LQG/LTR with NMP Plant

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0. Abstract

In this paper we present a method of reducing controller design problem from LQG/LTR approach to H∞ optimization. The condition of the existance of the optimal solution is derived. In order to derive the controller equation for NMP plant we reduce the H∞ LTR problem to Nehari's extension problem and derive the optimal controller equation which is best approximation for this problem. Furthermore, we show that the controller obtained by presented method guarantee the asymptotic LTR condition and stability of closed loop system.

I. Introduction

The Linear Quadratic Regulator(LQR) of Anderson and Moore [1] and Kwakernaak and Sivan[2] is a straight foward state feedback design approach which results in a simple controller consisting of a state feedback gain. The best known example of formal mathematical synthesis is the LQG optimal control problem. This problem formalize a specific design situation, namely the construction of feedback compensators for finite-dimensional linear time invariant plant models with stability and performance under additive disturbance as design objectives. However, the LQG design may have poor loop properties, i.e., low stability margin(robustness) at the plant input and/or output[3,6,11].

The loop transfer recovery(LTR) method was developed by Doyle and Stein[4,5] to improve the robustness of LQG regulators. In their work, they address the problem of finding the steady state observer gain which assures the recovery of the loop transfer function resulting full state feedback. This method is successfully applied to the minimum phase plant. But when the plant have right half plane zeros exact recovery is not possible and is required some restrictions for loop transfer recovery due to the RHP zeros[6]. Stein and Athan suggest that the plant with RHP zeros has it's approximation obtained by multiplying the finite Blaske product and LTR can be obtained by using the approximated plant. Since their method for MIMO case is no more valied because we cannot select the finite Blaske product for MIMO plant which cannot subdivide into SISO form. Also by using approximated plant, LTR error is always exist for some frequency. In 1989, Moor and Tay[7] presented the H∞ LTR controller scheme and its design algorithm which can obtain the LTR by obtaining the sensitivity recovbery. For the plant with single RHP zero, the NMP condition is replaced by MP system pre-(or post-) multiplying the frequency valued scalar parameter and the solution of H∞ minimization can be obtained. This method also valied on the plant with single RHP zero and SISO plant only.

In this paper we present the H∞ LTR for NMP plant based on Nehari's extension problem and show that the stability and H∞ LTR condition are satisfied. In order to obtain the optimal solution we derive bounded value of LTR error, present a method of reducing from the H∞ LTR problem to Nehari's extension problem and the reconstructing the target loop and derive the optimal solution in H∞ matrix form. Finally, we show that the controller suggested in this paper guarantee the optimality, stability and satisfy the H∞ LTR condition.

II. Mathematical preliminary

(1) Notations

In this section we present some notations which are used over the paper.

$R_{L\infty}$: real-rational, strictly proper transfer matrices functions which have no poles on $j\omega$-axis.

RH_∞ : proper transfer matrices function which is stable.

$\| \cdot \|_\infty \equiv \sup \{ \| \quad \|_2 \}$

G_o : outer factor matrix of G

G_i : inner factor matrix of G

X^{-L} : left inverse of X

X^{-R} : right inverse of X

Γ : Hankel operator

L_o, L_c : observability and controllability gramians

(2). Inverse

Consider the transfer function G(s) denoted by

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

the inverse of G(s) can be obtained by one of following method

① . D is not singular

$$G^{-1}(s) = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{bmatrix} \qquad --(1)$$

② D is singular with all zero element

$$G^{-1}(s) = F s + \begin{bmatrix} (I - BFC)A & AFF \\ - FCA & - FCABF \end{bmatrix}$$

where $F = (CB)^{-1}$ ---- (2)

(3) Inner-outer factorization and its properties

In this section we present the inner-outer factorization of any RH_∞ matrix which have one or more right plan zeros. The inner matrix is defined as a matrix G_i in RH_∞ which satisfy

$$G_i^* G_i = I$$

where G_i^* is complex conjugate transpose of G_i. One useful property of inner matrix is that any RH_∞ matrix preserves its norm and inner product by pre-(or posr-) multipling inner matrix. Another matrix used in this paper is outer matrix G_o which can be defined as a matrix that have full row or column rank for all frequency. The outer matrix G_o has a property that its inverse have no poles in RHP. By definitions of inner and outer function matrix, we can derive the inner and outer factorization for any transfer matrix in RH_∞[8]. The state space formulas for inner-outer factorization has been developed by Doyle when the given transfer matrix has full columnm rank for all frequency. This result is extended by Chen and Francis when the given transfer matrix G has full row rank for all $0 \le \omega \le \infty$. In 1989, Zhang and Freudenberg suggest that the functios which are allowed to have less then full rank on the extended imaginary axis or to have the state space realization of D=0 case. This results are summarise in the following lemma.

