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평행사변형 유한요소의 강성 매트릭스와 하중 매트릭스.

(Stiffness and Load Matrix for Finite Parallelogrammic Plate Bending Element)

조 병 완 *

요 지

최근에 건설된 여러 토목구조물에서 평행사변형 모양으로 이루어진 구조 요소들이 점차 생겨나고있다. 특히 미국의 일부 I-75 고속도로에서 쓰여진 평행사변형 모양의 슬라브구조나 지형학적 또는 구조역학적인 이유로 인한 고량의 평행사변형 모양 슬라브는 하중의 편심재하로 인한 구조역학의 정확한 해석의 필요성을 야기시켰다. 이에 본 연구는 유한요소법에 의한 평행사변형 요소의 강성매트릭스와 하중매트릭스를 유도하여 평행사변 유한요소를 해석할수있는 구조해석용 서브루틴 프로그램(Subroutine Program)을 만들고 특히 경사 죠인트를 가진 도로구조 해석용 쇼프트웨어(Software)를 만드는데 목적이있다.

Abstract

In the recent years, parallelogrammic shape structures were often introduced in the Civil Engineering. Especially parallelogrammic shape highway slabs, which were used in the portions of Interstate-75, U.S.A. , were a unique one to result in only one wheel crossing the joint at any one time. In this reserch, major efforts were made to provide an appropriate subroutine program and to study an analytical behavior of parallelogrammic plate bending element by developing the stiffness and load matrices.

1. Parallelogrammic Element.

The finite element formulation of the parallelogram plate bending element is presented based on the MZC rectangular plate bending element. The four node element as shown in Figure 1 has three independent displacements at each node, which are a vertical deflection, w, and two rotations about the X and Y axes, for a total of twelve degrees of freedom per element.

The array of nodal displacements at the ith node is:

$$q_i = \begin{Bmatrix} q_{i1} \\ q_{i2} \\ q_{i3} \end{Bmatrix} = \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} = \begin{Bmatrix} w_i \\ -dw_i/dy \\ dw_i/dx \end{Bmatrix}$$

The array of corresponding nodal forces is:

$$p_i = \begin{Bmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{Bmatrix} = \begin{Bmatrix} F_{zi} \\ M_{xi} \\ M_{yi} \end{Bmatrix}$$

(i = 1,2,3,4 node number)

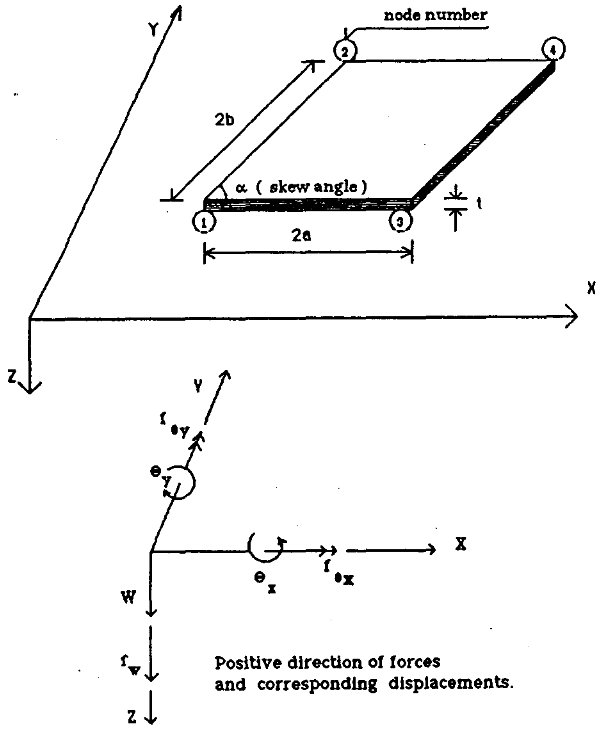

Fig. 1 Dimensions and Positive direction of a Parallelogram Plate Bending Element.

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1.1 Coordinate Mapping.

