측정 잡음 하에서의 최적 이산 퓨리에 변환

권 욱현, 이 기원[0], 이규승

서울대학교 자동화 시스템 공동 연구소 및 제어계측과

OPTIMAL DISCRETE FOURIER TRANSFORMATION IN THE PRESENCE OF MEASUREMENT NOISES

Wook Hyun Kwon, Gi Won Lee[O] and Kyu Seung Lee

Dept. of Control and Instrumentation Engr.
and Automation Systems Research Institute
Seoul National University

요약

본 논문은 상태 변수 모델 하에서 유도된 최적 FIR 필터의 해를 이용한 새로운 이산 퓨리에 변환 방법을 제시한다. 이 방법은 측정 잡음 하에서 시변 하모닉스 성분을 추정하는데 특히 유용하여, 기존의 이산 퓨리에 변환 보다 훨씬 효과적인 노이즈 억제성능을 얻을 수 있으며 또한 칼만 필터를 사용한 하모닉스 추정 방법에서 발생하는 발산 문제를 해결할 수 있다. 그것은 유한한 필터 구간 내에서 BIBO 안정도를 항상 보장하는 FIR필터의 구조에서 야기된다. 한편 기존의 하모닉스 추정 방법들과 여러가지 측면의 성능 비교가 있었고 시뮬레이션 예제를 통해 본 논문에서 제시한 방법의 효용성을 입증하였다.

1. INTRODUCTION

It is always a major topic in modern signal processing to estimate the spectral components more accurately and fastly. The normal method for performing this task is a discrete Fourier Transformation(DFT) to sampled signals together with a suitably chosen windowing function. Recently, this conventional block processing method to the DFT calculation was extended to a recursive DFT method proposed by some researchers [1,2,3]. This recursive DFT method, termed spectral observation, performs one iteration per sample spectral updating, incorporating a new sample and discarding all effects of the oldest sample affecting the previously calculated spectrum. This technique uses the concept of observer used in the control theory for the calculation of the DFT.

Then it is preassumed for these DFT methods that the measurement signals are generated from deterministic signal models. However there exists, in practice, a number of situations for which measurement signals are polluted by gfstochastic noises. In these cases, conventional DFT methods exhibit poor harmonic estimations since the DFT methods do not have any noise suppression properties which may be desirable in many applications.

In a recent, new harmonic estimation methods to overcome the effects of noises were proposed which is based on state estimation theory under quadratic optimization techniques [4]. They use Kalman filters to extract harmonic components of signals from noises. But it is inappropriate to apply the Kalman filter for this problem since the uncontrollability of the state variable signal model may destroy the stability of the Kalman filter and cause the filter to be degenerated. So they provided several techniques to avoid the filter divergence, which are as follows : state noise insertion, exponential data weighting and covariance setting. Even though these methods have useful effects on the filter divergence, they may destroy the optimality of the filter derived under the given optimal performances and provide the free-choice in the selection of the an appropriate filter gain.

On the other hand, it has been shown in the state observation approach to the DFT evaluation that the recursive DFT is implemented to the deadbeat observer in which the correct estimate will be obtained after at most n steps, where n means the order of the system. Thus we can conclude that the conventional DFT in deterministic signal models is in fact a FIR filter since the deadbeat observer can be represented to the FIR filter which products the n state vector from a linear combination of the measurements. It can be expected from the above results that the DFT in stochastic signal models can be also represented to the FIR filter derived under some optimal performances.

Our aim in this paper is to develop an optimal DFT method for noisy measurements via a state model based optimal FIR filter. Optimal FIR filters for continuous and discrete time state space models are suggested by Kwons [5,6], which have both the limited memory and the structure of the finite impulse response and, due to the finite observations, alleviates the divergence problem potentially occurred in the standard Kalman filter. In this paper, these optimal FIR filters are used for the DFT in the presence of noisy measurements, where its forms are modified according to a state space formulation of this problem. This method do not require any block processing of signals as the standard DFT, represent poor performances due to the existence of the measurement noises as the conventional DFT, and yield the filter divergence or the optimality destruction in the implementation of the filter as the Kalman filter. Thus, by using this method, the DFT can be successfully extended to the real applications where the extraction of the harmonic components from stochastic noise signals is required.

