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The Discrete-Time H_2/H_∞ Control Synthesis : State Feedback Case

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Abstract

A synthesis of feedback control-law with combined H_2/H_∞ performance criteria is proposed for discrete-time systems, under the assumption that the state is available for feedback. An auxiliary minimization problem is defined to enforce the H_∞ disturbance attenuation constraint while minimizing the H_2 performance bound. The design equation is presented in terms of a modified Riccati equation which leads to the standard LQ solution when the H_∞ constraint is completely relaxed. The results of the paper clarify the correspondences between H_2/H_∞ results in discrete-time systems and their continuous-time counterparts.

1.Introduction

Recently, much interest has been devoted to the design of feedback controllers for linear systems that minimizes the H_∞-norm of a specified closed-loop transfer function[4,5,6]. H_∞ approach has been known to be a powerful method in dealing with the problem of robust stability,i.e., obtaining closed-loop stability in the presence of system uncertainty. More recently, the design of feedback controllers with the combined H_2/H_∞ performance criteria has received a great deal of attention. The greater interests in the latter problem have been encouraged by the fact that it represents a problem of optimal nominal performance with guaranteed robust stability[1,12,13,14]. In this paper, a such combined H_2/H_∞ control synthesis in discrete-time systems is considered.

Continuous-time solution to the combined H_2/H_∞ control problem has been previously derived by Bernstein and Haddad[1], where a dynamic reduced order controller is given in terms of solutions to the modified Riccati equations. This result has also been extended to a more general problem. The discrete-time H_2/H_∞, however, has received a little attention. There is, of course, the well-known bilinear transformation which can be used to convert between continuous-time and discrete-time models. The application of this transformation to the results of continuous-time case, however, will not give satisfactory results for the corresponding discrete-time problems[3]. Consequently, the discrete-time H_2/H_∞ control problem is of interest in its own right.

Independently with the work of Bernstein et. al. who recently also consider the discrete-time case in the decentralized setting[15], we are able to derive the H_2/H_∞ state feedback controller. Compared to their results, first, we note that our Riccati equation which contains additional quadratic term, is directly related to the discrete-time LQ solution, and corresponds to the continuous-time H_∞ solution. Secondly, our Riccati equation is of the form that arises in discrete-time games and LEQG(Linear Exponential Quadratic Gaussian) solution, the problems to which H_∞ design is known to has some interesting connections[1,4,6,8]. Finally, our results provide relations with some existing results of the standard H_∞ control problem[2,3]. We note also that, there might be some difficulties in the computational algorithms for obtaining the solution to the design equation derived in [15]. The modified Riccati equation of our results, on the other hand, can be easily solved using the eigen vector method via the associated Hamiltonian matrix, as illustrated in a numerical example.

In [3], solution to the standard discrete-time H_∞ problem in state feedback case was solved using the saddle point solution to the corresponding discrete-time games. However, no H_2 interpretation was provided in [3]. Thus, the present paper attempts to provide H_2 interpretation in the context of H_∞ design constraints as done for continuous-time case in [1]. Moreover, we shall remove the orthogonality assumptions used in [3]. In this paper, however, we consider a certain class of the combined H_2/H_∞ control problem where the H_2 and H_∞ weighting matrices are equalized and we assume that the states are available for feedback.

2. Problem Formulation and Preliminaries

Consider time invariant discrete-time systems described by

$$x(k+1) = Ax(k) + B_2u(k) + B_1w(k) \quad (1)$$
$$z(k) = C_1x(k) + D_{12}u(k) \quad (2)$$

where $w(k) \in R^m$ is the disturbance vector; $u(k) \in R^p$ is the control vector; $x(k) \in R^n$ is the state vector; and $z(k) \in R^l$ is the controlled-output vector which may also be interpreted as the performance variable. Next, the following standard assumptions on the plant are made,

1. (A, B_2) and (A, C_1) are completely controllable and observable, respectively;
2. $D_{12}'[C_1 \quad D_{12}] = [0 \quad I]$.