The basic relationships between regular X,Y coordinate system and the inclined R,S coordinate system normalized in the range of (-1,1) for the Gauss Quadrature method are:

$$X = aR + Z$$
$$Y = bS \sin(\alpha)$$

Eliminating Z,

$$\tan(\alpha) = Y/Z,$$
$$Z = Y \cot(\alpha) = bS \sin(\alpha) \cot(\alpha) = bS \cos(\alpha)$$

Therefore, $X = aR + bS \cos(\alpha)$, $Y = bS \sin(\alpha)$

Or, $R = (X - Y \cot(\alpha))/a$, $S = Y \operatorname{cosec}(\alpha)/b$

The chain rule of differentiation of a general function of R and S, f(R,S), with respect to X and Y leads to

$$df(R,S)/dx = df/dR\ dR/dx + df/dS\ dS/dX = df(R,S)/dR\ 1/a$$
$$df(R,S)/dY = df/dR\ dR/dY + df/dS\ dS/dY = df(R,S)/dR\ (-\cot(\alpha)/a) + df(R,S)/dS\ (\operatorname{cosec}(\alpha)/b)$$

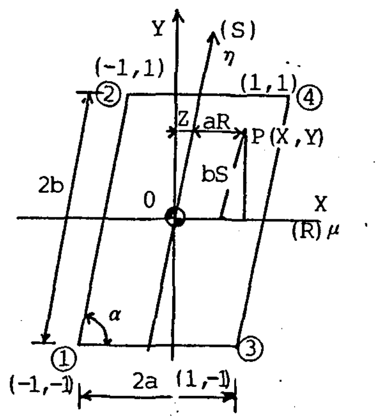

Fig. 2. Coordinate Mapping for Parallelogram Element.

1.2 Displacement Shape Function.

The displacement shape functions, [fi], are defined to relate generic displacements, U = { u, v, w }, to nodal displacements, q = {qi} = [wi, θxi, θyi]. Let the generic displacement function, vertical displacement w, with an element be:

$$w = C_1 + C_2X + C_3Y + C_4X^2 + C_5XY + C_6Y^2 + C_7X^3 + C_8X^2Y + C_9XY^2 + C_{10}Y^3 + C_{11}X^3Y + C_{12}XY^3$$

$$(-dw/dy)i = \theta Xi = -C_3 + C_5X + \dots$$
$$(dw/dx)i = \theta Yi = C_2 + 2C_4X + \dots$$

where the coordinates of the ith node are substituted for X and Y. Listing all twelve equations we can write, in matrix form,

$$\{ qi \} = \begin{Bmatrix} Wi \\ \theta xi \\ \theta yi \end{Bmatrix} = [C] \{ V \}$$

Where [C] is a twelve-by-twelve matrix which is a function of nodal coordinates, and { V } is a vector of the twelve unknown constants. Inverting

$$\{ V \} = [C]^{-1} \{ qi \}$$

substituting $w = [g][V]$

where [g] = { 1,X,Y,X2, XY,Y2 ... } 1x12

$$\text{Now, } w = [g][C]^{-1} \{ qi \} = \{ f \}\{ qi \}$$

where { f } is a twelve-shape functions which can be obtained by inverting a [C] matrix and then multiplied by [g]. Displacement shape functions are given in following Table.

Node	Coordinate R, S	Degree of Freedom #	Shape Function.
1	R=-1, S=-1	fi1=1 fi2=2 fi3=3	$f1 = (1-R)(1-S)(2-R-S-R^2-S^2)/8$ $f2 = -b(1-R)(1+S)(1-S)^2/8$ $f3 = a(1+R)(1-S)(1-R)^2/8$
2	R=-1, S=1	fi1=4 fi2=5 fi3=6	$f4 = (1-R)(1+S)(2-R+S-R^2-S^2)/8$ $f5 = b(1-R)(1-S)(1+S)^2/8$ $f6 = a(1+R)(1+S)(1-R)^2/8$
3	R=1, S=-1	fi1=7 fi2=8 fi3=9	$f7 = (1+R)(1-S)(2+R-S-R^2-S^2)/8$ $f8 = -b(1+R)(1+S)(1-S)^2/8$ $f9 = -a(1-R)(1-S)(1+R)^2/8$
4	R=1, S=1	fi1=10 fi2=11 fi3=12	$f10 = (1+R)(1+S)(2+R+S-R^2-S^2)/8$ $f11 = b(1+R)(1-S)(1+S)^2/8$ $f12 = -a(1-R)(1+S)(1+R)^2/8$

2. Displacement - Strain - Stress Relationships.

Displacement shape functions {f} relate generic displacements (U) to nodal displacements {q}.

$$U = \{ f \}\{ q \} \quad (2.1)$$

Strain can be obtained by differentiation of the generic displacements(U),

$$\{ \epsilon \} = d\ U. \quad (2.2)$$

where, $\epsilon x = dU/dx$, $\epsilon y = dV/dy$, $\epsilon xy = dU/dy + dV/dx$

Substituting (2.1) to (2.2),
$\{\epsilon\} = d\,U = d\{f\}\{q\} = B\,\{q\}$

where B Strain-Displacement operator ($B = d\,\{f\}$) relates nodal displacements at each point to strain matrix and can be obtained by the differentiation of shape functions.

