-1}B \cdots - q^{-n}B\ q^{-1}A \cdots q^{-n}A]\, z(t) \\ &+ [-q^{-1} \cdots - q^{-n}\ 0 \cdots 0]\, \eta(t). \end{aligned} \quad (4.8)$$

As noted in [3], there exists a $2n \times 4n$ matrix $H_1$ of full rank such that

$$\phi_1(t) = H_1 x(t) + N_1(t) \tag{4.9}$$

where $\quad N_1(t)^T = [\,-q^{-1}, \cdots, -q^{-n}, 0, \cdots, 0\,]\,\eta(t)$.

From (2.1) and (2.3), there exists a positive constant $k_x$ such that $\|x(t)\| \leq k_x m(t)$. In [3], it is proved that if the proposition is true for $x(t)$ it is also true for $\phi_1(t)$ and $\phi_2(t)$. Thus, in this proof we will show that the proposition for $x(t)$ is true.

Let $-2n \leq k \leq 6n-2$ be arbitrary. For $t \geq 3n$, (4.5) can be rewritten after some computations [6] as

$$\sum_{i=2n-1}^{4n-1} \gamma_{4n-i-1}(t) x(t+k+i+1)$$

$$= M[F(t+k) + Q_1(q^{-1})Y^*(t+k) + N(t+k)] + \sum_{i=2n-1}^{4n-1} \gamma_{4n-i-1}(t)$$

$$\times \{\sum_{j=0}^{i} G(t)^{i-j}[\,G(t+k+j) - G(t)\,]\, x(t+k+j)\} \tag{4.10}$$

where $\gamma_0(t) = 1$ , and $M =$

$$[g \; G(t)g \cdots G(t)^{4n-1}g] \begin{bmatrix} 0 & \cdots & 0 & \gamma_{2n}(t) & \cdots & \gamma_1(t) & 1 \\ \vdots & \cdot & \cdot & & \cdot & \cdot & \\ 0 & \cdot & & \cdot & \cdot & & \\ \gamma_{2n}(t) & & \cdot & \cdot & & & \\ \vdots & \cdot & \cdot & & & & \\ \gamma_1(t) & \cdot & & & & \mathbf{0} & \\ 1 & & & & & & \end{bmatrix}$$

$$= \begin{bmatrix} & \mathbf{0} & & \cdot & 1 \\ & & \cdot & & \\ & \cdot & & & \\ & \cdot & & \mathbf{0} & \\ 1 & & & & \end{bmatrix}$$

(4.10a)

$F(t+k)^T = [\,f(t+k) \cdots f(t+k+4n-1)\,]$ (4.10b)

$Y^*(t+k)^T = [\,y^*(t+k) \cdots y^*(t+k+4n-1)\,]$ (4.10c)

$N(t+k)^T = [\,R(t+k, q^{-1})\eta(t+k) \cdots$
$R(t+k+4n-1, q^{-1})\eta(t+k+4n-1)\,]$. (4.10d)

(4.10a) is derived in the Appendix.

From proposition 1 there exists a constant $k_\gamma > 0$ such that $|\gamma_i(t)| \leq k_\gamma$ $(1 \leq i \leq 2n)$ for all t. And $\|G(t)\|$ is bounded and $\|G(t+k+j) - G(t)\| \to 0$ as $t \to \infty$. Thus, the last term of (4.10) has the following upper bound :

$$\|\sum_{i=2n-1}^{4n-1} \gamma_{4n-i-1}(t)\{\sum_{j=0}^{i} G(t)^{i-j}[\,G(t+k+j) - G(t)\,]\, x(t+k+j)\}\|$$

$$\leq (2n+1)k_\gamma \delta_1(t) \max_{-2n \leq \tau \leq 10n-3} \|x(t+\tau)\|$$

$$\leq (2n+1)k_\gamma k_x \delta_1(t) m(t+10n-3) \tag{4.11}$$

where $\delta_1(t)$ is a positive function such that $\delta_1(t) \to 0$ as $t \to \infty$.