Assumption 1 guarantees that the set of stabilizing controllers is nonempty, while assumption 2 amounts to orthogonality of $C_1x(k)$ and $D_{12}u(k)$ in the output. In section 5, assumption 2 will be removed.

The combined H_2/H_∞ considered in this paper is as follows : for the plant given by (1)-(2), determine control-law

$$u(k) = Kx(k) \quad (3)$$

such that the following design criteria are satisfied,

1. the closed-loop system constructed by equations (1)-(3) is asymptotically stable,i.e. all eigen values of $\tilde{A} := A + B_2K$ lie inside the open unit circle;
2. the closed-loop transfer function from $w(k)$ to $z(k)$
$$T_{zw} := (C_1 + D_{12}K)(zI - A - B_2K)^{-1}B_1 \quad (4)$$
satisfies the H_∞ disturbance attenuation constraint
$$\|T_{zw}\|_\infty \leq \gamma \quad (5)$$
with $\gamma > 0$ a given constant;
3. the H_2 performance criteria defined by
$$J := \lim_{k\to\infty} \mathcal{E}[x'(k)R_1x(k) + u'(k)R_2u(k)] \quad (6)$$
is minimized, where $\mathcal{E}$ denotes the expectation.

Note that the standard H_∞ problem as considered in[3,4], does not deal with H_2 performance criterion.

The closed-loop system constructed by (1)-(3) can be written as

$$x(k+1) = \tilde{A}x(k) + \tilde{B}w(k) \quad (7)$$
$$z(k) = \tilde{C}x(k) \quad (8)$$

where

$$\tilde{A} := A + B_2K, \quad \tilde{B} := B_1, \quad \tilde{C} := C_1 + D_{12}K \quad (9)$$

and the H_2 performance criterion defined in (6) is also given by

$$J := \lim_{k\to\infty} \mathcal{E}[x'(k)(R_1 + K'R_2K)x(k)] \quad (10)$$

Since, in the present paper, we consider the equalized H_2/H_∞ control problem, we may further assume that $R_1 = C_1'C_1$ and $R_2 = D_{12}'D_{12}$, so we can rewrite (10), using assumption 2, as

$$J := \lim_{k\to\infty} \mathcal{E}[x'(k)\tilde{C}'\tilde{C}x(k)] \quad (11)$$

Moreover, if $\tilde{A}$ is asymptotically stable for a given controller K, the performance criterion (11) can be further expressed by [9]

$$J = tr(\hat{Q}\tilde{B}\tilde{B}') \quad (12)$$

where $\hat{Q}$ is the solution to the following Liapunov equation

$$\hat{Q} = \tilde{A}'\hat{Q}\tilde{A} + \tilde{C}'\tilde{C} \quad (13)$$

Remark 2.1
The performance criterion described in (10) is also given, alternatively, by

$$J = tr(\hat{P}\tilde{C}'\tilde{C})$$

where $\hat{P}$ is the solution to the Liapunov equation

$$\hat{P} = \tilde{A}\hat{P}\tilde{A}' + \tilde{B}\tilde{B}'$$

and is the steady-state closed-loop covariance defined by

$$\hat{P} = \lim_{k\to\infty} \mathcal{E}[x(k)x'(k)]$$

The following lemmas are useful in deriving the main result of the paper.

Lemma 2.1 *Let $G(z) := C(zI - A)^{-1}B$, with A a stable matrix. If there exists a symmetric positive definite solution to the following quadratic matrix equality (QME)*

$$X = A'XA + XB(\gamma^2I + B'XB)^{-1}B'X + C'C \quad (14)$$

then $\|G\|_\infty \leq \gamma$.