Lemma. ①. Every RH_∞ matrices have a unique I-O factorization. ②. Any RH_∞ matrix preserve its norm and inner product by pre-(or post-) multipling inner matrix. ③. Any outer matrix which is proper has left or right inverse in RH_∞. ④. Any outer matrix which is strictly proper has left or right inverse as the form of $Fs + RH_\infty$ matrix.

Since the axioms given in this lemma are obvious, we omit proofs. The axiom ③ and ④ can be prooved by using the equations given in section 2.

(4) Nehari problem

Let us deffine the distance from L_∞ matrix R to any matrix X in H_∞ be

$$\mathrm{dist}(R, X) = \min \{ \| R - X \|_\infty : X \in H_\infty \}$$

---(3)

Nehari's theorem states that the shortest distance from a unstable matrix to stable one equals to the L_∞ norm of its Hankel operator which maps a L_∞ matrix to H_∞ matrix. Let the Hankel operator for unstable matrix R be Γ_R. Then $\|\Gamma_R\|_\infty$ is a lower bound for the distance from R to H_∞.

Theorem 2. There exists a closet H_∞ matrix X to a given L_∞-matrix R, and $\| R - X \| = \|\Gamma_R\|$.

Generally, there is many X's nearest to R Now the problem to finding X which is apart from Γ_R can be solved by defining the following observability gramian L_o and controllability gramian L_c for state space realization of R.

$$R = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}$$

Then the controllability gramian L_c and the observability gramian L_o satisfying the following Lyapunov equations.

$$A_r L_c + L_c A_r^T - B_r B_r^T = 0 \qquad \text{------(4.a)}$$

$$A_r^T L_o + L_o A_r - C_r^T C_r = 0 \qquad \text{---(4.b)}$$

The L_∞-norm of Hankel operator equals to L_∞-norm of $L_o L_c$[1]. Thus the shortest distance from L_∞ matrix R to any H_∞ matrix, γ, is

$\gamma = \max\{\lambda_i(L_o L_c)\}^{1/2}$ -------(5)

Let ω be the corresponding eigenvectors of Eq.(5) and define f(s) and g(s) be

$$f(s) = \begin{bmatrix} A_g & \omega \\ C_g & 0 \end{bmatrix}, \qquad g(s) = \begin{bmatrix} -A_g^T & \lambda^{-1} L_o \omega \\ B^T & 0 \end{bmatrix}$$

Then the best approximation of R is $X = R - \lambda f/g$ and the state space formulars for $R-\lambda f/g$ becomes

$$R-\lambda f/g = \begin{bmatrix} A_g & B_g \\ C_g & 0 \end{bmatrix} - \begin{bmatrix} A_g & \lambda\omega \\ C_g & 0 \end{bmatrix} \begin{bmatrix} -A_g^T & \lambda^{-1} L_o \omega \\ B^T & 0 \end{bmatrix}^{-1}$$

where

$$M \equiv (B^T \lambda^{-1} L_o \omega)^{-1}$$
$$A_a \equiv -(I - B_i M B^T) A_g^T$$
$$B_a \equiv -A_g \lambda^{-1} L_o \omega M$$
$$C_a \equiv M B^T A_g^T$$
$$D_a \equiv M B^T A_g^{-1} B_i M$$
$$B_i \equiv \lambda^{-1} L_o \omega \qquad \text{---------------(6)}$$

By simple algebric manipulation and removing unobservable and/or uncontrollable modes the Eq.(6), then

$$R-\lambda f/g = \begin{bmatrix} -[I + \lambda^{-1} L_o\omega (B^T L_o \omega)^{-1} B^T] A_g^T & -A_g L_o \omega (B^T L_o \omega)^{-1} \\ -C_g & -\lambda \omega C_g \end{bmatrix} \qquad \text{------(7)}$$

Now we review the multivarible case solution of X which are nearest to L_∞ matrix R. Let us define the following matrices.

$$L_1(s) = \begin{bmatrix} A & -L_c C^T \\ C & I \end{bmatrix}$$
$$L_2(s) = \begin{bmatrix} A & N^T B \\ C & 0 \end{bmatrix}$$
$$L_3(s) = \begin{bmatrix} -A^T & N C^T \\ -B^T & 0 \end{bmatrix}$$
$$L_4(s) = \begin{bmatrix} -A^T & N L_o B \\ B^T & I \end{bmatrix}$$

Select Y in RH_∞ with $\|Y\|_\infty \le 1$ then X is

$$X = R - (L_1 Y - L_2)(L_3 Y + L_4)^{-1} \qquad \text{----(8)}$$

III. H∞ LTR

(1) H∞ LTR problem

A H∞ LQG controller structure is shown in Fig.1.

