H_∞ CONTROLLER DESIGN VIA LQ GAME PROBLEM FOR DISCRETE TIME SYSTEMS

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ABSTRACT

In this paper, a state space solution to the discrete time H_∞ control problem is presented. It is shown that there exist LQ game problems corresponding to H_∞ control problems and the H_∞ controller can be obtained by solving the LQ game problem. Explicit state space formulae are given for the state feedback H_∞ controllers and output feedback H_∞ controllers.

I. INTRODUCTION

The H_∞ theory has been developed in both the input-output operator and the state space frame work[1,4]. Furthermore, a number of recent papers has shown that certain H_∞ control problems for continuous time systems can be simply solved by using Riccati equation based approaches[2,3,5,6,7]. However, discrete time H_∞ optimal control has not been studies extensively. Furuta and Phoojaruenchanachai propose a discrete time H_∞ controller using Bounded Real Lemma [8]. Bilinear transformation is used to get the discrete time H_∞ controllers by Gu *et al.*[9].

In this paper, we propose a state space method for the discrete time H_∞ control problem. This method is motivated from the relations between the H_∞ control problem and the LQ game problem. This relations are reported in several papers[6,7,10] for continuous time systems. And using this relation, H_∞ controller for general time varying continuous system is obtained [10]. To author's knowledge, for the discrete time systems there exists no paper dealing with the connection between the H_∞ control problem and the LQ game problem. We generalize these relations to the discrete time linear systems and get an H_∞ controller from the LQ game problem. The solution of the LQ game problem is obtained by using a modified Riccati equation.

In Section II, some definitions and facts are presented. In Section III, we generalize the relations between the H_∞ control problem and the LQ game problem to the discrete time linear systems. In Section IV, we will give the explicit state feedback solutions to the discrete time H_∞ control problems. In Section V, an output feedback solution is given.

II. PRELIMINARY

For the discrete time linear system G

$$
\begin{aligned}
x(i+1) &= Ax(i) + Bw(i) \\
z(i) &= Cx(i)
\end{aligned}
\tag{2.1}
$$

over the time interval $0 \le i \le N$.

Each norm of signals is defined by

$$
\begin{aligned}
\|w\|_2^2 &:= \sum_{i=0}^{N} w^T(i)w(i) \\
\|z\|_2^2 &:= \sum_{i=0}^{N} z^T(i)z(i).
\end{aligned}
\tag{2.2}
$$

Let $g(\cdot)$ be the impulse response of the system (2.1). Define two norm and infinite norms of the system as

$$\|G\|_2^2 := Trace\left\{\sum_{i=0}^{N} g^T(i)g(i)\right\} \tag{2.3}$$

where

$$x(0) = 0, \quad w(i) = 0 \quad for \quad 0 \le i \le N$$

and

$$\|G\|_\infty := \max_{\|w\|_2 \neq 0}\left\{\frac{\|z\|_2}{\|w\|_2}\right\} \tag{2.4}$$

where $x(0) = 0$.

The system used in this paper is described by

$$
\begin{aligned}
x(i+1) &= Ax(i) + B_1w(i) + B_2u(i) \\
z(i) &= C_1x(i) + D_2u(i) \\
y(i) &= C_2x(i) + D_1w(i).
\end{aligned}
\tag{2.5}
$$

The signal w contains all external inputs, including disturbances, sensor noise, and commands; the output z is an error signal; y is the measured variables; and u is the control input. The resulting closed-loop transfer function from w to z is denoted by T_{zw}.

III. THE RELATIONS BETWEEN THE LQ GAME PROBLEM AND THE OPTIMAL H_∞ PROBLEM

The H_∞ problem(3.1) and the LQ game problem with parameter γ (3.2) are defined as

$$\min_u \sup_{\|w\|_2 \neq 0}\left\{\frac{\|z\|_2}{\|w\|_2}\right\} = \gamma^* \tag{3.1}$$

$$\min_u \max_w \{\|z\|_2^2 - \gamma^2\|w\|_2^2\} = \alpha(\gamma) \quad 0 \le \gamma \tag{3.2}$$

where $\gamma \ge 0$.