Proof.
First, we rewrite QME (14) as

$$X = A'XA + L'RL + C'C$$

where

$$L := (\gamma^2I + B'XB)^{-1}B'X, \quad R = (\gamma^2I + B'XB),$$

from which we obtain

$$C'C = (z^{-1}I - A)'X(zI - A) + (z^{-1}I - A)'XA$$
$$+ A'X(zI - A) - L'RL$$

Using the last equation and employing the identity $(U^{-1} + VZ^{-1}W)^{-1} = U - UV(Z + WUV)^{-1}WU$, we finally obtain the following spectral factorization of $\gamma^2 I - G^*(z)G(z)$,

$$\gamma^2 I - G^*(z)G(z) = [I - Lz^{-1}(z^{-1}I - A)^{-1}B]'R$$
$$[I - Lz(zI - A)^{-1}B].$$

Now, with * denoting the complex conjugate, the right hand side of the last equation is of the form $P^*(z)RP(z)$. Using the assumed existence of positive definite solution to the QME (14) and the fact that $P^*(z)RP(z) \geq 0$,on noting that $R \geq 0$, the equation implies that $G^*(z)G(z) \leq \gamma^2 I$, and thus, $\|G\|_\infty \leq \gamma$. Q.E.D.

Lemma 2.2 *If $Q \geq 0$ and A is stable, then the Liapunov equation $P = A'PA + Q$ has a unique solution P, and $P \geq 0$.*

Lemma 2.3 *Suppose $P \geq 0$, $Q \geq 0$, $(A, Q^{1/2})$ is detectable and $A'PA - P + Q = 0$. Then A is stable.*

The disturbance attenuation constraint (5) is enforced by replacing the algebraic Liapunov equation (13) by a quadratic matrix equality which gives an upperbound to the H_2 performance criterion. We note that this step slightly differs from its continuous-time counterpart where an algebraic Liapunov equation is replaced by the usual Riccati equation[1].

Proposition 2.1 *Let the controller K be given and assume there exists a symmetric positive definite solution Q to the following QME*

$$Q = \tilde{A}'Q\tilde{A} + Q\tilde{B}(\gamma^2 I + \tilde{B}'Q\tilde{B})^{-1}\tilde{B}'Q + \tilde{C}'\tilde{C} \qquad (15)$$

Then

$$(\tilde{A}, \tilde{C}) \text{ is detectable} \qquad (16)$$

if and only if

$$\tilde{A} \text{ is asymptotically stable.} \qquad (17)$$

In this case, the following conditions hold,

$$1.\ \ \|T_{zw}\|_\infty \leq \gamma \qquad (18)$$
$$2.\ \ Q \geq \hat{Q} \qquad (19)$$
$$3.\ \ \mathcal{J} \geq J \qquad (20)$$

where

$$\mathcal{J} := tr(Q\tilde{B}\tilde{B}') \qquad (21)$$

Proof:

Using the results of [7,11], it can be shown that (16) implies that $(\tilde{A}, V^{1/2})$ is also detectable, where $V = Q\tilde{B}(\gamma^2 I + \tilde{B}'Q\tilde{B})^{-1}\tilde{B}'Q + \tilde{C}'\tilde{C}$. As it is assumed that there exists a positive definite solution to (15), it now follows that $\tilde{A}$ is asymptotically stable by lemma 2.3, on noting that $V \geq 0$. The converse is immediate [1,15]. Now, using lemma 2.1 we immediately obtain the disturbance attenuation constraint $\|T_{zw}\|_\infty \leq \gamma$. (19) is proved by subtracting (13) from (15) which gives

$$Q - \hat{Q} = \tilde{A}'(Q - \hat{Q})\tilde{A}$$
$$+ Q\tilde{B}(\gamma^2 I + \tilde{B}'Q\tilde{B})^{-1}\tilde{B}'Q \qquad (22)$$

Applying lemma 2.2, on noting that $Q\tilde{B}(\gamma^2 I + \tilde{B}'Q\tilde{B})^{-1}Q\tilde{B}$ is positive semidefinite and the fact that $\tilde{A}$ is asymptotically stable, it now follows that

$$Q - \hat{Q} \geq 0 \qquad (23)$$

which prove (19). Finally, using (19) and the definition of $\mathcal{J}$ given in (21), we immediately obtain (20). Q.E.D.

