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NECESSARY CONDITIONS IN THE OPTIMAL CONTROL OF NONLINEAR INTEGRAL EQUATIONS

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A Class of nonlinear distrubuted parameter control problems is first stated in a partial differential equation form in multi-index notaion and then converted into an integral equation form. Necessary conditions for optimality in the form of maximum principle are then derived in Sobolev space $W^{\ell,p}(1 \le p \le \infty)$.

1. Introduction

Derivations of optimal control theory and algorithms in the past for lumped as well as distributed parameter systems have been mostly based on ordinary or partial differential equations rather than integral equations (for example, [1-4]). In this paper, we present how an optimal control theory in the form of maximum principles based on nonlinear integral equations can be rigorously derived in Sobolev space $W^{\ell,p}(1 \le p \le \infty)$. The control problem is first stated in a nonlinear partial differential equation form in multi-index notation and then converted into an integral equation by means of Green's function technique. Then the necessary conditions for optimality in the form of maximum principles are then derived. Techniques for nonlinear differential equations in Sobolev space [5] are used in treating the equations appearing in the course of treatment.

2. Statement of the Problem

We consider a control system described by a nonlinear partial differential equation of the form

$$\frac{\partial v}{\partial t} = L\,v + N\,v + S_1 \tag{1}$$

subject to initial and boundary conditions

$$\text{I.C.} \qquad v(x,0)=v_o(x) \ (=\text{specified}) \tag{2}$$

$$\text{B.C.} \quad A_i(\delta_{\bar{\ell}} v;\partial x_i,t)=C_i(\delta_k v,u_{3i};u_2(t);\partial x_i,t) \tag{3}$$

where L is a linear partial operator with order ℓ and N is a nonlinear operator with order 2k ($k<\ell$) given by the multi-index notation

$$L\,v = \sum_{|\beta|\le \ell} (-1)^{|\beta|} D^{\beta} v \tag{4}$$

$$N\,v = \sum_{|\alpha|\le k} (-1)^{|\alpha|} D^{\alpha} a_{\alpha}(x,t;\delta_k v,u_1(x,t),u_2(t)) \tag{5}$$

Here $x = x(x_1,x_2,\ldots,x_N) \in R^N$ is the spatial coordinate vector and $t \in [0,t_f] \equiv T$ is time; $v \in R$ the state variable, $u_1(x,t)$ the domain control, $u_2(t)$ time dependent control, $u_{3i}(\partial x_i,t)$ $(i=1,2,\ldots,s)$ the boundary control, ∂x_i a point on the boundary $\partial\Omega_i$ and $\partial\Omega = \bigcup_{i=1}^{s} \partial\Omega_i$, $\bar{\ell} = \max(\ell-1, 2k-1)$, $|\alpha| \le k$, $|\beta| \le \ell$ where $\alpha = (\alpha_1, \alpha_2,\ldots, \alpha_N)$ $\beta = (\beta_1, \beta_2,\ldots, \beta_N)$ with $|\alpha| = \sum_{j=1}^{N}\alpha_j$, $|\beta| = \sum_{j=1}^{N}\beta_j$. S_1 is an algebraic function of x and t satisfying a condition imposed later.

The optimal control problem can be stated: Find controls $u_1(x,t) \in U_1$, $u_2(t) \in U_2$, $u_{3i}(\partial x_i,t) \in U_{3i}$ $(i=1,2,\ldots,s)$ that minimize the objective function

$$J_1 = \int_0^{t_f}\int_{\Omega} K(v,u_1;x,t)\,dx\,dt + \int_{\Omega} R(v;x,t_f)\,dx + \sum_{i=1}^{s}\int_0^{t_f}\int_{\partial\Omega_i} F_i(v,u_{3i},u_2(t),\partial x_i,t)\,d\partial x_i\,dt + \int_0^{t_f} V(u_2,t)\,dt \tag{6}$$

where U_1, U_2, U_{3i} are admissible control sets and K, R, F_i $(i=1,2,\ldots,s)$, V are scalar functions of respective arguments. In order to recast this optimal control problem into an integral equation formulation, certain conditions must be met. Further additional conditions must be met to guarantee the existence of necessary conditions for optimality in the form of maximum principles in various Sobolev Spaces $W^{k,p}[\Omega \times T]$ for $1 \le p \le \infty$. We state these conditions.

