Existence of a Nash Equilibrium to Differential Games With Nonlinear Constraints

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§1. Introduction

The differential game is a mathematical decision-aid tool in a conflict situation which evolves over time. The model has been applied widely in managerial problems including investment, production, marketing. Jorgensen(1985) provides an excellent survey for applications of the differential game model. In order to apply the theory to the analysis of economic competition, where mutual interests play a significant role, nonzero-sum formulations are mostly appropriate. Zero-sum game models usually rule out the possibility of mutual benefits between the conflicting parties [Ciletti and Starr(1970)].

Although necessary conditions for an optimal solution of the differential game problems may be derived by an application of the maximum principle of optimal control theory, it is, in general, very difficult to find an analytical solution for the necessary conditions. The set of necessary conditions of an optimal solution to differential game problems requires solving a system of differential equations. Solution Methods and existence proofs for simple differential games are found in Starr and Ho(1969), Friedman(1971), and Neese and Pindyck(1984). Simman and Takayama(1978) gives an example for a single state linear-quadratic game with constraints. For open-loop nonzero-sum differential games, Scalzo(1974) proved existence for any finite duration. Scalzo's work has been extended by Wilson(1977) and Williams(1980) to games with incomplete information and by Scalzo and Williams(1976) to games with nonlinear state equations. All three extensions dealt with the finite horizon case.

With the assumptions of continuity, convexity, and compactness, this paper shows the existence of a Nash equilibrium for a more general class of differential games. To prove it, the Kakutani theorem is applied after discretization of the problem. The rationale for a period-by-period operation of multi-period problems is provided, using a dynamic programming approach.

§2. Differential game problem

We first define a general type of the nonzero-sum two-person differential game. Let subscript t denote time, and superscript i on functions and subscript for variables or parameters stand for player i. The state variable at time t, $z_t=(z_{1t},z_{2t})$, is governed by a system of the first order differential equations,

$$(1)\qquad \dot{z}_{it} = g^i(z_t,u_t) \qquad i=1,2,$$

where $u_t=(u_{1t},u_{2t})$ are the control variables. The initial states at time 0 are assumed to be known. The players set the control variables to achieve a desired state. They may have a limitation in setting the control variables. We assume the realized states at time t determine the control space for each player,

$$(2)\qquad (u_{it}) \in \Omega^i(t,z_t), \qquad i=1,2.$$

A strategy, (u_t) is admissible if it belongs to the space defined by the incumbent state, z_t.

The game begins at some initial time and state $(0,z_0)$, and terminates at (T,z_T). The terminal time T can be chosen freely. If T is a pre-specified point in time, the payoff to player i over the finite horizon is given by

$$(3)\qquad J^i(z_t,u_t) = S^i(T,z_T) + \int_0^T e^{-rt}f^i(t,z_t,u_t)dt, \qquad i=1,2$$

where S^i is a real valued function representing player i's salvage value at T if the terminal state of the game is z_T, and f^i is a real valued function on (t,z,u)-space. The functions S^i, f^i are also assumed to be of class C^3 with respect to their own arguments. The objective for each player is to select a control strategy which maximizes J^i.

In (2), we denoted the set of admissible values for the control variables by $\Omega^i(t,z_t)$. For the problem to be defined in a more manageable setting, we shall further assume that the control variables must satisfy the following constraints,

$$(4)\qquad h^i(t,z_t,u_t) \geq 0, \qquad i=1,2,$$

where h^i is a real-valued, third order differential function with respect to all the arguments. The admissible control spaces are now assumed to be expressed by a form of (4).