In Section 2, the optimal FIR filter will be applied for the harmonic estimation problem and the solutions of the FIR filter to this problem are obtained in the case of known and unknown initial conditions. In Section 3, the proposed FIR filtering method will be compared with the conventional DFT methods and the Kalman filtering methods in the various aspects and discussed. The improved characteristics of the proposed method will be demonstrated via simulations in Section 4. We conclude in Section 5.

2. APPLICATION OF FIR FILTER TO THE OPTIMAL DFT

We consider a periodic or quasi-periodic signal z(i) corrupted by measurement noises. Here we assume that the period T of this signal is known and there are 2N+1 samples per period. Then the fundamental frequency is $\omega_0 = 2\pi/T$. This discrete time signal z(i) is

$$z(i) = \alpha_0(i) + \sum_{k=1}^{N} [\alpha_k(i) \cos(\frac{2\pi k i}{2N+1})$$

$$+\beta_k(i)\sin(\frac{2\pi ki}{2N+1})]+v(i) \quad (2.1)$$

where v(i) is a zero mean white noise process, and the coefficients $\alpha_k(i)$ and $\beta_k(i)$, k=0,1,...,N are slowly time-varying functions over time intervals of the order of one period (2N+1) samples.

Our aim here is to estimate the Fourier coefficients $\{\alpha_k(i), \beta_k(i): k=0,1,...,N\}$ as rejecting the measurement noise v(i) with time i. Conventionally this is done by performing the standard DFT with 2N+1 data window under assumption that the measurement noise v(i)=0. In the case of a nonzero v(i), it can be easily noted that the application of the conventional DFT may yield poor estimations of the harmonic components due to its noise signal v(i). (This will be extremely shown later in simulation results.)

The above sampled data sequence (2.1) can be modeled as the scalar output of a state equation of dimension 2N+1 with no input :

$$x(i+1)=Ax(i) \quad (2.2)$$
$$z(i)=Cx(i)+v(i)$$

where, writing $\theta=2\pi/(2N+1)$,

$$A=\text{block diag}\{\begin{bmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{bmatrix}, k=1,...,N\}+1 \quad (2.3)$$

$$C=[1\ 0\ 1\ 0\ ...\ 1\ 0\ 1] \quad (2.4)$$

and the state x(i) is given by

$$x(i)=\begin{bmatrix}x_1(i)\\ x_2(i)\\ \cdot\\ \cdot\\ \cdot\\ x_{2N+1}(i)\end{bmatrix}=\begin{bmatrix}\alpha_1(i)\cos i\theta+\beta_1(i)\sin i\theta\\ -\alpha_1(i)\sin i\theta+\beta_1(i)\cos i\theta\\ \cdot\\ \cdot\\ \alpha_N(i)\cos Ni\theta+\beta_N(i)\sin Ni\theta\\ -\alpha_N(i)\sin Ni\theta+\beta_N(i)\cos Ni\theta\\ \alpha_0\end{bmatrix} \quad (2.5)$$

In (2.2), we presume that the initial time i_0 is taken as $i_0=0$, the initial state x(0) is a zero-mean random variable with $E[x(0)x'(0)]=P_0$, the measurement noise v(i) is zero mean gaussian white, and $E[v(i)v'(i)]=R\delta_{ij}$, $E[v(i)x'(0)]=0$ for all i and j where R>0. We can see that the 2N+1 harmonic components of z(i) are related to the (2N+1) dimension state x(i) in (2.2). The magnitude of the k th harmonic component is

$$m_k=(\alpha_k^2+\beta_k^2)^{1/2}$$
$$=[(\alpha_k\ \cos ki\theta\ +\ \beta_k\ \sin ki\theta)^2+(-\alpha_k\ \sin ki\theta\ +\ \beta_k\ \cos ki\theta)^2]^{1/2}$$
$$=[(x_{2k-1}(i))^2+(x_{2k}(i))^2]^{1/2} \quad (2.6)$$

The phase of the k th component is

$$\phi_k=\arctan\left[\frac{\beta_k}{\alpha_k}\right]$$
$$=\arctan\left[\frac{-\alpha_k\ \sin ki\theta+\beta_k\ \cos ki\theta}{\alpha_k\ \cos ki\theta+\beta_k\ \sin ki\theta}\right]_{i\theta=2\pi n} \quad (2.7)$$

where n is an integer. The individual harmonic components of z(i) can be directly obtained from the state vector x(i) at any specific time k.