3. The Class of Functions and Conditions

Condition 1 Green's function $Gr(x,t;\xi,\tau)$ exists for the linear equation

$$\frac{\partial v}{\partial t} = L\,v\,, \qquad \text{B.C.} \quad A_i(\delta_{\bar{\ell}} v;\partial x_i,t) = 0 \tag{7}$$

and satisfies the conditions in $W^{k,p}$, $2 \le p \le \infty$:

$$\int_0^{t_f}\int_{\Omega} |D_{\xi}^{\varphi} D_x^{\alpha} Gr(x,t;\xi,\tau)|^p\,dx\,dt < \infty$$

$$\int_0^{t_f}\int_{\Omega} |D_{\xi}^{\varphi} D_x^{\alpha} Gr(x,t;\xi,\tau)|^p\,d\xi\,d\tau < \infty$$

$$\int_0^{t_f}\int_{\partial\Omega_i} |D_\xi^{\varphi} D_x^{\alpha} Gr\,(\partial x_i, t; \xi, \tau)|\; d\partial x_i\; dt < \infty$$

$$\int_\Omega |D_x^{\alpha} Gr\,(x,t;\xi,0)|^p\, d\xi < \infty \qquad (8)$$

$$\int_0^{t_f}\int_{\partial\Omega_i} |D_x^{\alpha} Gr\,(x,t;\partial\xi_i,\tau)|^p\, d\partial\xi_i d\tau < \infty$$

where $|\varphi|, |\alpha| \leq k$ and, in $W^{k,p}$, $1 \leq p < 2$

$$|D_\xi^{\varphi} D_x^{\alpha} Gr(x,t;\xi,\tau)| \leq M \text{ on } [\Omega x T] \times [\Omega x T] \qquad (9)$$

except perhaps at $t=\tau=0$, for a positive constant M.

Condition 2 a_α in (5) satisfies the so-called Caratheodory conditions in $W^{1,p}(p \geq 1)$ (see [5] for definition).

$$a_\alpha \in CAR(p),\ \in CAR^{*}(p),\ \in CAR^{**}(p) \qquad (10)$$

Condition 3 For S_1 in (1) satisfies

$$\int_0^{t_f}\int_\Omega Gr(x,t;\xi,\tau)\; S_1(\xi,\tau)\; d\xi\, d\tau \quad \in W^{k,p} \qquad (11)$$

for $1 \leq p \leq \infty$. This holds whenever $S_1 \in L_{p''}$ for some p'' >1 and $L_p \subset L_{p''}$. This guarantees the existence of weak solution of (1).

Condition 4 Functions in (1) - (6) have partial derivatives with respect to argument functions and satisfy Lipschitz conditions.

Condition 5 Control functions u_1, u_2, and u_{3i} belong to admissible control sets U_1, U_2 and U_{3i}, respectively.

4. Conversion to Integral Equation Formulation

Under the conditions stated above, the partial differential equation in (1) can be recast into an integral equation:

$$\begin{aligned} v(x,t) = &\sum_{|\alpha|\leq k}\int_0^{t_f}\int_\Omega a_\alpha(\xi,\tau;\delta_k v, u_1, u_2(\tau))\; D_\xi^{\alpha} G\,(x,t;\xi,\tau)\; d\xi\, d\tau \\ &+ \sum_{i=1}^{s}\int_0^{t_f}\int_\Omega \Phi_i^T D_\xi^{\alpha} Gr\,(x,t;\partial\tau_i,\tau)\; C_i(v,u_2(\tau),u_{3i}(\partial\xi_i,\tau))\; d\partial\xi_i d\tau \\ &+ \int_\Omega Gr(x,t,\xi,0)\; v_0\, d\xi + S\,(x,t) \end{aligned} \qquad (12)$$

where Φ_i is an w_i-dimensional vector function and

$$S\,(x,t) = \int_0^{t_f}\int_\Omega Gr\; S_1\,(\xi,\tau)\; d\xi\, d\tau \;\in W^{k,p} \qquad (13)$$