Premultiplication of (4.10) by an arbitrary unit $4n$-vector $v$ yields the following lower bound using (4.11) :

$$|\sum_{i=2n-1}^{4n-1} \gamma_{4n-i-1}(t) v^T x(t+k+i+1)\,| \geq |v^T M\, F(t+k)|$$

$$-|v^T M Q_1(q^{-1}) Y^*(t+k)| - |v^T\, M\, N(t+k)|$$
$$-(2n+1)k_\gamma k_x \delta_1(t) m(t+10n-3) \tag{4.12}$$

and the following upper bound for a positive constant $k_1$ :

$$|\sum_{i=2n-1}^{4n-1} \gamma_{4n-i-1}(t) v^T x(t+k+i+1)|$$

$$\leq \sum_{i=2n-1}^{4n-1} k_\gamma m(t+k+i+1)\,|\,v^T \bar{x}(t+k+i+1)\,|$$

$$\leq (2n+1)\, k_\gamma\, k_1\, m(t+10n-2) \max_{0 \leq \tau \leq 10n-2} |v^T \bar{x}(t+\tau)|. \tag{4.13}$$

Let $2n-1 \leq l \leq 6n-2$ be such that $t+l$ is an integer multiple of $4n$. Then, for at least one $-4n+1+l \leq k \leq l$, using (4.10a)

$$F(t+k)^T = [\,0 \cdots 0\; 1_{l-k+1}\; 0 \cdots 0\,]\,(\alpha + \beta m(t+l)\,).$$

Thus, for at least one $-4n+1+l \leq k \leq l$,

$$\begin{aligned} |v^T M\, F(t+k)| &= |\,v_{4n-l+k}\,|\,(\alpha + \beta m(t+l)\,) \\ &\geq \frac{k_v}{2\sqrt{n}}\,(\alpha + \beta m(t+l)\,) \end{aligned} \tag{4.14}$$

where $\quad v^T = [v_1\; v_2\; \cdots\; v_{4n}]$

$$k_v = \frac{v_{i,\min}}{v_{i,\max}}, \qquad 1 \leq i \leq 4n.$$

Also, since $|y^*(t)| \leq y^*$, we have from (4.10b)

$$\begin{aligned} |v^T M\, Q_1(q^{-1}) Y^*(t+k)| &= |\,[\,v_{4n} \cdots v_2\; v_1\,] Q_1(q^{-1}) Y^*(t+k)\,| \\ &\leq 2\sqrt{n}(1 + \frac{n(a-1)}{a})\, y^* \\ &= \frac{k_v\, \alpha}{2\sqrt{n}.} \end{aligned} \tag{4.15}$$

Finally, from (4.10d), assumption A3, and proposition 1,

$$\begin{aligned} |v^T M\, N(t+k)| &= \|N(t+k)\| \\ &\leq 2\sqrt{n}\,\rho_2'\,\mu\, m(t+k+4n-1). \end{aligned} \tag{4.16}$$

Substituting (4.14)-(4.16) in (4.12), then from (4.12) and (4.13), we get

$$\max_{0 \leq \tau \leq 10n-2} |v^T \bar{x}(t+\tau)| \geq \frac{k_v \beta}{2\sqrt{n}(2n+1)k_\gamma k_1} \frac{m(t+l)}{m(t+10n-2)}$$

$$-\frac{2\sqrt{n}\,\rho_2'\,\mu}{(2n+1)k_\gamma k_1} \frac{m(t+k+4n-1)}{m(t+10n-2)} - \frac{k_x \delta_1(t)}{k_1} \frac{m(t+10n-3)}{m(t+10n-2).} \tag{4.17}$$

