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Complementary Sensitivity Characteristics in Digital Control Systems

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Abstract

We derive an integral-type constraint on the complementary sensitivity function in digital control systems. Some design guidances are proposed for the pole assignment of digital controller with computational-time delay to improve the complementary sensitivity characteristics.

1 Introduction

In feedback control systems, the complementary sensitivity function is closely related to the feedback properties such as sensor-noise attenuation and robust stability to the plant uncertainty. Some integral-type constraints on the complementary sensitivity function have been proposed for continuous-time systems [1],[2] and discrete-time control systems [3],[4]. In the digital control system, where the continuous-time plant is stabilized by the digital controller, the corresponding constraint has been derived for the system of which the relative degree of the open-loop transfer function is one [5]. However, no constraints have been proposed for the system of which the relative degree of the open-loop transfer function is greater than one.

In this paper, we derive an integral-type constraint on the complementary sensitivity function for SISO digital control system with delay-time and/or computational-time delay. This gives some guidances to the digital control system design based on the pole assignment of the digital controller.

2 Integral-type constraints

Consider an SISO digital control system with zero-order hold and the sampler with sampling period τ as shown in Fig.1, where $P(s)$ and $C(z)$ denote the continuous-time plant and the digital controller, respectively.

We note that the pulse-transfer function of the sampled plant model $P_\tau(z)$ can be expressed as

$$P_\tau(z) = (1 - z^{-1})\mathcal{Z}[P(s)/s] \tag{2.1}$$

where $\mathcal{Z}$ denotes the Z-transformation. Then the complementary sensitivity function $T(z)$ and the sensitivity

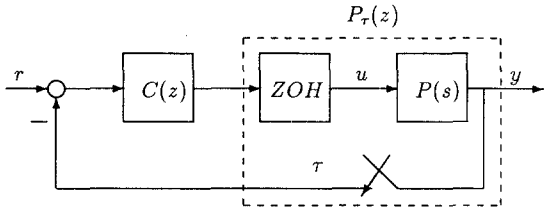


Figure 1: Digital Control System

function $S(z)$ of the closed-loop system of Fig.1 are defined by

$$T(z) := \frac{L(z)}{1 + L(z)}, \quad S(z) := \frac{1}{1 + L(z)} \tag{2.2}$$

where $L(z) := P_\tau(z)C(z)$ denotes the open-loop transfer function. Define $\hat{L}(\lambda) := L(1/\lambda)$ by using the transformation of $z = 1/\lambda$, we obtain the following Lemmas [5].

Lemma 1 *If $L(z)$ is strictly proper and the closed-loop system shown in Fig.1 is stable, then we have*

$$\frac{1}{\pi}\int_0^\pi \log|S(e^{j\phi})|\, d\phi = \sum_{i=1}^{\nu} \log|\alpha_i| \tag{2.3}$$

where $\alpha_i (i = 1, \cdots, \nu)$ are strictly unstable poles of $L(z)$.

Lemma 2 *Suppose the relative degree of $L(z)$ is $m(\geq 1)$ and its strictly unstable zeros are $\beta_i (i = 1, \cdots, \mu)$. If the closed-loop system shown in Fig.1 is stable, then we have*

$$\begin{aligned} &\frac{1}{\pi}\int_0^\pi \log|T(e^{j\phi})|\, d\phi \\ &= \sum_{i=1}^{\mu} \log|\beta_i| + \log\left|\frac{\hat{L}^{(m)}(0)}{m!}\right| \\ &= \sum_{i=1}^{\mu} \log|\beta_i| + \log|b_k| \end{aligned} \tag{2.4}$$

where b_k is the leading coefficient of the numerator of $L(z)$ when the denominator of $L(z)$ is monic.

Considering the relation between the leading coefficient of $L(z)$ and the poles of the closed-loop and open-loop systems, we have another formula for the constraint of $T(z)$.