It follows from proposition 2.1 that the existence of positive definite solution to (15) and asymptotic stability of $\tilde{A}$ leads automatically to the satisfaction of H_∞ disturbance attenuation constraint. Furthermore all such solutions give an upperbound on the H_2 performance criterion.

3. Auxiliary Minimization Problem and Necessary Conditions for Optimality

In this section we define an auxiliary minimization problem and derive the necessary conditions for optimality which explicitly synthesize the desired controller. From proposition 2.1, it follows that the existence of positive definite Q satisfying (15), together with the generic condition (16), leads to the following,

1. closed-loop stability;
2. prespecified H_∞ disturbance attenuation;
3. an upperbound for the H_2 performance criterion.

Hence, recalling the problem formulated in section 2, it remains to determine K which minimizes $\mathcal{J}$, and thus provides an optimized bound for the actual H_2 performance J. Therefore, $\mathcal{J}$ can be interpreted as an auxiliary cost which leads to the following minimization problem.

Auxiliary Minimization Problem: *Find $\{K, Q\}$ which minimizes $\mathcal{J}$ subject to (15), with Q positive definite.*

Solving the auxiliary minimization problem, we obtain the following necessary conditions for optimality.

Theorem 3.1 *If $\{K, Q\}$ solves the auxiliary minimization problem, then the controller K is given by*

$$K = -(I + B_2'QB_2)^{-1}B_2'QA \qquad (24)$$

where Q is positive definite satisfying

$$Q = A'QA - A'QB_2(I + B_2'QB_2)^{-1}B_2'QA$$
$$+ QB_1(\gamma^2 I + B_1'QB_1)^{-1}B_1'Q + C_1'C_1 \qquad (25)$$

Furthermore, the auxiliary cost is given by

$$\mathcal{J} = tr(QB_1B_1') \tag{26}$$

Conversely, if there exists a positive definite Q satisfying (25), then $\{K, Q\}$ given by (24) and (25) satisfies (15) with the auxiliary cost (21) given by (26).

Proof:
To optimize (21) subject to (15), we construct the following Lagrangian

$$\mathcal{L}(K,Q,P) = tr\{Q\tilde{B}\tilde{B}' + [-Q + \tilde{A}'Q\tilde{A} + Q\tilde{B} \\ (\gamma^2 I + \tilde{B}Q\tilde{B})^{-1}\tilde{B}'Q + \tilde{C}'\tilde{C}]P\} \tag{27}$$

where P is a symmetric matrix. The stationary conditions are obtained by setting $\frac{\partial \mathcal{L}}{\partial Q} = 0$ and $\frac{\partial \mathcal{L}}{\partial K} = 0$. Now, we can apply the rules for calculating gradient matrix proposed in [10] to obtain

$$\frac{\partial \mathcal{L}}{\partial Q} = \tilde{B}\tilde{B}' - P + \tilde{A}P\tilde{A}' + PQ\tilde{B}(\gamma^2 I \\ +\tilde{B}Q\tilde{B})^{-1}\tilde{B}' + \tilde{B}(\gamma^2 I + \tilde{B}'Q\tilde{B})^{-1}\tilde{B}'QP \\ -\tilde{B}(\gamma^2 I + \tilde{B}Q\tilde{B})^{-1}\tilde{B}QPQ\tilde{B} \\ (\gamma^2 I + \tilde{B}Q\tilde{B})^{-1}\tilde{B}' = 0 \tag{28}$$

Using (9) and assumption 2 of section 2, we have

$$\frac{\partial \mathcal{L}}{\partial K} = B_2'QAP + B_2'Q'AP' + B_2'QB_2KP \\ +B_2'Q'B_2KP' + KP + KP' = 0 \tag{29}$$