From proposition 2, since $2n-1 \leq l \leq 6n-2$, the lower bound is

$$\frac{m(t+l)}{m(t+10n-2)} \geq \frac{1}{(\nu_0 + \nu_1\mu + \beta)^2 (\nu_0 + \nu_1\mu)^{8n-3}.} \tag{4.18}$$

and since $-2n \leq k \leq 6n-2$, the upper bound is

$$\frac{m(t+k+4n-1)}{m(t+10n-2)} < \frac{1}{\sigma^{8n-1}} \tag{4.19}$$

and

$$\frac{m(t+10-3)}{m(t+10n-2)} < \frac{1}{\sigma.} \tag{4.20}$$

Therefore, the substitution of (4.18)-(4.20) in (4.17) yields

$$\max_{0 \leq \tau \leq 10n-2} |v^T \bar{x}(t+\tau)| \geq h(\beta, \mu) - k_x'\, \sigma^{-1}\, \delta_1(t) \tag{4.21}$$

where $\quad k_x' = k_x / k_1$

$$h(\beta,\mu) = \frac{1}{(2n+1)k_\gamma k_1}\left[\frac{k_v\beta}{2\sqrt{n}(\nu_0+\nu_1\mu+\beta)^2(\nu_0+\nu_1\mu)^{8n-3}} -2\sqrt{n}\,\rho_2'\,\mu\,\sigma^{-8n+1}\right].$$

The rest part of this proof can be similarly shown as in [3]. ∎

Based on this self-excitation capability of the adaptive control system, it is shown in the following proposition that identification mismatch error converges to the stability interval $[0,\epsilon]$ in a finite time. For this purpose, let us define the following constants :

$$\begin{aligned} \epsilon(\beta) &= 2k_e d(\beta)/\delta(\beta)\lambda_e \\ d(\beta) &= \max\{d_1(\mu(\beta))+\nu_1\mu(\beta),\ d_2(\mu(\beta)+\nu_2\mu(\beta))\} \end{aligned}$$

and $0<\lambda_e<1$ is a constant arbitrarily chosen which will be used in the proof.

**Proposition 4** *For all $0\le\mu\le\mu(\beta)$, if $0\le\epsilon\le\epsilon(\beta)$, then there exists a finite time $t_1(\beta)\ge t_0(\beta)$ such that $e(t)\le\epsilon$ for all $t\ge t_1(\beta)$. Therefore, $f(t)=0$ for all $t\ge t_1(\beta)$.*

The proof is given in [3].

### 4.2 Robust Stability

Using propositions in the previous section, the following robust stability theorem can be established.

**Theorem 1** *Consider the adaptive control system consisting of the system (2.1), the estimation algorithm (3.1)-(3.5), (3.6), and (3.18), and the control law (3.22) subject to the assumption A1-A4. Then there exist positive scalars $\mu_0$ and $\epsilon_0$ such that for all $0\le\mu\le\mu_0$ and $0<\epsilon\le\epsilon_0$ and arbitrary initial conditions the signals in the closed-loop control system are all bounded and there exist a positive constant $k$ and a finite integer $t_s\ge t_1(\beta)$ such that for all $t\ge t_s$ one has*

$$|\,Q_1(q^{-1})\,(\,A^*(q^{-1})y(t)-B(t,q^{-1})y^*(t)\,)\,|\,/m(t) = k_0\mu_0+\sigma^{-2n}\epsilon_0 \qquad (4.22)$$

$$|\,Q_1(q^{-1})\,(\,A^*(q^{-1})u(t)-A(t,q^{-1})y^*(t)\,)\,|\,/m(t) = k_0\mu_0+\sigma^{-2n}\epsilon_0. \qquad (4.23)$$

**Proof :** From (2.2) and (3.6), we get

$$A(t,q^{-1})y(t)-B(t,q^{-1})u(t) = e_1(t)+\phi_1(t)^T(\theta_1(t-1)-\theta_1(t)). \quad (4.24)$$

From (3.22)

$$(Q_1(q^{-1})-S(t,q^{-1}))u(t)-R(t,q^{-1})y(t) = Q_1(q^{-1})y^*(t)+f(t). \qquad (4.25)$$