Theorem 1 *Let $\alpha_i (i = 1, \cdots, n)$ and $\beta_j (j = 1, \cdots, \mu)$ be the poles and unstable zeros of $L(z)$, respectively. If the poles of the closed-loop system shown in Fig.1 are assigned to p_i(where $|p_i| < 1 : i = 1, \cdots, n$) then the following relation holds:*

$$\frac{1}{\pi}\int_0^{\pi} \log|T(e^{j\phi})|\, d\phi$$

$$= \sum_{i=1}^{\mu} \log|\beta_i| + \log\left|\sum_{i=1}^{n}{}^{m}\alpha_i - \sum_{i=1}^{n}{}^{m}p_i\right| \tag{2.5}$$

$$= \sum_{i=1}^{\mu} \log|\beta_i| + \log\frac{1}{m}\left|\sum_{i=1}^{n}\alpha_i^m - \sum_{i=1}^{n}p_i^m\right| \tag{2.6}$$

where the notation $\sum_{i=1}^{n}{}^{j}\alpha_i$ is the sum of all the j-combinations of $\alpha_i (i = 1, \cdots, n)$ and it is also the coefficient of $(-1)^j z^{n-j}$ in polynomial $\prod_{i=1}^{n}(z - \alpha_i)$.

(Proof) (2.5) is derived from the comparison of the open-loop and closed-loop characteristic equations. For simplicity, we use the notation $\sum_i$ or $\sum$ instead of $\sum_{i=1}^{n}$. Define $L(z)$ as

$$L(z) = \frac{b_{n-m}z^{n-m} + \cdots + b_1 z + b_0}{\prod_{i=1}^{n}(z - \alpha_i)} \tag{2.7}$$

then the pole-assignment condition yields

$$z^n - \sum p_i z^{n-1} + \cdots + (-1)^m \sum {}^m p_i z^{n-m} + \cdots + (-1)^n \sum {}^n p_i$$

$$= z^n - \sum \alpha_i z^{n-1} + \cdots + (-1)^m \sum {}^m \alpha_i z^{n-m} + \cdots + (-1)^n \sum {}^n \alpha_i$$

$$+ b_{n-m}z^{n-m} + \cdots + b_1 z + b_0 \tag{2.8}$$

The comparison of the coefficient of both sides in (2.8) leads to (2.5) and

$$\sum {}^k\alpha_i = \sum {}^k p_i \quad (k = 1, \cdots, m-1) \tag{2.9}$$

In order to complete the proof of (2.6), we will prove

$$\left|m(\sum {}^m\alpha_i - \sum {}^m p_i)\right| = \left|\sum \alpha_i^m - \sum p_i^m\right| \tag{2.10}$$

For $m = 1$, (2.10) is obvious. For $m \geq 2$, we consider the following relations: $(\mu \geq 2)$

$$n\sum_i^{\mu}\alpha_i = \alpha_1 \sum_{j\neq 1} {}^{\mu-1}\alpha_j + \cdots + \alpha_n \sum_{j\neq n} {}^{\mu-1}\alpha_j + (n-\mu)\sum_i {}^{\mu}\alpha_i$$

$$= \sum_i \left(\alpha_i \sum_{j\neq i} {}^{\mu-1}\alpha_j\right) + (n-\mu)\sum_i {}^{\mu}\alpha_i \tag{2.11}$$

and

$$\alpha_i \sum_{j\neq i} {}^k\alpha_j = \alpha_i\left(\sum_{j\neq i} {}^k\alpha_j + \alpha_i \sum_{j\neq i} {}^{k-1}\alpha_j - \alpha_i \sum_{j\neq i} {}^{k-1}\alpha_j\right)$$

$$= \alpha_i\left(\sum_j {}^k\alpha_j - \alpha_i \sum_{j\neq i} {}^{k-1}\alpha_j\right) \tag{2.12}$$