Simple continum mechanics shows

$$\sigma = [E]\,(\epsilon - \epsilon_0) \qquad (2.4)$$

Substituting (2.3) to (2.4),

$$\sigma = [E]\,\{(Bq) - \epsilon_o\} \qquad (2.5)$$

in which the matrix product E B, which is called stress matrix, gives stresses at a generic point.

2.1 Virtual Work Principle.

By the definition, the virtual work of external actions is equal to the virtual strain energy of internal stresses.

$$dUi = dWe \qquad (2.6)$$

Assume a set of nodal virtual displacements, $\{q^v\}$.
The resulting displacements and strains within the element are

$$[\,U^v\,] = [\,f\,]\;\{q^v\}$$
$$[\,\epsilon^v\,] = [\,B\,]\;\{q^v\}$$

Equating the external work with the total internal work and integrating over the volume of the element,

$$\{q^v\}^T\{F\} = \int_v [\epsilon^v]^T\{\sigma\}\,dv = \{q^v\}^T \cdot (\int [B]^T\{\sigma\}\,dv - \int [f]^T w\,dv)$$

Substituting Eqs.(2.1) through (2.5),

$$\{F\} = (\int_v [B]^T[E][B]\,dv)\{q\} - \int_v [B]^T[E]\{\epsilon_o\}\,dv - \int_v [f]^T w\,dv \qquad (2.7)$$

The stiffness matrix [K] becomes,

$$[\,K\,] = \int_v [B]^T\,[E]\,[B]\,dV \qquad (2.8)$$

Nodal forces due to distributed loads are

$$\{F_w\} = -\int_v [f]^T w\,dv \qquad (2.9)$$

Nodal forces due to initial strains are

$$\{F_{\epsilon o}\} = -\int_v [B]^T[E]\,\{\epsilon_o\}\,dv \qquad (2.10)$$

3. Stiffness Matrix

From equation (2.8), stiffness matrix becomes

$$K = \int_v B^T E\;B\;dV$$
$$= t\;|J|\;\iint B^T\;E\;B\;dRdS$$

(where $dV = dXdYdZ$)

(where -1 to 1 integration)

By the differentiation of f(X,Y) with respect to R and S, Chain rule produces the Jacobian matrix J as follows:

$$\begin{bmatrix} df/dR \\ df/dS \end{bmatrix} = \begin{bmatrix} dX/dR & dY/dR \\ dX/dS & dY/dS \end{bmatrix} \begin{bmatrix} df/dX \\ df/dY \end{bmatrix}$$

$$\text{Jacobian } [J] = \begin{bmatrix} dX/dR & dY/dR \\ dX/dS & dY/dS \end{bmatrix} = ab\;\sin(\alpha)$$

Therefore, the stiffness matrix K becomes

$$K = t\;ab\;\sin(\alpha)\iint B^T E\;B\;dRdS$$

where $Bi = df$

$$= \begin{bmatrix} -fi1,xx & -fi2,xx & -fi3,xx \\ -fi1,yy & -fi2,yy & -fi3,yy \\ 2fi1,xy & 2fi2,xy & 2fi3,xy \end{bmatrix}$$

After finding B operator by the second differentiation of shape function, Numerical integration of (BT E B) from -1 to 1 gives Stiffness matrix K of parallelogrammic plate bending element as shown in Appendix.