Employing the fact that Q and P are symmetric matrices, we obtain the following expression for controller K from equation (29),

$$K = -(I + B_2'QB_2)^{-1}B_2'QA \tag{30}$$

Next, substituting (9) into (15), on noting assumption 2, yields

$$Q = A'QA + K'B_2'QA + A'QB_2K + K'(I \\ +B_2'QB_2)K + QB_1(\gamma^2 I + B_1'QB_1)^{-1}B_1'Q \\ +C_1'C_1$$

Now, substituting K given in (30) into the last equation and performing some manipulations to the resulting terms, gives

$$Q = A'QA - A'QB_2(I + B_2'QB_2)^{-1}B_2'QA \\ +QB_1(\gamma^2 I + B_1'QB_1)^{-1}B_1'Q + C_1'C_1 \tag{31}$$

Next, equation (26) follows immediately from (21). Toward this end, it should be noted that equation (28) is superfluous for the present problem. This equation, however, might be required in deriving solution for a more general problem, such as output feedback controller for nonequalized H_2/H_∞ problem (see [1], for comparison).
We proceed to prove the converse part of Theorem 3.1. Now, suppose $\{K, Q\}$ satisfy (24) and (25), where Q is positive definite. We can rewrite (25), using (24), as

$$Q = (A + B_2K)'Q(A + B_2K) + QB_1(\gamma^2 I \\ +B_1'QB_1)^{-1}B_1'Q + C_1'C_1 + K'K$$

which, in view of (9) and assumption 2, is equivalent to

$$Q = \tilde{A}Q\tilde{A} + Q\tilde{B}(\gamma^2 I + \tilde{B}'Q\tilde{B})^{-1}\tilde{B}Q + \tilde{C}'\tilde{C} \tag{32}$$

In this case, the auxiliary cost given by (21), which is equivalent to (26), holds. Q.E.D.

4. Sufficient Conditions for the H_∞ Disturbance Attenuation

Similarly to the sufficient conditions derived in [1] for continuous-time case, we now combine proposition 2.1 with the converse part of theorem 3.1 to obtain the conditions underwhich the closed-loop stability, the H_∞ disturbance attenuation, and an optimized H_2 performance bound are guaranteed.

Theorem 4.1 *Suppose there exists a positive definite Q satisfying (25) and let K be given by (24). Then $(\tilde{A}, \tilde{C})$ is detectable if, and only if, $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function T_{zw} satisfies*

$$\|T_{zw}\|_\infty \leq \gamma \tag{33}$$

and the actual H_2 performance criterion (6) satisfies the bound given by

$$J \leq tr(QB_1B_1') \tag{34}$$

Proof:
Using the assumed existence of positive definite Q satisfying (25) with K given by (24), the converse part of Theorem 3.1 implies that Q satisfies (15). In view of Proposition 2.1, it now follows that the detectability condition (16) is equivalent to asymptotic stability of $\tilde{A}$. In this case, H_∞ disturbance attenuation (33) and H_2 performance bound (34) are satisfied. Q.E.D.

Remark 4.1
It is interesting to note that the modified Riccati equation (25) corresponds to the results of [3] which is obtained via the solution to the corresponding discrete-time games. Now, applying matrix identity $(U^{-1} + VZ^{-1}W)^{-1} = U - UV(Z+WUV)^{-1}WU$ and denoting $P^{-1} := Q^{-1}+\gamma^{-2}B_1B_1'$, where positive definite Q satisfies (25), straightforward manipulations to equation (25) yields [3]

$$P = C_1'C_1 + A'P\Lambda^{-1}A \\ \Lambda = I + (B_2B_2' - \gamma^{-2}B_1B_1')P$$

which is the Riccati equation arising in discrete-time games [2]. Some manipulations to the above equation lead further to the one arising in LEQG control problem[16].