Substituting (2.12) into (2.11) recursively, we obtain

$$n\sum_i {}^{\mu}\alpha_i = \sum_i \left(\sum_{k=1}^{\mu-1}(-1)^{k+1}\alpha_i^k \sum_j {}^{\mu-k}\alpha_j\right)$$

$$+ (-1)^{\mu+1}\sum_i \alpha_i^{\mu} + (n-\mu)\sum_i {}^{\mu}\alpha_i \tag{2.13}$$

and hence we have

$$\mu\sum_i {}^{\mu}\alpha_i = \sum_i \left(\sum_{k=1}^{\mu-1}(-1)^{k+1}\alpha_i^k \sum_j {}^{\mu-k}\alpha_j\right) + (-1)^{\mu+1}\sum_i \alpha_i^{\mu}. \tag{2.14}$$

By replacing α_i by p_i in (2.14) and subtracting from (2.14), we obtain

$$\mu\left(\sum {}^{\mu}\alpha_i - \sum {}^{\mu}p_i\right) = (-1)^{\mu+1}(\sum \alpha_i{}^{\mu} - \sum p_i^{\mu})$$

$$+\sum_i \left(\sum_{k=1}^{\mu-1}(-1)^{k+1}(\alpha_i^k \sum_j {}^{\mu-k}\alpha_j - p_i^k \sum_j {}^{\mu-k}p_j)\right) \tag{2.15}$$

For $\mu \leq m-1$, using the condition (2.9), (2.15) is reduced to

$$\mu\left(\sum {}^{\mu}\alpha_i - \sum {}^{\mu}p_i\right) = (-1)^{\mu+1}(\sum \alpha_i{}^{\mu} - \sum p_i{}^{\mu})$$

$$+\sum_i \left(\sum_{k=1}^{\mu-1}(-1)^{k+1}(\alpha_i^k - p_i^k)\sum_j {}^{\mu-k}p_j\right) \tag{2.16}$$

Since the left-hand side of (2.16) is zero for $\mu \leq m-1$, (2.16) yields another condition

$$\sum \alpha_i^k = \sum p_i^k \quad (k = 1, \cdots, m-1) \tag{2.17}$$

For $\mu = m$, (2.16) is written by

$$m\left(\sum {}^m\alpha_i - \sum {}^m p_i\right) = (-1)^{m+1}(\sum \alpha_i{}^m - \sum p_i^m)$$

$$+\sum_i \left(\sum_{k=1}^{m-1}(-1)^{k+1}(\alpha_i^k - p_i^k)\sum_j {}^{m-k}p_j\right) \tag{2.18}$$

Therefore, we can see from the conditions (2.9) and (2.17) that (2.6) holds . □

Theorem 1 is a natural extension of the result of Theorem 2.3 in [5], which gives the constraint for $m = 1$. We note that the constraint depends on the stable and unstable poles of $L(z)$ as well as the unstable zeros of $L(z)$ for the complementary sensitivity function, while it only depends on the unstable poles of $L(z)$ for the sensitivity function [4].

3 Pole-assignment of digital controller

In this section, we consider the pole-assignment of digital controller with 1-sample computational-time delay

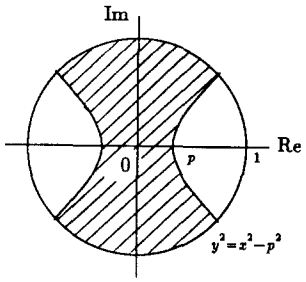
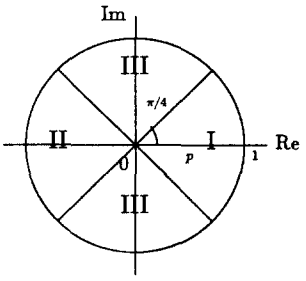


Figure 2: Region for pole-assignment



I: $\left(\frac{-\pi}{4} < \theta < \frac{\pi}{4}\right)$,
II: $\left(\frac{3\pi}{4} < \theta < \frac{5\pi}{4}\right)$ and
III: elsewhere in the unit disk

Figure 3: Pole-assignment for dead-beat control

based on the result for the complementary sensitivity function, Theorem 1.