Then we have

$$\begin{aligned} D_x^{\varphi} v(x,t) = &\sum_{|\alpha|\leq k}\int_0^{t_f}\int_\Omega a_\alpha(\xi,\tau;\delta_k v,u_1,u_2(\tau))\; D_x^{\varphi} D_\xi^{\alpha} G(x,t;\xi,\tau)\; d\xi\, d\tau \\ &+ \sum_{i=1}^{s}\int_0^{t}\int_{\partial\Omega_i} D_x^{\varphi}[\,\Phi_i^T(D_\xi^{\alpha} G)\; C_i]\; d\partial\xi_i\; d\tau \\ &+ \int_\Omega D_x^{\varphi} Gr\; v_0(\xi)\; d\xi + D_x^{\varphi} S\,(x,t) \end{aligned} \qquad (14)$$

where $1 \leq |\varphi| \leq k$.

If we denote $\underset{\sim}{v} = [D_x^{\varphi} v]$, $\underset{\sim}{S} = [D_x^{\varphi} S]$ for $|\varphi| \leq k$, etc., (12) can be written in expanded matrix and vector notation as

$$\begin{aligned} \underset{\sim}{v}(x,t) = &\int_0^{t_f}\int_\Omega \underset{\sim}{G}(x,t;\xi,\tau)\underset{\sim}{f}(\underset{\sim}{v},\underset{\sim}{u}_1,\underset{\sim}{u}_2(\tau);\xi\tau)\; \sigma(t-\tau)\; d\xi\, d\tau \\ &+ \int_\Omega \underset{\sim}{g}\,(x,t;\xi)\; \underset{\sim}{v}_0(\xi)\; d\xi \\ &+ \sum_{i=1}^{s}\int_0^{t_f}\int_{\partial\Omega_i} \underset{\sim}{g}_i(x,t;\partial\xi_i)\; \underset{\sim}{C}_i(\underset{\sim}{v},\underset{\sim}{u}_{3i};\underset{\sim}{u}_2(t);\partial\xi_i,\tau)\; \sigma(t-\tau)\; d\xi_i\, d\tau \\ &+ \underset{\sim}{S}\,(x,t) \end{aligned} \qquad (15)$$

where $\underset{\sim}{v} \in \mathbb{R}^n$ is the state; a_α is absorbed in $\underset{\sim}{f} \in \mathbb{R}^k$; $\underset{\sim}{G}$ is an n x k kernel matrix containing $D_x^{\gamma} D_\xi^{\alpha} Gr$ for $|\gamma|, |\alpha| \leq k$ as elements; $\underset{\sim}{g}\,(x,t;\xi) \in \mathbb{R}^n$ is a vector containing D_x^{α} Gr $(x,t;\xi,0)$; $\underset{\sim}{C}_i \in \mathbb{R}^{w_i}$ accounts for the boundary conditions; $\underset{\sim}{g}_i$ is an n x w kernel matrix containing $D_x^{\varphi} D_\xi^{\alpha} Gr\,(x,t;\partial\xi_i,0)$; and $\sigma(t-\tau)$ is the Heaviside unit step function.

5. Hamiltonian Functions and Costate Equations

We now maximize $(-J_1)$ in (6) subject to the converted nonlinear integral equation (15). We let $J = (-J_1)$. Then the augmented objective function $\bar{J}$ to be maximized is

$$\begin{aligned} \bar{J} = &\int_0^{t_f}\int_\Omega [\,-K(\underset{\sim}{v},\underset{\sim}{u}_1;\, x,t) + \underset{\sim}{\lambda}^T(x,t)\underset{\sim}{v} \\ &-\underset{\sim}{f}^T(\underset{\sim}{v},\underset{\sim}{u}_1,\underset{\sim}{u}_2(t);\, x,t)\underset{\sim}{Q}(x,t) - \underset{\sim}{\lambda}^T\underset{\sim}{S}(x,t)]dxdt \\ &+\sum_{i=1}^{s}\int_0^{t_f}\int_{\partial\Omega_i} [-F_i(\partial x_i,t) + \underset{\sim}{\psi}_i^T\underset{\sim}{v}(\partial x_i,t) \\ &- \underset{\sim}{C}_i^T(\underset{\sim}{v},\underset{\sim}{u}_{3i},\underset{\sim}{u}_2(t);\partial x_i,t)\,\underset{\sim}{\Phi}_i(\partial x_i,t) \\ &- \underset{\sim}{\psi}_i^T\underset{\sim}{S}(\partial x_i,t)]d\partial x_i\; dt \\ &+ \int_0^{t_f}[-V(t)]dt \end{aligned} \qquad (16)$$