From (4.24) and (4.25), the following closed-loop equations can be obtained :

$$Q_1(q^{-1})A^*(q^{-1})u(t)=\varsigma_1(t) \qquad (4.26)$$

where $\varsigma_1(t)=R(t,q^{-1})e_1(t)$
$+R(t,q^{-1})\phi_1(t)^T(\theta_1(t-1)-\theta_1(t))+A(t,q^{-1})Q_1(q^{-1})y^*(t)$
$+A(t,q^{-1})f(t)+Q(t,q^{-1})u(t)$ (4.26a)

and

$$Q_1(q^{-1})A^*(q^{-1})y(t)=\varsigma_2(t) \qquad (4.27)$$

where $\varsigma_2(t)=(Q_1(q^{-1})-S(t,q^{-1}))e_1(t)+(Q_1(q^{-1})$
$-S(t,q^{-1}))\phi_1(t)^T(\theta_1(t-1)-\theta_1(t))+B(t,q^{-1})Q_1(q^{-1})y^*(t)$
$+B(t,q^{-1})f(t)+Q(t,q^{-1})y(t).$ (4.27a)

From proposition 1, we get

$$\lim_{t\to\infty}\{|R(t,q^{-1})\phi_1(t)^T(\theta_1(t-1)-\theta_1(t))|\,/m(t)\}=0 \quad (4.28)$$

$$\lim_{t\to\infty}\{|(Q_1(q^{-1})-S(t,q^{-1}))\phi_1(t)^T(\theta_1(t-1)-\theta_1(t))|\,m(t)\}=0. \quad (4.29)$$

From (3.19), (3.20), (2.3) and proposition 4, it follows that for all $t\ge t_1(\beta)$,

$$\begin{aligned} &|Q(t,q^{-1})u(t)|+|Q(t,q^{-1})y(t)| \\ &\le \sum_{i=1}^{2n}|q_i(t)|\,(|u(t-i)|+|y(t-i)|) \\ &\le \sigma^{-2n}m(t)e(t)\ \le \epsilon\,\sigma^{-2n}m(t). \end{aligned} \qquad (4.30)$$

From (3.4) and (3.5) and proposition 1, there exists a positive function $\delta_1(t)$ such that $\delta_1(t)\to 0$ as $t\to\infty$ and

$$\begin{aligned} |R(t,q^{-1})e_1(t)|\,/m(t) &\le k_2\,\mu+\delta_1(t) \quad (4.31)\\ |(Q_1(q^{-1})-S(t,q^{-1}))e_1(t)|\,/m(t) &\le k_2\,\mu+\delta_1(t) \quad (4.32) \end{aligned}$$

where

$$k_2=(1+\sigma^{-n}(\frac{n(a-1)}{a}+\sqrt{n}\,\rho_2'))d_1(\mu_0)/\mu.$$

Finally, from proposition 4, for all $t\ge t_1(\beta)$

$$f(t)=0 \qquad (4.33)$$

and since $y^*(t)$ is bounded, there exists a positive constant $k_3$, such that

$$|A(t,q^{-1})Q_1(q^{-1})y^*(t)|+|B(t,q^{-1})Q_1(q^{-1})y^*(t)|\le k_3. \quad (4.34)$$

Combining (4.28)-(4.34), from (4.26a) and (4.27a) we get for all $t$

$$\{|\varsigma_1(t)|+|\varsigma_2(t)|\}/m(t)\le 2\,k_2\,\mu+\epsilon\,\sigma^{-2n}+\frac{k_3}{m(t)}+\delta_2(t) \quad (4.35)$$

where $\delta_2(t)$ is a positive function such that $\delta_2(t)\to 0$ as $t\to\infty$. Following Giri et al. [3] for the rest part of this proof, we can obtain the following inequality as a stability criterion.