§3. Existence theorem

The problem described in the previous section can be written in a discretized form as follows,

$$(P_i) \quad \begin{array}{l} \underset{\{u_i\}}{\text{Max}} \quad J^i = \Sigma_{t=0}^{T-1} (1+r)^{-t} f^i(z_t, u_t) + (1+r)^{-T} S^i(z_T) \\ \text{subject to,} \\ z_{j,t+1} - z_{jt} = g^j(z_t, u_t) \quad , j=1,2, \quad t=0,\dots,T-1 \\ h^i(z_t, u_t) \geq 0 \quad , t=0,\dots,T-1. \end{array}$$

The problem is now to find a sequence of equilibrium control vectors, $\{u_t\}_{t=0}^{T-1}$, and hence a sequence of the state vectors, $\{z_t\}_{t=0}^{T}$, which maximizes the objective function. Define a function W^i as

$$W^i(z_t, u_t, \dots, u_{T-1}) = \Sigma_{\tau=t}^{T-1} (1+r)^{-\tau} f^i(z_\tau, u_\tau) + (1+r)^{-T} S^i(z_T)$$

and

$$(SP_{it}) \quad \begin{array}{l} V^i(z_t, t) = \underset{\{u_{it}\}}{\text{Max}} \quad (1+r)^{-t} f^i(z_t, u_t) + V^i[z_{t+1}(z_t, u_t), t+1] \\ \text{subject to,} \\ z_{j,t+1} - z_{jt} = g^i(z_t, u_t) \quad , j=1,2, \\ h^i(z_t, u_t) \geq 0. \end{array}$$

$V^i(z_t, t)$ represents the optimal objective value from t to the terminal time period if the incumbent state at t is z_t. We write $z_{t+1}(z_t, u_t)$ to show explicitly that z_{t+1} depends on z_t and u_t. The subproblem of player i at time t, (SP_{it}), is to maximize his payoffs from time t to T for z_t given. Note $V^i(.,T) = (1+r)^{-T} S^i(z_T)$.

The following theorem provides us with a basis for the claim that a period-by-period solution consists of a Nash equilibrium for the original problem.

THEOREM 1. If u_t^* is a Nash equilibrium to (SP_{it}), for $t=0,\dots,T-1$, then $\{u_t^*\}_{t=0}^{T-1}$ is a Nash equilibrium to the problem (P). Therefore, if (SP_{it}) has an equilibrium solution due to Nash for all the subperiods t, there exists a Nash equilibrium point to the problem (P).

PROOF. At $t=T-1$, $V^i[z_T(z_{T-1}, u_{T-1}), T] = (1+r)^{-T} S^i(z_T)$. Since u_{T-1}^* is a Nash equilibrium to $(SP_{i,T-1})$ for a given feasible z_{T-1},

$$(1+r)^{-(T-1)} f^i(z_{T-1}, u_{T-1}^*) + V^i[z_T(z_{T-1}, u_{T-1}^*), T] \geq W^i(z_{T-1}, u_{T-1})$$

where $u_{T-1} = (u_{i,T-1}, u^*_{-i,T-1})$, i.e., any admissible strategy for player i while the other player's Nash equilibrium policy remains fixed. From now on, we denote u_t for $(u_{it}, u^*_{-i,t})$. At any time,

$$\begin{aligned}
&(1+r)^{-t} f^i(z_t, u^*_t) + V^i[z_{t+1}(z_t, u^*_t), t+1] \\
&\geq (1+r)^{-t} f^i(z_t, u_t) + V^i[z_{t+1}(z_t, u_t), t+1] \\
&\geq (1+r)^{-t} f^i(z_t, u_t) + W^i(z_{t+1}(z_t, u_t), u_{t+1}, \ldots, u_{T-1}) \\
&= W^i(z_t, u_t, u_{t+1}, \ldots, u_{T-1}).
\end{aligned}$$

But we know that

$$\begin{aligned}
&(1+r)^{-t} f^i(z_t, u^*_t) + V^i[z_{t+1}(z_t, u^*_t), t+1] \\
&= (1+r)^{-t} f^i(z_t, u^*_t) + W^i[z_{t+1}(z_t, u^*_t), u^*_{t+1}, \ldots, u^*_{T-1}] \\
&= W^i(z_t, u^*_t, u^*_{t+1}, \ldots, u^*_{T-1}].
\end{aligned}$$