In this section, it is shown that the solution of LQ game problem is equivalent to an H_∞ sub-optimal controller which stabilizes the closed loop system and reduces the $\|T_{zw}\|_\infty$ to γ.

THEOREM 1. *If the LQ game problem with parameter γ has a solution then $\gamma^* \le \gamma$.*

Proof: Let $\tilde{u}$ and $\tilde{w}$ be the solution of the LQ game problem. Denote $\tilde{z} = z(\tilde{u}, \tilde{w})$.

Consider the case that $w = 0$ then clearly

$$\min_u \max_w \{\|z\|_2^2 - \gamma^2\|w\|_2^2\} := \alpha(\gamma) \ge 0. \tag{3.3}$$

By Equation (3.3) for all w

$$\|z(\tilde{u}, w)\|_2^2 - \gamma^2\|w\|_2^2 \le \|\tilde{z}\|_2^2 - \gamma^2\|\tilde{w}\|_2^2 \ge 0. \tag{3.4}$$

Hence, from Equation (3.3)for all $w \neq 0$

$$\frac{\|z(\tilde{u}, w)\|_2^2}{\|w\|_2^2} \le \frac{\alpha(\gamma)}{\|w\|_2^2} + \gamma^2. \tag{3.5}$$

And

$$\sup_{w\neq 0}\frac{\|z(\tilde{u},w)\|_2^2}{\|w\|_2^2}=\max_{\|w\|_2=M}\frac{\|z\|_2^2}{\|w\|_2^2}\leq\gamma^2+\frac{\alpha(\gamma)}{M^2} \tag{3.6}$$

where $M>0$ and $u=\tilde{u}$
Equation (3.6) must be satisfied for all $M>0$. Let $M->\infty$, and we obtain

$$\sup_{w\neq 0}\frac{\|z(\tilde{u},w)\|_2^2}{\|w\|_2^2}\leq\gamma^2. \tag{3.7}$$

Equation (3.7) means that $\gamma^*\leq\gamma$. ▮

THEOREM 2. *If $\gamma^*<\gamma$ then the LQ game problem with parameter γ has a solution.*

Proof: Before proving this theorem we prove two lemmas.

LEMMA 1. *If there exist a solution of LQ game problem then*

$$\frac{\partial^2 J_\gamma}{\partial u^2}(u,w)\geq 0 \tag{3.8}$$

$$\frac{\partial^2 J_\gamma}{\partial w^2}(u,w)\leq 0 \tag{3.9}$$

where

$$J_\gamma:=\|z\|_2^2-\gamma^2\|w\|_2^2 \tag{3.10}$$

Proof: Since J_γ is a quadratic functional of u and w ,so $\frac{\partial^2 J_\gamma}{\partial u^2}(u,w)$ and $\frac{\partial^2 J_\gamma}{\partial w^2}(u,w)$ are constant for all u and w. Especially at the solution of the LQ game problem, the above values satisfies

$$\frac{\partial^2 J_\gamma}{\partial u^2}(u,w)\geq 0 \tag{3.11}$$

$$\frac{\partial^2 J_\gamma}{\partial w^2}(u,w)\leq 0. \tag{3.12}$$

▮

LEMMA 2. *If LQ game problem with parameter with $\gamma 1$ is solved and $\gamma 1\leq\gamma 2$ then LQ game problem with parameter $\gamma 2$ can be solved.*

Proof: Since $J_{\gamma 2}$ is more negative than $J_{\gamma 1}$, LQ game problem with parameter $\gamma 2$ can be solved. ▮