In digital control system, the following assumptiona are satisfied in general.

Assumption A: $P(s)$ is an n_p-th order strictly proper rational function, then we have

(i) The relative degree of $P_\tau(z)$ is 1.

(ii) $\sum_{i=1}^{n_p} \alpha_{pi} > 0$ and $\sum_{i=1}^{n_p} \alpha_{pi}^2 > 0$ hold, where $\alpha_{pi}(i = 1, \cdots, n_p)$ are the poles of $P_\tau(z)$.

It is well-known that the assumption (i) holds for almost all sampling periods $\tau > 0$. We can easily verify that the assumption (ii) holds if the sampling period is selected as sufficiently small such that $\tau < \pi/2\omega_M$ (where ω_M is the maximum value of imaginary part of the poles in the continuous-time plant), since $|\angle\alpha_{pi}| < \pi/4$ holds in such a case. Hence, the assumption yields no restriction to the continuous-time plant $P(s)$ if the sampling period is chosen as small enough.

Assumption B: The controller $C(z)$ has 1-sample computational-time delay $1/z$, that is, $C(z) = \hat{C}(z)/z$ where the realtive degree of $\hat{C}(z)$ is zero.

In the design of the digital controller, a design guidance for the improvement of the sensitivity characteristics has been proposed as follows [4],[5]:

(a) Assign poles of $C(z)$ in the unit disk so that the controller is stable and minimum-phase.

On the ther hand, from Theorem 1, it is desirable to assign zeros of $C(z)$ in the unit disk for improvement of the complementary sensitivity characteristics.

From Assumptions A and B, we can assume the relative degree of the open-loop transfer function is 2. In this case, the second-term of (2.6) in Theorem 1 is written by

$$\log\frac{1}{2}\left|\sum p_i^2 - \sum \alpha_i^2\right| = \log\frac{1}{2}\left|\sum_{i=1}^{n} p_i^2 - \sum_{i=1}^{n_c} \alpha_{ci}^2 - \sum_{i=1}^{n_p} \alpha_{pi}^2\right|. \tag{3.1}$$

where $\alpha_{ci}(i = 1, \cdots, n_c)$ are the poles of the controller.

Based on the Theorem 1, we propose the following guidance:

Guidance for pole-assignment of digital controller $C(z)$

(1) Since $\Sigma\alpha_{pi} > 0$, $\Sigma\alpha_{pi}^2 > 0$ and $|p_i| < 1$, it is desirable to set $\Sigma p_i^2 - \Sigma\alpha_{ci}^2$ positive and the approximate value of $\Sigma\alpha_{pi}^2$ to improve the complementary sensitivity characteristics. When the closed-loop poles are real, it is desirable to assign the poles of the controller to the shaded region in Fig.2.
(2) For the dead-beat control, it is desirable to assign the poles of the controller so that the value of

$$\left|\sum_{i=1}^{n} \alpha_i^2\right| = \left|\sum_{i=1}^{n_c} \alpha_{ci}^2 + \sum_{i=1}^{n_p} \alpha_{pi}^2\right| \tag{3.2}$$

should be reduced. When poles of the plant are real, it is desirable to assign the poles of the controller to region III in Fig.3 (especially, if poles of the controller are also real, on real axis closer to the origin).

We confirm the above by an example.