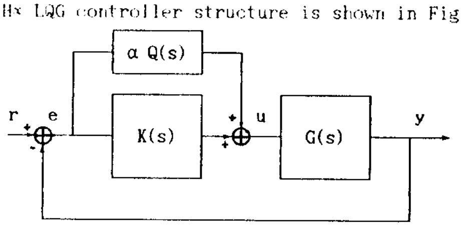

figure.1 H∞ LQG structure

The H∞ loop transfer recovery error is

$$\delta Q \equiv K_c \phi B - KQG \qquad \text{------(9)}$$

where $\phi = (sI - A)^{-1}$

Coprime factorization for plant G and general LQG controller K is given by

$$G = NM^{-1} \quad (\text{or} = \tilde{M}^{-1}\tilde{N})$$
$$K = UV^{-1} \quad (\text{or} = \tilde{V}^{-1}\tilde{U}) \qquad \text{------(10)}$$

where

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} A + B K_c & B & K_f \\ K_c & I & 0 \\ C + D K_c & D & I \end{bmatrix}$$

or

$$\begin{bmatrix} \tilde{M} & \tilde{U} \\ \tilde{N} & \tilde{V} \end{bmatrix} = \begin{bmatrix} A + K_f C & -(B + K_f D) & K_f \\ K_c & I & 0 \\ C & -D & I \end{bmatrix}$$

Substitude the Eq. (10) into (9), we can obtain δQ as

$$\delta Q = I - M - U(I + \alpha Q')V^{-1}NM^{-1} \qquad \text{------(11)}$$

Then the H∞ LTR problem is to find $Q' \in RH_\infty$ such that the H∞-norm of δQ small as possible as we can. This H∞ LQG/LTR problem can be reduced the well known H∞ model-matching problem by defining matrices T and P as follows.

$$T \equiv \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} I - M & U \\ V^{-1}NM^{-1} & 0 \end{bmatrix} \qquad \text{------(12)}$$

$$P \equiv \begin{bmatrix} M - I & K(Q)^{-1} \\ V^{-1}NM^{-1} & NM^{-1} \end{bmatrix} \qquad \text{------(13)}$$

The reduced model matching problem is find $Q \in RH_\infty$ which minimize the transfer matrix $(T_{11} - T_{12} Q T_{21})$

$$\min_{Q \in RH_\infty} \|T_{11} - T_{12} Q T_{21}\|_\infty \qquad \text{------(14)}$$

For the case of minum phase plant, i.e. T_{21} is full rank for all s>0 direct calculation of Q or interpolation is possible. But for the non-minimum phase case, i.e. T_{21} is not full rank for some s>0, there is no Q in RH_∞ which minimize the H∞-norm of loop tyransfer recovery error. This is due to the fact that the sufficient condition for existence of Q in RH_∞ for model matching problem is that the rank of T_{12} and T_{21} are full for all s>0.[8]

(2). Problem formulation for H∞ LQG/LTR for nonminimum phase plant

In this section we present a problem for solving H∞ LQG/LTR with NMP plant. Since H∞ LQG/LTR problem has no solution for the of NMP plant we must approximate to select the controller. The H∞

LQG/LTR error Eq.(3) is used in this approximation in which the plant model is directly employed. The sufficient condition for existance of this optimal problem is U or $V^{-1}NM^{-1}$ is full rank for all s > 0 including ∞. Which implies that U as $V^{-1}NM^{-1}$ has no zeros in RHP including at origin and infinity (∞). However, $V^{-1}NM^{-1}$ or U has zeros at 0 or ∞. The H∞ LQG/LTR problem can be restate as find Q ∈ RH∞ which minimize the H∞ loop transfer recovery error δQ with constraint that K(Q) stabilize G(s). By Eq.(2) in H∞ LQG/LTR problem the loop transfer recovery error $\mathcal{E}_Q$ can be denoted by

$$\mathcal{E}_Q = (K_C\phi B - K(Q)G(s))$$

By post multiplication of G^*, we can obtain

$$\mathcal{E}_{Q'} = K_C\phi BG_i^* - K(Q)G_o \qquad \text{------(15)}$$

Theorem 3: $\| \mathcal{E}_Q \|_\infty = 0$ if and only if $\| \mathcal{E}_{Q'} \|_\infty = 0$

proof)

Substitude Eq.. () into Eq.. () we can obtain

$$\mathcal{E}_{Q'} = \| K_C\phi BG_i^* - U_o(I + \alpha Q')V_o^{-1}N_oM_o^{-1} \| \qquad \text{------(16)}$$

Define the following matrices.

$$T_{11} = K_C\phi BG_i^{-1}$$

$$T_{12} = U_o$$

$$Q'' = (I + \alpha Q')$$

$$T_{21} = V_o^{-1}N_oM_o^{-1}$$

where $Q' = U_oQV_o$

then $\mathcal{E}_{Q'}$ becomes

$$\mathcal{E}_{Q'} = T_{11} - T_{12}Q''T_{21}$$

Hence we can restate the H∞ LQG/LTR problem for nonminimum phase plant as to find a H∞ LQG LTR controller equation K(Q) ∈ RH∞ which make the L∞-norm of $\mathcal{E}_{Q'}$ as possible as small.