Remark 4.2

For γ that approximates infinity, equation (25) clearly reduces to the standard Riccati equation of LQ problem. Therefore, solution Q of (25), under the assumption 1 of section 2, is guaranteed to be positive definite. Now, if condition (33) is also satisfied for enough small γ, it is of interest to know whether one such controller can be obtained by solving (25). Under a strengthen assumption, i.e. $\tilde{B}$ is of full rank and $(\tilde{A}, \tilde{C})$ is completely observable, lemma 4 and lemma 5 of [3] guarantees that (15) possesses a solution for any controller satisfying (33). Thus, as far as the auxiliary minimization has at least one extremal, the above sufficient condition will be necessary as well. When these conditions hold, we immediately obtain the following result.

Proposition 4.1 *Let γ^* denote the infimum of $\|T_{zw}\|_\infty$ over all stabilizing controllers and suppose that the auxiliary minimization problem has a solution for all $\gamma > \gamma^*$. Then for all $\gamma > \gamma^*$, there exists positive definite Q satisfying (25).*

5. Removing the Orthogonality Assumption

In this section we remove the orthogonality assumption employed in the previous sections and immediately obtain the following results.

Theorem 5.1 *If $\{K, Q\}$ solves the auxiliary minimization problem, then the controller K is given by*

$$K = -(D_{12}'D_{12} + B_2'QB_2)^{-1}(B_2'QA + D_{12}'C_1) \tag{35}$$

where Q is positive definite satisfying

$$\begin{aligned} Q = A'QA - (A'QB_2 + C_1'D_{12})(D_{12}'D_{12} \\ +B_2'QB_2)^{-1}(B_2'QA + D_{12}'C_1) + QB_1(\gamma^2 I \\ +B_1'QB_1)^{-1}B_1'Q + C_1'C_1 \end{aligned} \tag{36}$$

Furthermore, the auxiliary cost is given by

$$\mathcal{J} = tr(QB_1B_1'). \tag{37}$$

Conversely, if there exists a positive definite Q satisfying (36), then $\{K, Q\}$ given by (35) and (36) satisfies (15) with the auxiliary cost (21) given by (37).

Proof:

Construct the following Lagrangian

$$\begin{aligned} \mathcal{L}(K, Q, P) = tr\{Q\tilde{B}\tilde{B}' + [-Q + \tilde{A}'Q\tilde{A} \\ +Q\tilde{B}(\gamma^2 I + \tilde{B}Q\tilde{B})^{-1}\tilde{B}'Q + \tilde{C}'\tilde{C}]P\} \end{aligned}$$

and obtain

$$\begin{aligned} \tfrac{\partial \mathcal{L}}{\partial Q} = \tilde{B}\tilde{B}' - P + \tilde{A}P\tilde{A}' + PQ\tilde{B}(\gamma^2 I + \tilde{B}Q\tilde{B})^{-1} \\ \tilde{B}' + \tilde{B}(\gamma^2 I + \tilde{B}'Q\tilde{B})^{-1}\tilde{B}'QP - \tilde{B}(\gamma^2 I \\ +\tilde{B}Q\tilde{B})^{-1}\tilde{B}QPQ\tilde{B}(\gamma^2 I + \tilde{B}Q\tilde{B})^{-1}\tilde{B}' = 0 \end{aligned}$$

$$\begin{aligned} \tfrac{\partial \mathcal{L}}{\partial K} = B_2'QAP + B_2'Q'AP' + B_2'QB_2KP \\ +B_2'Q'B_2KP' + D_{12}'C_1P + D_{12}'C_1P' \\ +D_{12}'D_{12}KP + D_{12}'D_{12}KP' = 0. \end{aligned}$$

The proof of the theorem then proceeds as in the proof of Theorem 3.1. Q.E.D.

As in section 4, we readily obtain the following.