For the case when the measurement niose v(i) in the system equation (2.2) is zero, we can construct the deadbeat observer in order to obtain the state estimate $\hat{x}(i)$ as

$$\hat{x}(i+1)=(A-KC)\hat{x}(i)+Kz(i) \quad (2.8)$$

where K is a (2N+1) vector observer gain and chosen to place all eigenvalues of A-KC at the origin. By using this observer, $\hat{x}(i+1)$ converges exactly to x(i) in 2N+1 steps. It is well known that the deadbeat observer approach to harmonic estimation (which is denoted to the recursive DFT) is exactly equivalent to a standard DFT method using a rectangular window. By iterating the system (2.2), the state estimate $\hat{x}(i)$ can be represented to the form of FIR filter as follows

$$\hat{x}(i)=\sum_{k=0}^{2N}G(k)z(i-k) \quad (2.9)$$

where the filter gain is given by $G(k)=A^{2N}[W_o^{-1}]_{k\text{ th column}}$ and W_o is the observability matrix.

$$W_o=\begin{bmatrix}C\\ CA\\ \cdot\\ \cdot\\ CA^{2N}\end{bmatrix}$$

Thus the DFT in the deterministic system model can be implemented to the FIR filter as mentioned before.

Now we consider the case when the measurement noise signal v(i) is not zero. As described before, Bitmead, Tsoi, and Parker [4] have utilized the Kalman filter in order to extract the harmonic components of signals from noises but the destruction of the optimality in this case has been caused by the modification which make the filter gain a nonzero value.

In this paper, we propose a new optimal harmonic estimation method using a state model based optimal FIR filter

$$\hat{x}(i\,|\,i:M)=\sum_{k=0}^{M}H(i,k:M)z(k) \quad (2.10)$$

under the performance criterion

$$J=E[x(i)-x(i\,|\,i:M)]'[x(i)-x(i\,|\,i:M)] \quad (2.11)$$

In (2.10), the index M is the information interval and the notation $x(i|i:M)$ and H(i,j:M) denotes that the filter and its impulse response are dependent on M.

The filter solution in (2.10) is provided by Kwons [5,6], where the impulse response H(i,j:M) of the FIR filter (2.10) is obtained from the orthogonality principle and the discrete Schumitzky algorithm. Its solution to this harmonic estimation problem is given in the following theorem. The proof is omitted for brevity.

THEOREM **1:** If N is fixed and P(i-M-1,i-M-1) is not zero, the impulse response H(i,j:M) of the FIR filter (2.10) in the system (2.2) is uniquely determined by $H(i,j:M)=H(i,j:n)|_{n=M}$ for fixed i and M, where

$$H(i,j:n+1)=[I-U(i,n+1)C'R^{-1}C]AH(i,j:n) \quad (0\le M-i+j\le n\le M-1) \quad (2.12.a)$$

$$H(i,j:M-i+j)=U(i,M-i+j)C'R^{-1} \quad (2.12.b)$$

$$U(i,n+1)=V(i,n)-V(i,n)C'R^{-1}[I+CV(i,n)C'R^{-1}]^{-1}CV(i,n) \quad (-1\le n\le M-1) \quad (2.12.c)$$

$$U(i,-1)=P(i-M-1,i-M-1) \quad (2.12.d)$$

$$V(i,n)=AU(i,n)A' \quad (2.12.e)$$

where P(i,j) is the state covariance matrix and

$$P(i,j)=E[x(i)x'(i)]=\begin{cases}A^{i-j}P(i,j), & i\ge j\\ P(i,i), & i\le j\end{cases} \quad (2.13)$$

$$P(i+1,i+1)=AP(i,i)A',\quad P(0,0)=P_0$$

From the above results, it is immediately noted that the DFT implementation via (2.10) in the stochastic signal model is identical to the deadbeat observer form (2.8) except that the time invariant gain vector G(i) is replaced by the time varying gain vector H(i,j:M) computed according to (2.12) and (2.13). This time variation of the filter gain is caused by the requirement of optimality at each time instant.