Example 1: Consider a digitalized plant

$$P_\tau(z) = (1 - e^\tau)/(z - e^\tau)$$

and a digital controller

$$C(z) = (\gamma z^2 + \delta z + \epsilon)/z(z - \alpha_1)(z - \alpha_2)$$

The parameters of $C(z)$ are chosen so that the closed-loop system is dead-beat for $\tau = 0.1$ and $\alpha_1 + \alpha_2 = -e^\tau$. Fig.4 shows that $|T(e^{j\phi})|$ for a)$\alpha_1 = -0.6$ ($\alpha_2 = -0.5052$), b)$\alpha_1 = 0.1$ ($\alpha_2 = -1.2052$) and c)$\alpha_1 = 0.5$ ($\alpha_2 = -1.6052$). It can be seen from Fig.4 that if we set the poles of the controller closer to the origin, we have the better complementary sensitivity characteristics. It is also confirmed from the values of $|\Sigma\alpha_i^2| =$ a)0.6152, b)1.4625 and c)2.8267. Since $\alpha_1 + \alpha_2 = -e^\tau$ and the minimum value of $\Sigma\alpha_i^2 = (\alpha_1 + \alpha_2)^2 - 2\alpha_1\alpha_2$ is obtained by the maximum value of $\alpha_1\alpha_2$, we obtain the best complementary sensitivity characteristic at $\alpha_1 = \alpha_2 = -0.5526$.

In the practical design point of view, both the complementary sensitivity and sensiticity characteristics must

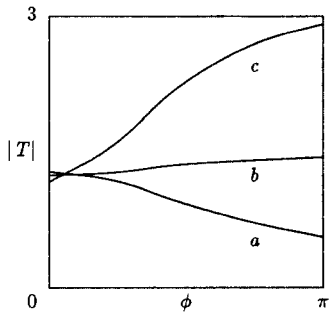


Figure 4: $|T(e^{j\phi})|$ in Example 1

be considered. The following shows the pole-assignment of digital controller considering both the sensitivity and complementary sensitivity characteristics for $m = 1$ and $m = 2$.

Example 2: Consider a plant

$$P_1(z) = 1/(z-1)$$

and a controller

$$C(z) = (bz - c)/(z - \alpha)$$

i.e., $m = 1$. If the closed-loop system is dead-beat, we have $b = 1 + \alpha$ and $c = \alpha$. Fig.4 shows that $|T|$ and $|S|$ for the value of the pole of controller, α. Secondly, in the case of $m = 2$, we consider a plant

$$P_2(z) = 1/(z-1)$$

and a controller

$$C(z) = (bz - c)/(z - \alpha)(z - \beta)$$

then $\alpha + \beta = -1$ and $b = 1 + \alpha + \alpha^2$ and $c = \alpha + \alpha^2$. The properties of $|T|$ and $|S|$ are illustrated in Figs. 5 and 6.

Note that for the dead-beat control, it is desirable to assign the poles of digital controllers to the left(right)-hand side in the unit circle to improve the complementary sensitivity (sensitivity) characteristics for $m = 1$[5]. On the other hand, for $m = 2$, the poles of the controller must assign closer to the origin to improve the complementary sensitivity characteristics [5]. Example 2 shows that it is desirable to assign to $\alpha = -0.5$ for both cases to improve the complementary sensitivity characteristics, however, for the sensitivity characteristic, $-1 \leq \alpha \leq 0$ for $P_2(z)$ ($-1 \leq \alpha \leq 1$ for $P_1(z)$). Therefore, in the practical design, we need to consider their tradeoffs.

4 Conclusion

Some integral-type constraints on the complementary sensitivity function have been developed for SISO digital control systems. Based on the results, we have proposed several design guidances for the pole-assignment of the digital controller in the digital control system with 1-sample computational-time delay.

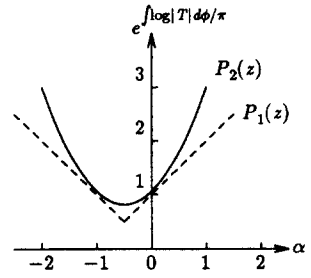


Figure 5: $e^{\int \log|T| d\phi/\pi}$ in Example 2

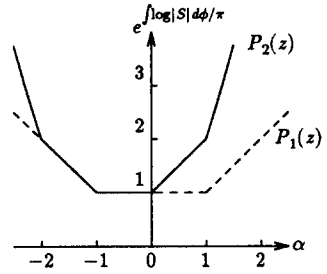


Figure 6: $e^{\int \log|S| d\phi/\pi}$ in Example 2

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