4. Equivalent Nodal Loads.

4.1 Uniformly Distributed Loads.

When an uniformly distributed load of magnitude w acts over an element, the equivalent nodal loads:

$$Fw = A\int f^T\,w\,dA = w\;|J|\;\iint f^T\,dR\,dS$$
$$= w\;ab\;\sin(\alpha)\;\iint f^T\,dR\,dS$$

Numerical integration using three point Gauss Quadrature for each of the two axes of integration leads to:

$$F_w = 4\;w\;ab\;\sin(\alpha)\cdot \begin{Bmatrix} 1/4 \\ -b/12 \\ a/12 \\ 1/4 \\ b/12 \\ a/12 \\ 1/4 \\ -b/12 \\ -a/12 \\ 1/4 \\ b/12 \\ -a/12 \end{Bmatrix}$$

4.2 Thermal Gradients.

Assuming that the temperature

varies linearly from the top to the bottom of the plate with a temperature differential of dT, the curvatures of the unrestrained plate element will be :

$$\epsilon_0 = \begin{bmatrix} -d^2w/dx^2 \\ -d^2w/dy^2 \\ 2d^2w/dxdy \end{bmatrix} = \begin{bmatrix} -\alpha\, dT/t \\ -\alpha\, dT/t \\ 0 \end{bmatrix}$$

α = Coeff. of thermal expansion.
dT= Temperature differential.
t = Thickness of plate element.

The virtual work theory yields to the equivalent nodal loads due to the uniform temperature variations as follows:

$$F^T = \int_A B^T E\, \epsilon_0\, dA$$
$$= ab\, \mathrm{Sin}(\alpha) \iint B^T E\, \epsilon_0\, dR\, dS$$
$$= ab\, \mathrm{Sin}(\alpha) \iint B^T \begin{bmatrix} Dx & D1 & 0 \\ D1 & Dy & 0 \\ 0 & 0 & Dxy \end{bmatrix} \cdot (-\alpha \cdot dT/t) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot dRdS$$

$$= \frac{E\, t^2\ (\alpha \cdot dT)}{12(1-PR)} \begin{bmatrix} -2\,\mathrm{Cot}(\alpha) \\ a/\mathrm{Sin}(\alpha) \\ -b/\mathrm{Sin}(\alpha) \\ 2\,\mathrm{Cot}(\alpha) \\ -a/\mathrm{Sin}(\alpha) \\ -b/\mathrm{Sin}(\alpha) \\ -2\,\mathrm{Cot}(\alpha) \\ a/\mathrm{Sin}(\alpha) \\ -b/\mathrm{Sin}(\alpha) \\ 2\,\mathrm{Cot}(\alpha) \\ -a/\mathrm{Sin}(\alpha) \\ -b/\mathrm{Sin}(\alpha) \end{bmatrix}$$

5. Conclusions

Comparisons of deflection and moment between Timoshenko's analytical equation and this F.E.M. formula show that skew angles up to 45° provide great accuracy.

Skew angle	Deflections at the center Timoshenko's	Jo's	Error
90	0.8298"	0.8323"	0.3%
60	0.8069"	0.8022"	0.6%
45	0.7864"	0.7367"	4.1%

Skew angle	Moments at the center Timoshenko's	Jo's	Error
90	143.86	144.55	0.5%
60	139.39	138.74	0.5%
45	129.31	122.90	4.9%

In common with other methods, the accuracy of results obtained using the finite element method to solve the structural response of concrete pavements system decreases with increase in skewed angles. At high skew some improvement can be obtained by meshing a larger number of elements or using a higher order displacement function.

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APPENDIX : STIFFNESS MATRIX OF PARALLELOGRAMMIC PLATE BENDING ELEMENT
(12×12 SYMMETRIC)