where

$$\begin{aligned} \underset{\sim}{Q} = &\int_0^{t_f}\int_\Omega [\,\underset{\sim}{\lambda}^T(\xi,\tau)\; \underset{\sim}{G}\,(\xi,\tau;x,t)]^T\; \sigma(\tau-t)d\xi d\tau \\ &+ \int_\Omega [\,\underset{\sim}{\lambda}_f^T(\xi)\; \underset{\sim}{G}\,(\xi,t_f;x,t)]^T\; d\xi \\ &+ \sum_{i=1}^{s}\int_0^{t_f}\int_{\partial\Omega_i} [\,\underset{\sim}{\psi}_i^T(\partial\xi_i,\tau)\; \underset{\sim}{G}(\partial\xi_i,\tau;x,t)]^T\; \sigma(\tau-t)d\partial\xi_i\, d\tau \end{aligned} \qquad (17)$$

$$\begin{aligned} \underset{\sim}{\Phi}_i = &\int_0^{t_f}\int_\Omega [\underset{\sim}{\lambda}^T(\xi,\tau)\; \underset{\sim}{g}_i(\xi,\tau;\partial x_i,t)]^T\; \sigma(\tau-t)d\xi d\tau \\ &+ \int_\Omega [\,\underset{\sim}{\lambda}_f^T(\xi)\; \underset{\sim}{g}_i(\xi,t_f;\partial x_i,t)]^T\; d\xi \end{aligned}$$

$$+\sum_{j=1}^{s}\int_0^{t_f}\int_{\partial\Omega_j}[\psi_j^T(\partial\xi_j,\tau)\, g_i(\partial\xi_j,\tau;\partial x_i,t)]^T \sigma(\tau-t)\, d\partial\xi_j\, d\tau \quad (18)$$

and λ, λ_f, ψ_i are costate vectors.

For the perturbed changes Δu_1, Δu_2 and Δu_{3i}, the increment of the augmented objective function is

$$\Delta\bar{J} = \int_0^{t_f}\int_{\Omega}\{\Delta v^T(x,t)[-\frac{\partial K}{\partial v} + \lambda - \frac{\partial f^T}{\partial v}Q]$$

$$-\Delta_{u_1}K - \Delta_{u_1}f^T Q - \Delta_{u_2}f^T Q\}\, dx\, dt$$

$$+ \int_{\Omega}\Delta v^T(x,t_f)[-\frac{\partial R}{\partial v} + \lambda_f]\, dx$$

$$+\sum_{i=1}^{s}\int_0^{t_f}\int_{\partial\Omega_i}\{\Delta v^T(\partial x_i,t)[-\frac{\partial F_i}{\partial v} + \psi_i$$

$$-\frac{\partial C_i^T}{\partial v}\Phi] - \Delta_{u_{3i}}F_i - \Delta_{u_{3i}}C_i^T\Phi - \Delta_{u_2}C_i^T\Phi\}\, d\partial x_i\, dt$$

$$+ \int_0^{t_f} -\Delta_{u_2}V\, dt + \eta \quad (19)$$

where η is the remainder term and $\Delta_u B$ is defined as $\Delta_u B = B(u+\Delta u,.,.) - B(u,.,.)$. In (16), we define the domain Hamiltonian function as

$$H(u_1, u_2(t);x,t) = -K(u_1;x,t) - f^T(u_1, u_2;x,t)Q \quad (20)$$

We also define the boundary Hamiltonian functions as

$$h_i(u_{3i},u_2(t);\partial x_i,t) = -F_i(u_{3i};\partial x_i,t) - C_i^T(u_{3i},u_2;\partial x_i,t)\Phi_i \quad (21)$$

and the time dependent Hamiltonian function as

$$H_T(u_2;t) = -V(u_2) + \int_{\Omega} -f^T(u_1,u_2)Q\, dx$$

$$+\sum_{i=1}^{s}\int_{\partial\Omega_i} -C_i^T(u_{3i},u_2)\Phi_i\, d\partial x_i \quad (22)$$