$$k_h(2\,k_2\,\mu+\epsilon\,\sigma^{-2n})<(1-\rho)(1-\sigma) \qquad (4.36)$$

with $|h(t)|\le k_h\rho^t,\quad 0<k_h<\infty$ and $0\le\rho<1$

where $h(t)$ is the impulse response of the system with $1/Q_1(q^{-1})A^*(q^{-1})$ as a transfer function. Thus, there always exist positive scalars $\mu_0$ and $\epsilon_0$ satisfying (4.36), and therefore boundedness of $m(t)$ as well as boundedness of all the closed-loop signals follows. The second part of the theorem directly follows from (4.26) and (4.27) using (4.28)-(4.33), with $k_0=\lambda_0 k_2$ where $\lambda>1$ is arbitrary, and $t_s$ depends on $\lambda_0$. ∎

## 5. Conclusions

A model reference direct adaptive pole placement control algorithm for not necessarily minimum phase systems is proposed, which guarantees robust stability of the resulting closed-loop adaptive control system subject to unmodeled dynamics and bounded disturbances. The considered adaptive control algorithm has exponential data weighting properties for past data by the method of selecting the sensitivity function filters. The robustness is achieved under some weak assumptions such as upper bounds on the system and the controller parameters are known, and by employing two estimators for the system and the controller parameter estimation, respectively, and by using both a normalized least-squares algorithm with dead zone and an appropriate nonlinear feedback which ensures a sufficient amount of excitation.

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## Appendix

*Derivation of (4.10a)*

Defining

$$G(t)^{j-1}g = [g_{j-1}\, g_{j-2} \cdots g_0 \overbrace{0 \cdots 0}^{4n-j}]^T \tag{A.1}$$

then

$$G(t)^{j}g = [g_{j}\, g_{j-1} \cdots g_0 \overbrace{0 \cdots 0}^{4n-j-1}]^T \tag{A.2}$$

with

$$g_j = \sum_{i=1}^{j}(-\gamma_i(t))g_{j-i},\ 1 \le j \le 4n-1 \tag{A.3}$$

where
$$\left.\begin{aligned} g_0 &= 1 \\ \gamma_i(t) &= 0 \quad \text{if } i > 2n. \end{aligned}\right\} \tag{A.3a}$$

Then,

$$M = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots & \cdots & g_{4n-1} \\ & g_0 & g_1 & \cdots & \cdots & g_{4n-2} \\ & & \ddots & & & \vdots \\ & & & g_0 & \cdots & g_{4n-l} \\ & \mathbf{0} & & & \ddots & \vdots \\ & & & & & g_0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & \cdots & 0 & \gamma_{2n}(t) & \cdots & \gamma_1(t) & 1 \\ \vdots & \cdot & \cdot & & \cdot & \cdot & \\ 0 & \cdot & & \cdot & \cdot & & \\ \gamma_{2n}(t) & & \cdot & \cdot & & & \\ \vdots & \cdot & \cdot & & & & \\ \gamma_1(t) & \cdot & & & & \mathbf{0} & \\ 1 & & & & & & \end{bmatrix}$$

$$= \begin{bmatrix} & \mathbf{M_{lm}} & & & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & \cdot & & \mathbf{0} & \\ 1 & & & & \end{bmatrix} \tag{A.4}$$

where $M_{lm}$ are the elements of the matrix M and

$$1 \le l, m \le 4n.$$

In general,

$$\begin{aligned} M_{lm} &= g_{4n-l-k+1} + \gamma_1(t)\, g_{4n-l-k} + \cdots + \gamma_{2n}(t)\, g_{2n-l-k+1} \\ &= g_{4n-l-k+1} + \sum_{i=1}^{4n-l-k+1} \gamma_i(t)\, g_{4n-l-k+1-i} \\ &= 0. \end{aligned} \tag{A.5}$$

The 2nd and the last equalities follow from (A.3a) and (A.3), respectively.

Therefore, we get

$$M = \begin{bmatrix} & & & & 1 \\ & \mathbf{0} & & \cdot & \\ & & \cdot & & \\ & \cdot & & \mathbf{0} & \\ 1 & & & & \end{bmatrix}$$

Note that the eigenvalues of the above matrix are $\pm$ 1s.