Proof of theorem 2: Let (u^*, w^*) be the solution of the H_∞ problem. We will show that (u^*, w^*) is one solution of LQ game problem with parameter γ^*. And then from the Lemma 2, LQ game problem with parameter $\gamma 2$ has a solution.
Denote $J_1:=\|z\|$ and $J_2:=\|w\|$. Since (u^*,w^*) is a solution of H_∞ problem

$$\delta\left(\frac{J_1}{J_2}\right)(u^*,w^*)=0. \tag{3.13}$$

Namely

$$\frac{\delta J_1 J_2-\delta J_2 J_1}{J_2^2}=0 \tag{3.14}$$

and

$$\delta J_1-\left(\frac{J_1}{J_2}\right)\delta J_2=0. \tag{3.15}$$

Since

$$\left(\frac{J_1}{J_2}\right)(u^*,w^*)=(\gamma^*)^2 \tag{3.16}$$

we obtain

$$\delta J_1-(\gamma^*)^2\delta J_2(u^*,w^*)=0. \tag{3.17}$$

Hence (u^*,w^*) may be a solution of the LQ game problem with parameter γ^*. We must show that the followings.

$$\frac{\partial^2 J_\gamma^*}{\partial u^2}(u^*,w^*)\geq 0 \tag{3.18}$$

$$\frac{\partial^2 J_\gamma^*}{\partial w^2}(u^*,w^*)\leq 0 \tag{3.19}$$

Equation (3.18) is always satisfied. When $u=u^*$ and for all w Equation(3.19) is satisfied.

$$\frac{\|z(u^*,w)\|_2^2}{\|w\|_2^2}\leq(\gamma^*)^2 \tag{3.20}$$

And for all w

$$\|z(u^*,w)\|_2^2-(\gamma^*)^2\|w\|_2^2\leq 0. \tag{3.21}$$

Hence we get

$$J_{\gamma^*}(u^*,w)\leq J_{\gamma^*}(u^*,w^*)\quad for\quad all\quad w. \tag{3.22}$$

Equation (3.22) means (3.19). From this result and Lemma 1, the theorem is proven. ▮

COROLLARY 1. *Let $\hat{u}$ be the solution of LQGP(γ). With the control $\hat{u}$ the closed loop system satisfies*

$$\|z\|_2^2\leq\alpha(\gamma).$$

Corollary 1 states the guaranteed H_2 performance bound of the sub-optimal H_∞ controller.

IV. STATE FEEDBACK CONTROL

In this section the explicit solution of state feedback controller is proposed. The solution is obtained by solving the LQ game problem. We will solve the LQ game problem by the dynamic programming.

Let $C_1=C, D_2=D, C_2=I, D_1=0$.
Assume that $C^TD=0$, $R:=D^TD>0$, $Q:=C^TC\geq 0$.
Define the performance $J(\gamma)$ as follows.

$$\begin{aligned}J(\gamma)&:=\frac{1}{2}\{\|z\|_2^2-\gamma^2\|w\|_2^2\}\\&=\frac{1}{2}\left(x^T(N)Qx(N)+u^T(N)Ru(N)-w^T(N)w(N)\right)\\&+\frac{1}{2}\sum_{i=0}^{N-1}\{x^T(i)Qx(i)+u^T(i)Ru(i)\}-\frac{1}{2}\gamma^2\sum_{i=0}^{N-1}w^T(i)w(i)\end{aligned} \tag{4.1}$$

Then LQ game problem with parameter γ is equivalent to

$$\min_{u(i)}\max_{w(i)}J(\gamma). \tag{4.2}$$

We begin by defining

$$J_{NN}(x(N))=\frac{1}{2}x^T(N)P(N)x(N),\quad P(N):=Q,\quad u(N):=0,\quad w(N)=0$$

The cost over the final interval is given by

$$\begin{aligned}J_{N-1,N}\left(x(N-1),u(N-1),w(N-1)\right)&=\frac{1}{2}x^T(N-1)Qx(N-1)\\&+\frac{1}{2}u^T(N-1)Ru(N-1)-\frac{1}{2}\gamma^2w^T(N-1)w(N-1)\\&+\frac{1}{2}[Ax(N-1)+B_1w(N-1)+B_2u(N-1)]^T\\&\times P(N)[Ax(N-1)+B_1w(N-1)+B_2u(N-1)].\end{aligned} \tag{4.3}$$