The domain costate equation is defined as

$$\lambda(x,t) = \frac{\partial K}{\partial v} + \frac{\partial f^T}{\partial v}Q \quad (\equiv -\frac{\partial H}{\partial v}) \quad (23)$$

the boundary costate functions as

$$\psi_i(\partial x_i,t) = \frac{\partial F_i}{\partial v(\partial x_i,t)} + \frac{\partial C_i^T}{\partial v(\partial x_i,t)}\Phi_i$$

$$(\equiv -\frac{\partial h_i}{\partial v(\partial x_i,t)}) \quad (i = 1,2,\dots,s) \quad (24)$$

and the final-time costate function as

$$\lambda_f(x,t_f) = \frac{\partial R}{\partial v(x,t_f)} \quad (25)$$

Then (19) becomes

$$\Delta\bar{J} = \int_0^{t_f}\int_{\Omega}\Delta_{u_1}H\, dx\, dt$$

$$+\sum_{i=1}^{s}\int_0^{t_f}\int_{\partial\Omega_i}\Delta_{u_{3i}}h_i\, d\partial x_i\, dt$$

$$+ \int_0^{t_f}\Delta_{u_2}H_T dt + \eta \quad (26)$$

The remainder term η can be estimated for a change in u_1, u_2, and u_{3i}, respectively, over $\Delta\Omega$, Δt and $\Delta\partial\Omega_i$. It turns out to be

$$|\eta| \le \begin{cases} k\,[(\Delta\Omega \times \Delta t)^r + (\Delta\Omega \times \Delta t)^d] & \text{for a change in } u_1 \\ k\,[(\Delta t)^r + (\Delta t)^d] & \text{for a change in } u_2 \\ k\,[(\Delta\partial\Omega_i \times \Delta t)^r + (\Delta\partial\Omega_i \times \Delta t)^d] & \text{for a chang in } u_{3i} \end{cases} \quad (27)$$

where

$$\begin{cases} r = 2 & \text{for } 1 \le p < 2 \\ r = 2 + \frac{p}{2} - \frac{2}{p} & \text{for } 2 \le p \le \infty \end{cases} \quad (28)$$

$$\begin{cases} d = 2 & \text{for } 1 \le p < 2 \\ d = 2 - \frac{1}{p} & \text{for } 2 \le p \le \infty \end{cases} \quad (29)$$

and k is a constant.

6. Necessary Conditions for Optimality (Maximum Principles)

Once the derivation and the estimate of η available, we can now state the maximum principles for optimality for the nonlinear integral equation (15). The proof is straightfoward from (26)-(29) and it is omitted here.

Theorem 1. (Maximum principle for domain control u_1)

If $u_1^*(x,t)$ minimizes J_1 for given $u_2(t)$ and u_{3i} ($i = 1,2,\dots s$), then $H(x,t)$ must attain its absolute maximum with respect to $u_1(x,t)$ at $u_1^*(x,t)$ almost everywhere (a.e.) on $\Omega \times T$.

Theorem 2. (Maximum principle for boundary controls u_{3i})

If $u_{3i}^*(\partial x_i,t)$ minimizes J_1 for given $u_1(x,t)$, $u_2(t)$ and $u_{3j}(\partial x_j,t)$ $(j \ne i)$, then $h_i(\partial x_i,t)$ must attain its absolute maximum with respect to u_{3i} at u_{3i}^* a.e. on $\partial\Omega_i \times T$.