However the time invariant FIR filter can be derived for the system (2.2) under the assumption of the stationarity or unknown initial conditions since the system (2.2) is completely observable and the system matrix is nonsingular. Now let us consider the case of the system with unknown initial conditions. In this case, the FIR filter solution to the system (2.2) is as follows. The proof is omitted for brevity.

THEOREM 2 : In system (2.2), the FIR filter (2.10) becomes time invariant supposing that $P(i-M-1, i-M-1)=\infty I$. Its impulse response is determined as follows :

$$H(i:M)=[\sum_{j=0}^{M} A'^{-j}C'CA^{-j}]^{-1}A'^{-i}C' \tag{2.14}$$

It is noted that the FIR filter solution (2.14) can be directly obtained from the system matrices A and C, where the noise covariance R is not required for the calculation of (2.14). For the case when the filter interval M is equal to the system order M (which means the number of the harmonic components), its filter impulse response is exactly equivalent to the gain of the recursive DFT.

3. FILTER PERFORMANCE IN THE HARMONIC ESTIMATION PROBLEM

In this paper, we consider some performances of the proposed FIR filters for the implementation of the DFT in comparison with the conventional DFT and the Kalman filtering method, where filter performances to be considered are the stability, the noise performance, the tracking ability, the computational efficiency and the frequency sampling property.

3.1 Stability

As mentioned before, it is noted that the state variable signal model (2.2) is uncontrollable since the measurement noise v(i) is zero. Thus from a discrete Riccati equation

$$P(i+1)=A[P(i)-P(i)C'(CP(i)C'+R)^{-1}CP(i)]A' \tag{3.1}$$

From (3.1), we can see that the Riccati solution $\underline{P}$ approaches to zero as time i tends to infinity. In this case, the Kalman filter gain $\underline{K}$ becomes zero, so that the filter ignores the incoming measurement and is asymtotically degenerated. Thus the Kalman filter is stable in the Lyapunov sense but not expotentially asymptotically stable at this problem , due to the uncontrollability of the state in the system (2.2). This degeneracy of the filter is well known in the filtering theory as 'filter divergence'. Even though several technique have been proposed to solve the filter divergence, all of these techniques have been based on the methods to maintain a steady state Riccati solution to non-zero value by force, which it come to the suboptimal filter eventually.

On the other hand, the proposed FIR filtering method possesses BIBO stability due to its FIR structure. Moreover,it is noted that the FIR filter gain given in Theorem 1 is nonvanishing as time i tends to infinity under the assumption that the interval M is fixed and the initial covariance P(i-M-1,i-M-1) is not zero. The reason is that the FIR filter discards all the measurment data before i-M but the Kalman filter do not so since the latter is a IIR filter. It is also noted from Theorem 2 that its impulse response is always nonzero even for the case of an infinite interval 2N in system (2.2). Thus, without any destruction of the filter optimality, the FIR filter can maintain the asymptotic stability at the problem of the extraction of harmonic components from noise.

It is well known that, in the recursive DFT (2.8), its stability is always guaranted since there exists the gain K of the deadbeat observer which causes A-KC to have all its eigenvalues at zero, due to the observability of the pair (A,C).

3.2 Noise Performance and Tracking Ability

As remarked earlier, the conventional DFT (which includes the recursive DFT) suffer a fatal degradation of performances in the case of the existence of the random measurement noises, since the measurement noises are not considered in the signal model for the DFT. Thus it has been a very serious problem in many applications to extract the harmonic components from the noisy measurements.

If we adopt the Kalman filter for this purpose, the selection of the filter gain has an considerable effect on the noise performance as described [5]. That is, one would choose the filter gain to balance a noise rejection versus a tracking speed since a fast convergence rate of the state estimator implies poor noise smoothing properties and, conversely, a slow convergence rate implies a good noise rejection but a poor tracking of the time variations of the harmonic components. In [4], the design parameter is ε ,where the Riccati solution is choosed to $P=\varepsilon I$.

The proposed FIR filtering method not only represent a better performance in the noise suppression than the conventional DFT, but also do not suffer the trade-off in the selection of an appropriate filter gain since the best filter gains are selected by optimal techniques under the given performance criterion such as (2.11).