$A_4+C_1C_1+C_2C_2+1.9C_3C_3$ $+PR(2C_1+0.5C_2)$ $+D_{xy}(4H_{11}+\frac{7}{5}H_{22})$											
$-0.7C_1C_7-C_2C_7$ $-PR(0.5C_7)$ $-0.2D_{xy}H_{33}$	$\frac{8}{15}C_6C_6+\frac{4}{3}C_7C_7-C_6C_7$ $+D_{xy}(\frac{8}{15}P_{11})$										
$A_3+C_1C_9+0.7C_3C_9$ $+PR(2C_1D+0.5C_3D)$ $+D_{xy}(4C_1D+0.2C_3D)$	$\frac{1}{6}C_6C_9-C_7C_9+PR(\frac{1}{6}DC_6$ $-DC_7)+DXY\frac{1}{3}P_{22}$	$\frac{4}{3}A_2+\frac{4}{3}C_4C_9-\frac{7}{15}C_9C_{10}$ $+\frac{1}{3}C_6C_{10}+PR\frac{8}{3}(Z_4-Z_5)$ $+DXY(\frac{16}{3}Q_{11}+\frac{8}{15}Q_{22}-2Q_{33})$									
$0.5A_4+0.5C_1C_1-C_2C_2$ $-1.9C_3C_3+PR(C_1-0.5C_2)$ $+DXY(2H_{11}-\frac{7}{5}H_{22})$	$0.2C_1C_7+C_2C_7$ $+DXY(0.2H_{33})$	$0.5A_3+0.5C_1C_9-C_1C_{10}$ $-0.7C_3C_9+PR(C_1D-C_2D$ $-0.5C_3D)+DXY(2C_1D$ $-2C_2D-0.2C_3D)$	$A_4+C_1C_1+C_2C_2+1.9C_3C_3$ $+PR(2C_1+0.5C_2)$ $+DXY(4H_{11}+\frac{7}{5}H_{22})$								
$-0.2C_1C_7-C_2C_7$ $+DXY(-0.2H_{33})$	$-\frac{2}{15}C_6C_6+\frac{2}{3}C_7C_7$ $+DXY(-\frac{2}{15})P_{11}$	$-\frac{1}{6}C_6C_9+PR(-\frac{1}{6})DC_6$ $+DXY(-\frac{1}{3})P_{22}$	$0.7C_1C_7+C_2C_7$ $+PR0.5C_7+DXY0.2H_{33}$	$\frac{8}{15}C_6C_6+\frac{4}{3}C_7C_7+C_6C_7$ $+DXY\frac{8}{15}P_{11}$							
$0.5A_3+0.5C_1C_9+C_1C_{10}$ $-0.7C_3C_9+PR(C_1D-0.5C_3D$ $+C_2D)+DXY(2C_1D+2C_2D$ $-0.2C_3D)$	$-\frac{1}{6}C_6C_9+PR(-\frac{1}{6})DC_6$ $+DXY(-\frac{1}{3})P_{22}$	$\frac{2}{3}A_2+\frac{2}{3}C_4C_9-\frac{8}{15}C_9C_{10}$ $+PR\frac{4}{3}Z_4+DXY(\frac{8}{3}Q_{11}$ $-\frac{8}{15}Q_{22})$	$A_3+C_1C_9+0.7C_3C_9$ $+PR(2C_1D+0.5C_3D)$ $+DXY(4C_1D+0.2C_3D)$	$\frac{1}{6}C_6C_9+C_7C_9+PR(\frac{1}{6}DC_6$ $+DC_7)+DXY\frac{1}{3}P_{22}$	$\frac{4}{3}A_2+\frac{4}{3}C_4C_9+\frac{8}{15}C_9C_{10}$ $+C_6C_{10}+PR(\frac{8}{3}Z_4+Z_5)$ $+DXY(\frac{16}{3}Q_{11}+\frac{8}{15}Q_{22}+2Q_{33})$						
$-A_4-C_1C_1+0.5C_2C_2$ $-1.9C_3C_3+PR(-2C_1-0.5C_2)$ $+DXY(-4H_{11}-\frac{7}{5}H_{22})$	$0.7C_1C_7+C_2C_6-0.5C_2C_7$ $+PR0.5C_7+DXY0.2H_{33}$	$-A_3-C_1C_9-0.2C_3C_9$ $+PR(-2C_1D)$ $+DXY(-4C_1D-0.2C_3D)$	$-0.5A_4-0.5C_1C_1-0.5C_2C_2$ $+1.9C_3C_3+PR(-C_1+0.5C_2)$ $+DXY(-2H_{11}+\frac{7}{5}H_{22})$	$0.2C_1C_7-C_2C_6-0.5C_2C_7$ $+DXY0.2H_{33}$	$-0.5A_3-0.5C_1C_9-C_1C_{10}$ $+0.2C_3C_9+PR(-C_1D-C_3D)$ $+DXY(-2C_1D-2C_2D)$ $+0.2C_3D)$	$A_4+(C_1C_1+C_2C_2+1.9C_3C_3$ $+PR(2C_1+0.5C_2)$ $+DXY(4H_{11}+\frac{7}{5}H_{22})$					