THis implies that $W^i(z_t, u^*_t, \ldots, u^*_{T-1}] \geq W^i(z_t, u_t, u_{t+1}, \ldots, u_{T-1})$. In other words, a strategy, $\{u^*_{i\tau}\}_{\tau=t}^{T-1}$, provides player i with the maximum payoff if the other player chooses a strategy of $\{u^*_{-i,\tau}\}_{\tau=t}^{T-1}$. This implies $\{u^*_\tau\}_{\tau=t}^{T-1}$, consists of a Nash equilibrium policy to (P). ∎

THEOREM 2. (SP_{it}) has a Nash equilibrium if (i) f^i and S^i are continuous in u_t, and concave in u_{it}, (ii) $g^j, j=1,2$, is linear in u_{it}, and the Hessian matrix of V^i with respect to z_{t+1} is negative semidefinite, and (iii) the functions h^i are convex in u_{it}, i=1,2, and $D_t = \{(z_t, u_t) | h^i(z_t, u_t) \geq 0\}$ is a nonempty compact convex set.

PROOF. The Kakutani Theorem is applied. Since f^i and S^i are continuous, the V^i is also a continuous mapping. The condition (ii) guarantees V^i to be concave in u_{it}. Therefore, the objective function of (SP_{it}) is continuous and concave in u_{it}. Define a point-to-set mapping $F^i(u_t)$ as

$$F^i(u_t) = \{u^*_{it} | \ u^*_{it} \text{ is optimal for } (SP_{it}) \text{ for given } u_{-i,t}\},$$

and, $F(u_t) = F^1 x F^2 = \{u^*_t\}$, where u^*_t is such that u^*_{it} maximizes (SP_{it}) for given $u_{-i,t}$, i=1,2. Since the objective functions are concave in their own control variables and D_t is convex, the set $F^i(u_t)$ is convex for any $u_t \in D_t$. To show this, suppose u^1_{it}, $u^2_{it} \in F^i(u_t)$. Consider $\tilde{u}_{it} = \beta u^1_{it} + (1-\beta) u^2_{it}$ for $0 \leq \beta \leq 1$. Then, $\tilde{u}_{it} \in D_t$ by convexity of D_t, i.e., $\tilde{u}_{it}$ is feasible. Furthermore, by convexity of the objective function, $\tilde{u}_{it}$ should be optimal. That is, $\tilde{u}_{it} \in F^i(u_t)$. Therefore, F is convex since the Cartesian product of convex sets is also convex. From the compactness of D_t and continuity of the objective functions, we can show that F is an upper hemicontinuous mapping that maps each point of the convex, compact set D_t into a closed convex subset of D_t.[1] Then, the Kakutani Theorem says that there exists a fixed point $u_t^* \in D_t$ such that $u_t^* = F(u_t^*)$.[2] By the definition of the function F, the fixed point u_t^* is a Nash equilibrium. ■

§4. Discussion

The above theorem states that much larger classes of differential games have an equilibrium. The most severe assumption is the second one. It requires that state dynamic equations be linear on his own control variables. But, the dynamic programming approach applied in the above is hardly implementable for the purpose of computation. It is very difficult to solve (SP_{it}) directly. Notice, however, the problem can be transformed into a Hamiltonian maximization problem which is easy to solve if initial conditions are given. In this way, it is possible to design a solution algorithm to problems with nonlinear constraints. The above two theorems provide a basis for such an algorithm.

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[1] Garcia and Zangwill(1981) proved upper hemicontinuity for an economic equilibrium problem. Following their work, the upper hemicontinuous property of the mapping, F, for our problem can be shown. For the proof, continuity and compactness should be assumed.

[2] See Garcia and Zangwill(1981) for the proof of the Kakutani Theorem by the path-following approach.

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