However, it is noted that, for the case of no measurement noises, the conventional DFT may represent a fastest response in the tracking to time varying harmonic components even though it has the worst noise performance among various harmonic estimation methods.

3.3 Computation Efficiency

In order to compare compuation burdens, we consider the problem to obtain 2N+1 harmonic components in 2N+1 data samples. If we apply the recursive DFT [2] to this problem, it requires $8N^3+16N^2+10N$ multiplications until the accurate harmonic estimation can be obtained. Also the Kalman filtering method by [4] requires $L_{KF}[4N^2+6N]$ multiplications, where L_{KF} is the effective iterating length of the Kalman filter provided an accurate harmonic estimation to this problem and the filter gain K is calculated by off-line in advance.

On the other hand, if the FIR filtering method by using Theorem 2 is applied to this problem, it requires only $L_{FIR}M$ multiplications, where L_{FIR} is the effective iterating length of the FIR filter provided an accurate harmonic estimation to this problem and time invariant filter gains are also calculated in advance.

It is noted from the above comparison that the proposed method given by Theorem 2 may be more efficient than the recursive DFT or the Kalman filtering method in computations since L_{FIR} is less than L_{KF} and M is less than $4N^2+6N$ in the most case. This comes from the fact that the proposed method has the FIR structure but the others do not so.

For the case when the desired harmonic components are not all of the 2N+1 frequency samples but only some of those samples, the proposed FIR filtering methods may be very useful for this purpose but, on the other hand, the conventinal DFT may not be done for this purpose, since the former takes only required harmonic components as the states but the latter is performed by 2N+1 block processing. Also if we apply the Kalman filtering method to this purpose, it is necessary to find all of the (2N+1)x1 components of the filter gain vector K but, on the other hand, the proposed FIR filtering method can find only the related components to the required harmonic components. Thus the latter requires less computations than the former. From the above comparisons, we can conclude that the FIR filtering method is more efficient than the others.

3.4 Frequency Sampling Properties

It was well known in the signal processing area that the frequency sampling filters for the FIR digital filter

$$y(i) = \sum_{k=0}^{N-1} h(k)x(i-k) \tag{3.2}$$

have the cascade structure of a comb filter having z-transfer function $N^{-1}(1-z^{-N})$, a complex oscillator $[1-\exp(j2\pi k/N)z^{-1}]^{-1}$, and a complex gain $\underline{H}_{FSF}(k)$. So its transfer function $\underline{H}_{FSF}(z)$ is written as follows

$$H_{FSF}(z) = \frac{(1-z^{-N})}{N} \sum_{k=0}^{N-1} \frac{1}{1-W_N^{-k}z^{-1}} \underline{H}_{FSF}(k)$$

$$= \frac{1}{N}(1+W_N^{k}z^{-1}+\ldots+W_N^{(N-1)k}z^{-N+1} \tag{3.3}$$

where W_N^{-k} is given by $W_N^{-k}=\exp(j2\pi k/N)$ and $\underline{H}_{FSF}(k)$ is k-th frequency sample value on the unit circle.

It was shown in Bitmead [2] that the deadbeat state observer approach to DFT evaluation is exactly equivalent to the non-recursive frequency sampling filter of a FIR digital filter design.

Now we consider the relation between the proposed FIR filtering method to the harmonic estimation and the above frequency sampling filter. It is expected that the proposed harmonic estimation method has a frequency sampling property as the recursive DFT. The frequeency sampling property of the proposed method is as follows. Here, we consider the FIR filters with unknown initial conditions. The proof is omitted for brevity.

THEOREM 3 : If an initial condition P(i-M-1,i-M-1) is given by P(i-M-1,i-M-1)=∞I and filter interval M is equal to 2N+1, then the FIR filtering method by Theorem 2 has the structure of the frequency sampling filter.

4. SIMULATION

In this paper, we justify the effectiveness of the proposed optimal DFT method in the presence of measurement noises via simulation examples.