$0.7C_1C_7-C_2C_6-0.5C_2C_7$ $+PR0.5C_7+DXY0.2H_{33}$	$-\frac{8}{15}C_6C_6+\frac{2}{3}C_7C_7$ $+DXY(-\frac{8}{15})P_{11}$	$-\frac{1}{6}C_6C_9+PR(-\frac{1}{6})DC_6$ $+DXY(-\frac{1}{3})P_{22}$	$-0.2C_1C_7+C_2C_6+0.5C_2C_7$ $+DXY(-0.2)H_{33}$	$\frac{2}{15}C_6C_6+\frac{1}{3}C_7C_7+C_6C_7$ $+DXY\frac{2}{15}P_{11}$	$\frac{1}{6}C_6C_9+PR\frac{1}{6}DC_6$ $+DXY\frac{1}{3}P_{22}$	$-0.7C_1C_7-C_2C_7$ $+PR(-0.5C_7)$ $+DXY(-0.2H_{33})$	$\frac{8}{15}C_6C_6+\frac{4}{3}C_7C_7+C_6C_7$ $+DXY\frac{8}{15}P_{11}$				
$A_3+C_1C_9+0.2C_3C_9$ $+PR2C_1D+DXY(4C_1D$ $+0.2C_3D)$	$-\frac{1}{6}C_6C_9+PR(-\frac{1}{6})DC_6$ $+DXY(-\frac{1}{3})P_{22}$	$\frac{2}{3}A_2+\frac{2}{3}C_4C_9-\frac{2}{15}C_9C_{10}$ $+PR\frac{4}{3}Z_4+DXY(\frac{8}{3}Q_{11}$ $-\frac{2}{15}Q_{22})$	$0.5A_3+0.5C_1C_9+C_1C_{10}$ $-0.2C_3C_9+PR(C_1D+C_3D)$ $+DXY(2C_1D+2C_2D$ $-0.2C_3D)$	$\frac{1}{6}C_6C_9+PR\frac{1}{6}DC_6$ $+DXY\frac{1}{3}P_{22}$	$\frac{1}{3}A_2+\frac{1}{3}C_4C_9+\frac{2}{15}C_9C_{10}$ $+C_6C_{10}+PR(\frac{2}{3}Z_4+Z_5)$ $+DXY(\frac{4}{3}Q_{11}+\frac{2}{15}Q_{22}+2Q_{33})$	$-A_3-C_1C_9-0.7C_3C_9$ $+PR(-2C_1D-0.5C_3D)$ $+DXY(-4C_1D-0.2C_3D)$	$\frac{1}{6}C_6C_9+C_7C_9$ $+PR(\frac{1}{6}DC_6+DC_7)$ $+DXY\frac{1}{3}P_{22}$	$\frac{4}{3}A_2+\frac{4}{3}C_4C_9+\frac{8}{15}C_9C_{10}$ $+C_6C_{10}+PR(\frac{8}{3}Z_4+Z_5)$ $+DXY(\frac{16}{3}Q_{11}+\frac{8}{15}Q_{22}+2Q_{33})$			
$-0.5A_4-0.5C_1C_1-0.5C_2C_2$ $+1.9C_3C_3+PR(-C_1+0.5C_2)$ $+DXY(-2H_{11}+\frac{7}{5}H_{22})$	$-0.2C_1C_7-C_2C_6+0.5C_2C_7$ $+DXY(-0.2H_{33})$	$-0.5A_3-0.5C_1C_9+C_1C_{10}$ $+0.2C_3C_9+PR(-C_1D$ $+C_3D)+DXY(-2C_1D$ $+2C_2D+0.2C_3D)$	$-A_4-C_1C_1+0.5C_2C_2$ $-1.9C_3C_3+PR(-2C_1-0.5C_2)$ $+DXY(-4H_{11}-\frac{7}{5}H_{22})$	$-0.7C_1C_7+C_2C_6+0.5C_2C_7$ $+PR(-0.5C_7)$ $+DXY(-0.2H_{33})$	$-A_3-C_1C_9-0.2C_3C_9$ $+PR(-2C_1D)$ $+DXY(-4C_1D-0.2C_3D)$	$0.5A_4+0.5C_1C_1-C_2C_2$ $-1.9C_3C_3+PR(C_1-0.5C_2)$ $+DXY(2H_{11}-\frac{7}{5}H_{22})$	$0.2C_1C_7+C_2C_7$ $+DXY0.2H_{33}$	$-0.5A_3-0.5C_1C_9-C_1C_{10}$ $+0.7C_3C_9+PR(-C_1D-C_2D$ $+.5C_3D)+DXY(-2C_1D$ $-2C_2D+0.2C_3D)$	$A_4+C_1C_1+C_2C_2+1.9C_3C_3$ $+PR(2C_1+0.5C_2)$ $+DXY(4H_{11}+\frac{7}{5}H_{22})$		