To minimize $J_{N-1,N}$ with respect to $u(N-1)$ and maximize $J_{N-1,N}$ with respect to $w(N-1)$ we need to evaluate the indicated partial derivatives.

$$\begin{aligned}\frac{\partial J_{N-1,N}}{\partial u}&=Ru(N-1)+B_2^TP(N)\\&\times[Ax(N-1)+B_1w(N-1)+B_2u(N-1)]=0\end{aligned} \tag{4.4}$$

$$\frac{\partial J_{N-1,N}}{\partial w} = -\gamma^2 w(N-1) + B_1^T P(N) \times [Ax(N-1) + B_1 w(N-1) + B_2 u(N-1)] = 0 \tag{4.5}$$

$$\frac{\partial^2 J_{N-1,N}}{\partial u^2} = R + B_2^T P(N) B_2 > 0 \tag{4.6}$$

$$\frac{\partial^2 J_{N-1,N}}{\partial w^2} = -\gamma^2 I + B_1^T P(N) B_1 < 0 \tag{4.7}$$

Equation (4.6) is always satisfied and Equation (4.7) is satisfied by suitable chosen γ. From Equation (4.4) and (4.5) we get a Equation (4.8)

$$\begin{bmatrix} R + B_2^T P(N) B_2 & B_2^T P(N) B_1 \\ B_1^T P(N) B_2 & -\gamma^2 I + B_1^T P(N) B_1 \end{bmatrix} \begin{bmatrix} u(N-1) \\ w(N-1) \end{bmatrix} = \begin{bmatrix} -B_2^T \\ -B_1^T \end{bmatrix} P(N) A x(N-1) \tag{4.8}$$

Solving (4.5) then we get the optimal control $u^*(N-1)$ and the worst case disturbance $w^*(N-1)$

$$\begin{bmatrix} u^*(N-1) \\ w^*(N-1) \end{bmatrix} = \begin{bmatrix} R + B_2^T P(N) B_2 & B_2^T P(N) B_1 \\ B_1^T P(N) B_2 & -\gamma^2 I + B_1^T P(N) B_1 \end{bmatrix}^{-1} \times \begin{bmatrix} -B_2^T \\ -B_1^T \end{bmatrix} P(N) A x(N-1) := \begin{bmatrix} F(N-1) \\ G(N-1) \end{bmatrix} x(N-1). \tag{4.9}$$

We must choose γ in order that the inverse in Equation (4.9) exists. Substituting the expression for $u^*(N-1), w^*(N-1)$ into the Equation (4.3) for $J_{N-1,N}$ gives $J^*_{N-1,N}$.

$$\begin{aligned} &J^*_{N-1,N}(x(N-1), u^*(N-1), w^*(N-1)) \\ &= \frac{1}{2} x^T(N-1)\{[A + B_2 F(N-1) + B_1 G(N-1)]^T P(N) \\ &\times [A + B_2 F(N-1) + B_1 G(N-1)] + F^T(N-1) R F(N-1) \\ &\quad - \gamma^2 G^T(N-1) G(N-1) + Q\} x(N-1) \\ &:= \frac{1}{2} x^T(N-1) P(N-1) x(N-1). \end{aligned} \tag{4.10}$$