We consider a periodic signal z(i) with added measurement noises

$$z(i) = 5+\cos(120\pi i/15)+\sin(120\pi i/15)+0.2v(i) \tag{4.1}$$

where v(i) is zero mean white and $E[v(i)v'(j)]=I\delta_{ij}$ for all i and j. The DFT, the Kalman filter and the FIR filtering method suggested in this paper are applied for the estimation of particular harmonic components in z(i) and compared with each other. For the FIR filtering method, we use the FIR filter with unknown initial conditions whose solution is given by Theorem 2, where the order of the filter is choosed as M=10. Figure 1 presents the estimate of the 60 HZ harmonic power for the measurement (4.1). Then we can know from this figure that the DFT converges in one period if there exists no measurement noises but, if not so, it represents very sensitive responses to measurement noises as expected. The noise performances in the Kalman filtering method seem to be improved to some extend in comparison with the DFT. But its harmonic response depends entirely on ε value where P=ε I. On the other hand, it is noted from this figure that the FIR filtering method suggested by this paper has a good noise suppression property and a fast tracking ability.

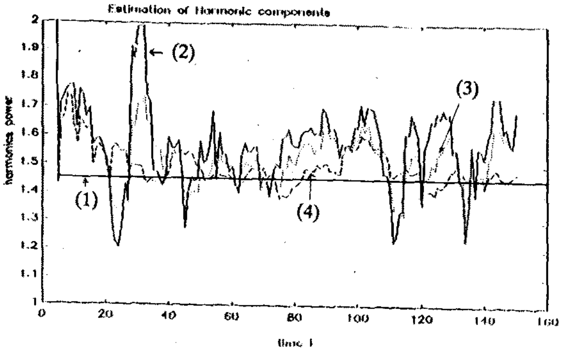

Figure 1. Response of the harmonic estimation in the presence of measurement noises
(1) DFT without noises
(2) DFT with noises
(3) Kalman filtering method
(4) FIR filtering method

5. CONCLUSIONS

In this paper, we introduced the new optimal methods for the harmonic estimation in the presence of measurement noises, which use state model based optimal FIR filters. Perhaps the major benefit of the this result is to extend the DFT evaluation on the deterministic models to stochastic models with random measurement noises. This procedure has been done via state estimation theory based on the optimal design criterion.

The proposd method is superior to the Kalman filtering method by [4] in the sense that the former may alleviate the divergence problem potentially occurred in the latter. The Kalman filtering method has the difficulty in the filter implementation since there exists free-choice at the design of the appropriate gains due to the destruction of the optimality of the filter but the proposed FIR filtering method does not so.

Moreover, the new DFT methods which utilizes a state model based optimal FIR filter have many desirable properties. They are exponentially asymptotically stable. For the case when the measurement signals are corrupted by the random noises, they exhibit better noise performances in comparison with other DFT methods. Their computations for the implementation of the filter are more efficient than the standard DFT, the recursive DFT method or the Kalman filtering method. Also, in Theorem 3, they were shown to be equivalent to the FIR frequency sampling filters to be used for the design of the FIR digital filter, which is a similar property to the recursive DFT. In these viewpoints, the FIR filtering method looks better than the conventional DFT method or the Kalman filtering method.

It is believed that the proposed optimal DFT method can be effectively utilized instead of the conventional DFT when measurement signals are corrupted by noises and when the required harmonic components are limited to some specific harmonic components, not all of the 2N+1 harmonic components.

REFERENCES

[1] Gene H. Hostetter, "Recursive Discrete Fourier Transformation," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-28, NO. 2, Apr. 1980.

[2] Robert R. Bitmead, "On Recursive Discrete Fourier Transformation," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-30, NO. 2, Apr. 1980.

[3] Gene H. Hostetter, "Recursive Discrete Fourier Transformation with unevenly spaced data," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-31, NO. 1, Feb. 1983.

[4] Robert R. Bitmead, Ah Chung Tsoi and Philip J. Parker, "A Kalman Filtering Approach to Short-Time Fourier Analysis," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-34, NO. 6, Dec. 1986.

[5] W.H. Kwon and O.K. Kwon, "FIR filters and recursive forms for continuous time-invariant state-space models," IEEE Trans. Automat. Contr., vol. AC-32, pp. 352-356, 1987.

[6] O.K. Kwon, W.H. Kwon and K.S. Lee, "FIR filters and recursive forms for discrete-time state space models," To be appeared in Automatica.