Theorem 5.2 *Suppose there exists a positive definite Q satisfying (36) and let K be given by (35). Then $(\tilde{A}, \tilde{C})$ is detectable if, and only if, $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function T_{zw} satisfies*

$$\|T_{zw}\|_\infty \le \gamma \tag{38}$$

and the actual H_2 performance criterion (6) satisfies the bound given by

$$J \le tr(QB_1B_1') \tag{39}$$

6. Numerical Example

The results in the previous sections are illustrated here using a numerical example. Consider the following time-invariant discrete-time system :

$$\begin{aligned} x(k+1) = \begin{bmatrix} -0.4 & 0.8 \\ 1.2 & 0.04 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \\ + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k) \end{aligned}$$

$$z(k) = \begin{bmatrix} 1 & 1.5 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

Actual $\|T_{zw}\|_\infty$, auxiliary performance criterion $\mathcal{J}$, actual H_2 performance criterion J and closed-loop eigenvalues, for several values of γ, are listed in Table 1. The positive definite solution Q of equation (3.2) is calculated by using the eigen-vector method via the associated Hamiltonian matrix given by [3]

$$\mathcal{H} = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{21} & \mathcal{H}_{22} \end{bmatrix}$$

where

$$\begin{aligned} \mathcal{H}_{11} &= A + B_2B_2'(A')^{-1}C_1'C_1 \\ \mathcal{H}_{12} &= \gamma^{-2}AB_1B_1' - B_2B_2'(A')^{-1} \\ & \quad [I - \gamma^{-2}C_1'C_1B_1B_1'] \end{aligned}$$

$$
\begin{aligned}
& [I-\gamma^{-2}C_1'C_1B_1B_1'] \\
\mathcal{H}_{21} &= -(A')^{-1}C_1C_1' \\
\mathcal{H}_{22} &= (A')^{-1}[I-\gamma^{-2}C_1'B_1B_1']
\end{aligned}
$$

Note that $\gamma \rightarrow \infty$ corresponds to the standard LQ control problem. Minimum value γ of the above example is found γ_{min}=2.31, below which there is no positive definite solution to (3.2) that makes the closed-loop system stable. The results show that the actual H_2 performance criterion is less than or equal to the auxiliary cost, justifying our technique which replace Liapunov equation by QME. It is also interesting to note the trade-off between H_∞ disturbance attenuation and H_2 performance criterion. As is well known, H_∞-norm bound implies a prespecified level of stability robustness under unstructured uncertainty (via small gain theorem). Thus, we can achieve the compromise between robustness and quadratic performance by specifying γ.

7. Conclusion

Control synthesis for discrete-time systems with combined H_2/H_∞ performance criteria has been presented. Solution to this problem is obtained by defining an auxiliary minimization problem which enforce H_∞ disturbance attenuation constraint while minimizing H_2 performance bound. Similarly to the corresponding continuous-time case, the solution is given in terms of a modified Riccati equation arising in discrete-time games and LEQG control problem. The results of the paper, therefore, clarify the correspondences between H_2/H_∞ properties of discrete-time system and their continuous-time counterparts.

H_∞ constraint(γ)	*actual* $\lVert T_{zw}\rVert_\infty$	*auxiliary* H_2-*cost*($\mathcal{J}$)	*actual* H_2-*cost*(J)
10000	2.6694	5.1005	5.1005
150	2.6693	5.1007	5.1005
100	2.6692	5.1032	5.1005
50	2.6687	5.1113	5.1005
10	2.6502	5.3864	5.1009
8	2.6394	5.5618	5.1015
5	2.5924	6.4766	5.1078
4	2.5491	7.6373	5.1192
3	2.4556	12.4761	5.1666
2.75	2.4150	17.2265	5.1991
2.5	2.3617	34.5406	5.2557
2.45	2.3491	45.3895	5.2718
2.4	2.3356	68.1585	5.2902
2.38	2.3300	86.2353	5.2983
2.35	2.3213	146.3319	5.312

Table 1

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