In general, the same derivation gives optimal solution $u^*(N-k), w^*(N-k)$

$$\begin{aligned} u^*(N-k) &= F(N-k) x(N-k) \\ w^*(N-k) &= G(N-k) x(N-k) \quad 1 \le k \le N \end{aligned} \tag{4.11}$$

where F and G are obtained by

$$\begin{bmatrix} F(N-k) \\ G(N-k) \end{bmatrix} = - \begin{bmatrix} R + B_2^T P(N-k+1) B_2 & B_2^T P(N-k+1) B_1 \\ B_1^T P(N-k+1) B_2 & -\gamma^2 I + B_1^T P(N-k+1) B_1 \end{bmatrix}^{-1} \times \begin{bmatrix} B_2^T \\ B_1^T \end{bmatrix} P(N-k+1) A x(N-k) \tag{4.12}$$

and

$$\begin{aligned} P(N-k) &= [A + B_2 F(N-k) + B_1 G(N-k)]^T P(N-k+1) \\ &\times [A + B_2 F(N-k) + B_1 G(N-k)] + F^T(N-k) R F(N-k) \\ &\quad - \gamma^2 G^T(N-k) G(N-k) + Q. \end{aligned} \tag{4.13}$$

And the optimal value is given by

$$J^*_{0,N}(x(0), u^*(0), w^*(0)) = \frac{1}{2} x^T(0) P(0) x(0).$$

For infinite time controller the following Riccati equation must be solved.

$$\begin{bmatrix} F_\infty \\ G_\infty \end{bmatrix} = - \begin{bmatrix} R + B_2^T P_\infty B_2 & B_2^T P_\infty B_1 \\ B_1^T P_\infty B_2 & -\gamma^2 I + B_1^T P_\infty B_1 \end{bmatrix}^{-1} \begin{bmatrix} B_2^T P_\infty A \\ B_1^T P_\infty A \end{bmatrix} \tag{4.15}$$

$$\begin{aligned} P_\infty &= [A + B_2 F_\infty + B_1 G_\infty]^T P_\infty [A + B_2 F_\infty + B_1 G_\infty] \\ &\quad + F_\infty^T R F_\infty - \gamma^2 G_\infty^T G_\infty + Q \end{aligned} \tag{4.16}$$

The parameter γ must be chosen in order that the inverse in Equation (4.15) exists and the parameter γsatisfies

$$-\gamma^2 I + B_1^T P_\infty B_1 < 0. \tag{4.17}$$

Then the optimal solutions are

$$\begin{aligned} u^*(i) &= F_\infty x(i) \\ w^*_{worst}(i) &= G_\infty x(i). \end{aligned} \tag{4.18}$$

Using the result in this section, we can obtain the output feedback form of H_∞ controler as in section V.

V. Output Feedback Control

To get the output feedback form of H_∞ controller, we following the Doyle's approach [7]. The infinite time system (2.3) has the realization of the transfer matrix taken to be of the form

$$G := \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_1 \\ C_2 & D_2 & 0 \end{bmatrix}.$$

The following assumptions are made

(i) (A, B_1) is stabilizable, (C_1, A) is dectectable.
(ii) (A, B_2) is stabilizable, (C_2, A) is detectable.
(iii) $D_1^T [C_1 D_1] = [0 R_1], R_1 > 0$

(iv)

$$\begin{bmatrix} B_1 \\ D_2 \end{bmatrix} D_2^T = \begin{bmatrix} 0 \\ R_2 \end{bmatrix} \quad , R_2 > 0$$

Then the H_∞ controller is given by the following steps. Define a matrix

$$A_{tmp} := A + B_1 G_\infty \tag{5.1}$$

$$G_{tmp} := \begin{bmatrix} A_{tmp} & B_1 & B_2 \\ -F_\infty & 0 & I \\ C_2 & D_2 & 0 \end{bmatrix}. \tag{5.2}$$

Where G_∞ is defined in Equation (4.15). Then a sub-optimal controller is

$$K_{sub} := \begin{bmatrix} A_{tmp} - L_{tmp} C_2 + B_2 F_\infty & -L_{tmp} \\ F_\infty & 0 \end{bmatrix} \tag{5.3}$$

where L_{tmp} is output estimation solution of G_{tmp}, which is defined in the followings. These problems are defined as in [6]. The problems are labeled as follows.