(3).H∞ LTR error boundary

The nessary and sufficient condition for LTR occur is $\|\delta Q_1\|_\infty$ approaches zero. But such a controller K(Q) is not exist for G(s) with RHP zeros. Here we note that T_{11} have RHP poles, hence it has factorization as

$$T_{11} = R_s + R_u \qquad \text{------(17)}$$

where R_s is the stable part of T_{11} and R_u is the unstable part of T_{11}. Let the best approximation of T_{11} be $T_{11'}$ then

$$T_{11'} = R_s + X \qquad \text{------(18)}$$

where X is any RH∞ matrix which is best approximation of R_u.

Define $\mathcal{E}_{Q''}$ be

$$\mathcal{E}_{Q''} = T_{11'} - T_{12}Q''T_{21}$$

Theorem 4. $\|\mathcal{E}_Q\|_\infty$ approaches γ as $\|\mathcal{E}_{Q''}\|_\infty$ approaches 0

proof) If $\|\mathcal{E}_{Q''}\|_\infty = 0$ then the direct solution of Q'' is

$$Q'' = T_{12}^{-1}T_{11'}T_{21}^{-1}$$

$$\mathcal{E}_{Q'} = T_{11} - T_{11'} \qquad \text{---------------(19)}$$

L∞ norm of Eq.(19) equal to γ, hence

$$\gamma = \| K_C\phi BG_i^* - K(Q)G_o \|_\infty$$

$$= \| K_C\phi B - K(Q)G \|_\infty$$

$$= \|\mathcal{E}_Q\|_\infty \qquad \text{----------(20)}$$

(4). H∞ LTR

As shown in previous section the LTR error approaches γ. We can obtain δQ as

$$\delta Q = (R_S + X) M_oN_o^{-1} - U V^{-1} \qquad \text{-----(21)}$$

then the controller equation K(Q) is

$$K(Q) = (R_S + X) M_oN_o^{-1} \qquad \text{---------(22)}$$

Theorem 5: The controller K(Q) obtained above stabilize the plant and make the LTR error approachs γ.

proof)

The stabilizability of plant can be reduced as the existance of the inverse in RH∞ of following matrix

$$\begin{bmatrix} I & -P_{12} & 0 \\ 0 & I & -K(Q) \\ 0 & -P_{22} & I \end{bmatrix}$$

Also it can be reduced as the existance of Q ∈ RH∞.

$$\delta Q = T_{11} - T_{11'}$$

$$= R_u - R_S$$

Take L∞ norm, we can obtain

$$\| \delta Q \|_\infty = ,$$

(5). summery of LTRO

In this section we present two theorem without proof which can explain the loop transfer recovery at plant output. The loop transfer recovery error could be defined at plant output as

$$\delta Q^o = GK(Q) - C\Phi K_f \qquad \text{------(23)}$$

where Φ(s) also defined by

$$\Phi(S) = (sI - A + K_fC) \qquad \text{------(24)}$$

Rewrite Eq. (23) in state space form and simplifying it, we can obtain the following equation.

$$\delta Q = \Sigma^{-1}(\Sigma\,(I + \alpha^{-1}Q'')^{-1} - I + \Sigma) \qquad \text{--------(25)}$$

Note that Σ^{-1} is not small for all frequency thus $\mathcal{E}_Q$ can reduce as

$$\delta Q = \Sigma\,(I + \alpha^{-1}Q'')^{-1} - I + \Sigma \qquad \text{--------(26)}$$

As $\|\mathcal{E}_Q\|_\infty$ equal to zero for all frequency we could obtain loop transfer recovery thus the controller quation Q and K(Q) is obtained by

$$Q = [illegible]^{-1}([illegible]-I) - [illegible]^{-1} \qquad ---------(27)$$

$$K(Q) = [illegible]^{-1}([illegible]-I) \qquad -------(28)$$

The stability and the exact loop transfer recovery results are summerized in the following theorem.

Theorem 6. K(Q) $\in$ RH_∞ and stabilize the plant, futher more the exact loop transfer recovery can be achieved, iff, the plant has no zeros in RHP.

proof) The proof is the same as theorem

VI. Referance

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