Theorem 3. (Maximum principle for spatially independent boundary controls u_{3i})

In Theorem 2, if $u_{3i}(\partial x_i,t)$ is spatially independent, i.e., it appears on the whole boundary uniformly as $u_4(t)$, then

$\sum_{i=1}^{s}\int_{\partial\Omega_i} h_i(\partial x_i,t)d\partial x_i$ must attain its absolute maximum with respect to $\underset{\sim}{u}_4(t)$ at $\underset{\sim}{u}_4^*(t)$ a.e. on T.

Theorem 4. (Maximum principle for time dependent control $\underset{\sim}{u}_2(t)$)

If the time dependent control $\underset{\sim}{u}_2^*(t)$ minimizes J_1 for given $\underset{\sim}{u}_1(x,t)$ and $\underset{\sim}{u}_{3i}(i = 1,2,\ldots, s)$, then H_T must attain its absolute maximum with respect to $\underset{\sim}{u}_2(t)$ at $\underset{\sim}{u}_2^*(t)$ a.e. on T.

7. Illustrative Example

We consider the following tubular reactor problem: The equation given is

$$\frac{\partial v}{\partial t} = a\frac{\partial^2 v}{\partial x^2} - b\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial x}\right) + u_1(x,t) \quad (30)$$

I.C. $v(x,0) = v_o(x)$

B.C.
$$\begin{cases} \left.\dfrac{\partial v}{\partial x}\right|_{x=0} = C(u_2, v(0,t),t) \\ \left.\dfrac{\partial v}{\partial x}\right|_{x=1} = 0 \end{cases}$$

$0 \le x \le x_f,\ 0 \le t \le t_f,\ m_1 \le u_1 \le M_1,\ m_2 \le u_2 \le M_2.$

The minimizing objective function is

$$J_1 = \frac{1}{2}\int_0^{t_f}\int_0^{x_f} \alpha_1 u_1^2(x,t)\,dx\,dt + \frac{1}{2}\int_0^{x_f}[v(x,t_f) - v_d(x)]^2 dx + \frac{1}{2}\int_0^{t_f}\alpha_2 u_2^2(t)\,dt \quad (31)$$

Equation (30) can be converted into the following integral equation:

$$v(x,t) = \int_0^{t_f}\int_0^{x_f} Gr\,(t-\tau,x;\xi)\left[-b\frac{\partial}{\partial\xi}\left(v\frac{\partial v}{\partial\xi}\right) + u_1(\xi,\tau)\right]\sigma(t-\tau)\,d\xi\,d\tau - a\int_0^{t_f} Gr\,(t-\tau,x;0)\,C(u_2,v(0,t))\,\sigma(t-\tau)\,d\tau + \int_0^{x_f} Gr\,(t,x;\xi)\,v_o(\xi)\,d\xi \quad (32)$$

where the Green's function is given by

$$Gr\,(t-\tau,x;\xi) = \sum_{i=1}^{\infty}\exp\,[\beta_i(t-\tau)]\,\varphi_i(x)\,\varphi_i(\xi)$$
$$\varphi_1(x) = 1 \quad (33)$$
$$\varphi_i(x) = \sqrt{2}\cos[(i-1)\pi x] \quad (i = 2,3,\ \ldots)$$
$$\beta_i = -a\,(i-1)^2\,\pi^2 \quad (i = 1,2,\ \ldots)$$

We let $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v \\ \frac{\partial v}{\partial x} \end{bmatrix}$ and integrating by parts to obtain

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \int_0^{t_f}\int_0^{x_f} \underset{\sim}{G}\,(t-\tau,x;\xi)\begin{bmatrix} u_1(\xi,\tau) \\ bv_1 v_2(\xi,\tau)\end{bmatrix}\sigma(t-\tau)\,d\xi\,d\tau + \int_0^{t_f}\underset{\sim}{g}(t-\tau,x;0)\,[-C(u_2,v_1(0,\tau),\tau) - bv_1(0,\tau)\,C(u_2,v_1(0,\tau))]\,\sigma(t-\tau)\,d\tau + \int_0^{x_f}\underset{\sim}{g}(t-\tau,x;\xi)\,v_o(\xi)\,d\xi \quad (34)$$

where

$$\underset{\sim}{G}\,(t-\tau,x;\xi) = \begin{bmatrix} Gr & \dfrac{\partial Gr}{\partial\xi} \\ \dfrac{\partial Gr}{\partial x} & \dfrac{\partial^2 Gr}{\partial x\,\partial\xi}\end{bmatrix} \quad (35)$$

$$\underset{\sim}{g}\,(t-\tau,x;\xi) = \begin{bmatrix} Gr \\ \dfrac{\partial Gr}{\partial x}\end{bmatrix} \quad (36)$$