$0.2C_1C_7+C_2C_6-0.5C_2C_7$ $+DXY0.2H_{33}$	$\frac{2}{15}C_6C_6+\frac{1}{3}C_7C_7-C_6C_7$ $+DXY\frac{2}{15}P_{11}$	$\frac{1}{6}C_6C_9+PR\frac{1}{6}DC_6$ $+DXY\frac{1}{3}P_{22}$	$-0.7C_1C_7-C_2C_6+0.5C_2C_7$ $+PR(-0.5C_7)$ $+DXY(-0.2H_{33})$	$-\frac{8}{15}C_6C_6+\frac{2}{3}C_7C_7$ $+DXY(-\frac{8}{15})P_{11}$	$-\frac{1}{6}C_6C_9+PR(-\frac{1}{6})DC_6$ $+DXY(-\frac{1}{3})P_{22}$	$-0.2C_1C_7-C_2C_7$ $+DXY(-0.2H_{33})$	$-\frac{2}{15}C_6C_6+\frac{2}{3}C_7C_7$ $+DXY(-\frac{2}{15})P_{11}$	$-\frac{1}{6}C_6C_9+PR(-\frac{1}{6})DC_6$ $+DXY(-\frac{1}{3})P_{22}$	$0.7C_1C_7+C_2C_7$ $+PR0.5C_7+DXY0.2H_{33}$	$\frac{8}{15}C_6C_6+\frac{4}{3}C_7C_7-C_6C_7$ $+DXY\frac{8}{15}P_{11}$	
$0.5A_3+0.5C_1C_9-C_1C_{10}$ $-0.2C_3C_9+PR(C_1D-C_3D)$ $+DXY(2C_1D-2C_2D-0.2C_3D)$	$\frac{1}{6}C_6C_9+PR\frac{1}{6}DC_6$ $+DXY\frac{1}{3}P_{22}$	$\frac{1}{3}A_2+\frac{1}{3}C_4C_9+\frac{2}{15}C_9C_{10}$ $-C_6C_{10}+PR(\frac{2}{3}Z_4-Z_5)$ $+DXY(\frac{4}{3}Q_{11}+\frac{2}{15}Q_{22}-2Q_{33})$	$A_3+C_1C_9+0.2C_3C_9$ $+PR2C_1D$ $+DXY(4C_1D+0.2C_3D)$	$-\frac{1}{6}C_6C_9+PR(-\frac{1}{6})DC_6$ $+DXY(-\frac{1}{3})P_{22}$	$\frac{2}{3}A_2+\frac{2}{3}C_4C_9-\frac{2}{15}C_9C_{10}$ $+PR\frac{4}{3}Z_4$ $+DXY(\frac{8}{3}Q_{11}-\frac{2}{15}Q_{22})$	$-0.5A_3-0.5C_1C_9+C_1C_{10}$ $+0.7C_3C_9+PR(-C_1D$ $+0.5C_3D+C_2D)+DXY$ $(-2C_1D+2C_2D+0.2C_3D)$	$-\frac{1}{6}C_6C_9+PR(-\frac{1}{6})DC_6$ $+DXY(-\frac{1}{3})P_{22}$	$\frac{2}{3}A_2+\frac{2}{3}C_4C_9-\frac{8}{15}C_9C_{10}$ $+PR\frac{4}{3}Z_4+DXY(\frac{8}{3}Q_{11}$ $-\frac{8}{15}Q_{22})$	$-A_3-C_1C_9-0.7C_3C_9$ $+PR(-2C_1D-0.5C_3D)$ $+DXY(-4C_1D-0.2C_3D)$	$\frac{1}{6}C_6C_9-C_7C_9$ $+PR(\frac{1}{6}DC_6-DC_7)$ $+DXY\frac{1}{3}P_{22}$	$\frac{4}{3}A_2+\frac{1}{3}C_4C_9-\frac{7}{15}C_9C_{10}$ $+PR(\frac{8}{3}Z_4-Z_5)$ $+DXY(\frac{16}{3}Q_{11}+\frac{8}{15}Q_{22}-2Q_{33})$