FI: Full information problem
FC: Full control problem
DF: Disturbance feedforward control problem
OE: Output estimation problem

A. Full Information Problem: In this problem, the transfer matrix G is taken to be of the form

$$G := \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_1 \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}.$$

The following assumptions are made.

(i) (A, B_1) is stabilizable, (C_1, A) is dectectable.

(ii) (A, B_2) is stabilizable.

(iii) $D_1^T[C_1 D_1] = [0 R_1]$, $R_1 > 0$

The H_∞ controller is given by

$$K = F_\infty .$$

B. Full Control Problem: In this problem, the transfer matrix G is taken to be of the form

$$G := \begin{bmatrix} A & B_1 & [I0] \\ C_1 & 0 & [0I] \\ C_2 & D_2 & [00] \end{bmatrix}$$

The following assumptions are made.

(i) (A, B_1) is stabilizable, (C_1, A) is dectectable.

(ii) (C_2, A) is detectable.

(iv)

$$\begin{bmatrix} B_1 \\ D_2 \end{bmatrix} D_2^T = \begin{bmatrix} 0 \\ R_2 \end{bmatrix} \qquad , R_2 > 0$$

The H_∞ controller is given by

$$K = L_\infty .$$

Where

$$\begin{bmatrix} L_\infty \\ M_\infty \end{bmatrix} = -\begin{bmatrix} R_2 + C_2 S_\infty C_2^T & C_2 S_\infty C^T \\ C_1 S_\infty C_2^T & \gamma^2 I + C_1 S_\infty C_1^T \end{bmatrix}^{-1} \begin{bmatrix} C_2 S_\infty A^T \\ C_1 S_\infty A^T \end{bmatrix} \tag{5.4}$$

and

$$S_\infty = [A^T + C_2^T L_\infty + C_1^T M_\infty]^T S_\infty [A^T + C_2^T L_\infty + C_1^T M_\infty] + L_\infty^T R_2 L_\infty - \gamma^2 M_\infty^T M_\infty + B_1 B_1^T . \tag{5.5}$$

The parameter γ must satisfy the following equation.

$$-\gamma^2 I + C_1 S_\infty C_1^T \leq 0 \tag{5.6}$$

C. Disturbance Feedforward Control Problem: In this problem, the transfer matrix G is taken to be of the form

$$G = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_1 \\ C_2 & I & 0 \end{bmatrix} .$$

Assume that $A - B_1 C_2$ is stable then a H_∞ solution is given by

$$K = \begin{bmatrix} A + B_2 F_\infty - B_1 C_2 & B_1 \\ F_\infty & 0 \end{bmatrix}$$

D. Output Estimation Problem: In this problem, the transfer matrix G is taken to be of the form

$$G = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & I \\ C_2 & D_2 & 0 \end{bmatrix} .$$

The following assumptions are made.

(i) (A, B_1) is stabilizable, $A - B_2 C_1$ is stable.

(ii) (A, B_2) is stabilizable, (C_2, A) is detectable.

(iv)

$$\begin{bmatrix} B_1 \\ D_2 \end{bmatrix} D_2^T = \begin{bmatrix} 0 \\ R_2 \end{bmatrix} \qquad , R_2 > 0$$

Then the H_∞ solution is given by

$$K = \begin{bmatrix} A + L_\infty C_2 & L_\infty \\ C_1 & 0 \end{bmatrix}$$

VI. Conclusion

In this paper, a H_∞ controller design method for the discrete time linear system is presented. It is shown that the solution of the LQ game problem is a H_∞ controller. From this fact, we can obtain the discrete time H_∞ controller in state space by solving the modified Riccati equations. Adjusting the parameter of the LQ game problem, we can get a controller which has the desired H_∞ performance and H_2 performance. And in order to otain the optimal H_∞ controller an iterative computation is needed. The results can be applied to the robust stabilization of linear discrete time systems including uncertain parameters.

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