The domain Hamiltonian function is

$$H = -\frac{\alpha_1}{2}u_1^2(x,t) - \begin{bmatrix} u_1(x,t) \\ bv_1 v_2(x,t)\end{bmatrix}^T \underset{\sim}{Q} \quad (37)$$

where

$$\underset{\sim}{Q} = \int_0^{t_f}\int_0^{x_f}\underset{\sim}{G}^T(\tau-t,\xi;x)\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix}\sigma(\tau-t)\,d\xi\,d\tau + \int_0^{x_f}\underset{\sim}{G}^T(t_f-t,\xi;x)\begin{bmatrix}\lambda_{1f} \\ \lambda_{2f}\end{bmatrix}d\xi + \int_0^{t_f}\underset{\sim}{G}^T(\tau-t,0;x)\left.\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}\right|_{\xi=0}\sigma(\tau-t)\,d\tau \quad (38)$$

The boundary Hamiltonian function is

$$h = -\frac{\alpha_2}{2}u_2^2(t) + [bv_1(0,t)+a]\,C(u_2,v(0,t),t)\,\Psi \quad (39)$$

where

$$\Psi = \int_0^{t_f}\int_0^{x_f}\underset{\sim}{g}^T(\tau-t,\xi;0)\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix}\sigma(\tau-t)\,d\xi\,d\tau + \int_0^{x_f}\underset{\sim}{g}^T(t_f-t,\xi;0)\begin{bmatrix}\lambda_{1f} \\ \lambda_{2f}\end{bmatrix}d\xi + \int_0^{t_f}\underset{\sim}{g}^T(\tau-t,0;0)\left.\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}\right|_{\xi=0}\sigma(\tau-t)\,d\tau \quad (40)$$

The costate equations are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & bv_2(x,t) \\ 0 & bv_1(x,t) \end{bmatrix} Q \qquad (41)$$

$$\begin{bmatrix} \lambda_{1f} \\ \lambda_{2f} \end{bmatrix} = \begin{bmatrix} v(x,t_f)-v_d(x) \\ 0 \end{bmatrix} \qquad (42)$$

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial}{\partial v_1(0,t)}\{[bv_1+a]C(u_2,v,t)\} \\ 0 \end{bmatrix} \Psi \qquad (43)$$

These are the equations to be used to find the optimal control by the use of the theorems of maximum principles derived above. For example, if we have $\alpha_1 = \alpha_2 = b = 0$, $u_1 = 0$, and $C = -\rho[u_2^4(t) - v^4(0,t)]$ then the objective function is

$$J = \frac{1}{2}\int_0^{x_f}[v(x,t_f) - v_d(x)]^2 \, dx \qquad (44)$$

The boundary Hamiltonian function is

$$h = -a\rho[u_2^4(t) - v^4(0,t)]\{\int_0^{x_f} Gr(\tau - t, \xi; 0)[v(\xi, t_f) - v_d(\xi)]d\xi$$
$$+ \int_0^{t_f} Gr(\tau - t, 0; 0)\psi_1 \sigma(\tau - t)d\tau\} \qquad (45)$$

and the costate equation becomes

$$\psi_1 = -4a\rho v^3(0,t)\{\int_0^{x_f} Gr(\tau - t, \xi; 0)[v(\xi, t_f) - v_d(\xi)]d\xi$$
$$+ \int_0^{t_f} Gr(\tau - t, 0; 0)\psi_1 \sigma(\tau - t)d\tau\} \qquad (46)$$

Now the maximum principle (Theorem 4) can be applied to obtain the optimal control $u_2(t)$, i.e., we seek $u_2(t)$ that maximizes the boundary Hamiltonian function $h(t)$. Some numerical results will be presented at the conference.

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