Element stiffness matrix

$SK(i,j)\ i=1,12 = ab\sin(\alpha)\,[SK_1\cdot D_x+SK_2\cdot D_y+SK_3\cdot D_1+SK_4\cdot D_{xy}]$
$j=1,12$

For isotrofic material $D_x=D_y=1$
$D_1=PR$ (Poisson's Ratio)
$D_{xy}=(-PR)/2$

Notations :

$A_1=1/A$ $B_1=1/B$
$A_2=1/A^2$ $B_2=1/B^2$
$A_3=1/A^3$ $B_3=1/B^3$
$A_4=1/A^4$ $B_4=1/B^4$
$C_2C_7=B_3\cdot\mathrm{cosec}^3(\alpha)$
$C_6C_9=A_3\cdot\cot^3(\alpha)\cdot\mathrm{cosec}(\alpha)$
$C_7C_9=A_1\cdot B_1\cdot\cot^2(\alpha)\cdot\mathrm{cosec}^2(\alpha)$
$C_6C_6=A_2\cdot\cot^4(\alpha)$

$C_1C_1=A_4\cdot\cot^4(\alpha)$
$C_2C_2=B_4\cdot\mathrm{cosec}^4(\alpha)$
$C_3C_3=A_2\cdot B_2\cdot\cot^2(\alpha),\ \mathrm{cosec}^2(\alpha)$
$C_4C_4=A_2\cdot\cot^2(\alpha)\cdot\mathrm{cosec}^2(\alpha)$
$C_{10}C_{10}=B_2\cdot\cot^2(\alpha)\cdot\mathrm{cosec}^2(\alpha)$
$C_9C_{10}=A_1\cdot B_1\cot^2(\alpha)\cdot\mathrm{cosec}(\alpha)$
$C_1C_9=A_3\cdot\cot^4(\alpha)$
$C_1C_{10}=A_2\cdot B_1\cdot\cot^2(\alpha)\cdot\mathrm{cosec}(\alpha)$
$C_7=A_3\cdot B_1\cdot\mathrm{cosec}^2(\alpha)$
$C_1D=A_3\cdot\cot^2(\alpha)$
$C_2D=A_1\cdot B_2\cdot\mathrm{cosec}^2(\alpha)$
$C_3D=A_2\cdot B_1\cdot\cot(\alpha)\cdot\mathrm{cosec}(\alpha)$

$C_7C_7=B_2\cdot\mathrm{cosec}^4(\alpha)$
$C_6C_7=A_1\cdot B_1\cot(\alpha)\cdot\mathrm{cosec}^3(\alpha)$
$C_1C_7=A_2\cdot B_1\cot^2(\alpha)\cdot\mathrm{cosec}^2(\alpha)$
$C_2C_6=A_1\cdot B_2\cot(\alpha)\cdot\mathrm{cosec}^3(\alpha)$
$C_2C_9=A_1\cdot B_2\cot^2(\alpha)\cdot\mathrm{cosec}^2(\alpha)$
$C_7C_{10}=B_1\cdot\cot(\alpha)\cdot\mathrm{cosec}^3(\alpha)$
$C_1=A_4\cdot\cot^2(\alpha)$
$C_2=A_2\cdot B_2\cdot\mathrm{cosec}^2(\alpha)$
$DC_6=A_2\cdot\cot(\alpha)\cdot\mathrm{cosec}(\alpha)$
$DC_7=A_1\cdot B_1\cdot\mathrm{cosec}^2(\alpha)$
$Z_4=A_2\cdot\cot^2(\alpha)$
$Z_5=A_1\cdot B_1\cdot\cot(\alpha)\cdot\mathrm{cosec}(\alpha)$
$H_{11}=A_4\cdot\cot^2(\alpha)$
$H_{22}=A_2\cdot B_2\cdot\mathrm{cosec}^2(\alpha)$
$H_{33}=A_3\cdot B_1\cdot\cot(\alpha)\cdot\mathrm{cosec}(\alpha)$
$H_{44}=A_3\cdot\cot(\alpha)\cdot\mathrm{cosec}(\alpha)$
$H_{55}=A_1\cdot B_1\cdot\mathrm{cosec}^2(\alpha)$
$P_{11}=A_2\cdot\mathrm{cosec}^2(\alpha)$
$P_{22}=A_2\cdot\cot(\alpha)\cdot\mathrm{cosec}(\alpha)$
$P_{33}=A_1\cdot B_1\ \mathrm{cosec}^2(\alpha)$
$Q_{11}=A_2\cdot\cot^2(\alpha)$
$Q_{22}=B_2\cdot\mathrm{cosec}^2(\alpha)$
$Q_{33}=A_1\cdot B_1\cdot\cot(\alpha)\ \mathrm{cosec}(\alpha)$