A Formal Linearization of Nonlinear Systems based on the Trigonometric Fourier Expansion

Hitoshi Takata*, Kazuo Komatsu**

*Department of Electrical, Electronic and Computer Engineering,
Kyushu Institute of Technology, Kitakyushu 804, Japan
**Department of Information and Computer Sciences,
Kumamoto National College of Technology, Kumamoto 861-11, Japan

Abstract

Most of systems are included nonlinear characteristics in practice. One might be faced with difficulties when problems of nonlinear systems are solved. In this paper we present a formal linearization method of nonlinear systems by using the trigonometric Fourier expansion on the state space considering easy inversion. An error bound, an application, and a compensation of this method are also investigated.

1. Introduction

When systems are nonlinear, we should need linearization for adapting the linear system theory. There are several methods of linearization though they are not for numerical computer methods[1]~[3]. It is significant in practice to linearize nonlinear systems with high accuracy by computers.

In this paper we present a formal linearization method of nonlinear systems based on the trigonometric Fourier expansion on the state space considering easy inversion.

A nonlinear system $\dot{x}(t)=Ax(t)+f(x(t))$ ($\cdot=d/dt$) is considered. Define that linearization function $\phi(x)$ which is comprised of sequence of trigonometric functions { x, $\sin(rx)$, $\cos(rx)$: $r=1,2,\cdots,N$ }. $\dot{\phi}(x)$ is expanded in Fourier series so that $\dot{\phi}(x(t))=B\phi(x)+C$ is acquired. Thus the given nonlinear system of $x(t)$ is transformed into the linear system of $\phi(x)$. The inversion is easily carried out by $x(t)=[1\ 0\ 0\ \cdots\ 0]\ \phi(x(t))$. The effectiveness of the method is confirmed through numerical examples.

Moreover we investigate an error bound of this linearization. A nonlinear observer is synthesized as an application of the method. A compensation of the linearization is also studied.

2. A Formal Linearization

We consider a scalar system, for a vector system is straightforward. Assume that a nonlinear system is given as

$$\Sigma_1: \dot{x}(t)=Ax(t)+f(x(t)) \quad (\cdot=d/dt) \qquad (2.1)$$
$$x(0)=x_0\in[0,2\ell]\subset R$$

where x is a state variable defined on $[0,2\ell]$, R is the set of all real-valued, f is a nonlinear square integrable function with the first continuous derivative.

We here define a formal linearization function

$$\phi(x)=[x\quad \sin\frac{\pi}{\ell}x\quad \cos\frac{\pi}{\ell}x\quad \sin\frac{2\pi}{\ell}x\quad \cos\frac{2\pi}{\ell}x\ \cdots\ \sin\frac{N\pi}{\ell}x\quad \cos\frac{N\pi}{\ell}x]^T$$
$$=[x\quad \phi_1(x)\quad \phi_2(x)\quad \phi_3(x)\quad \phi_4(x)\ \cdots\ \phi_{2N-1}(x)\quad \phi_{2N}(x)]^T. \qquad (2.2)$$

It is called the Nth-order linearization function. In this case the inverse transformation ϕ^{-1} is easily carried out by

$$x=[1\ 0\ 0\ \cdots\ 0]\ \phi(x). \qquad (2.3)$$

We transform the system Σ_1 into a linear system:

$$\Sigma_2: \dot{\phi}(x)=B\phi(x)+C \qquad (2.4)$$
$$\phi(x(0))=\phi(x_0)$$

as follows.

From Eqs.(2.1) and (2.2),

$$f(x)=g_0(x) \qquad (2.5)$$

$$\dot{\phi}_{2r-1}(x)=\frac{d}{dt}\sin\frac{r\pi}{\ell}x=(\frac{d}{dx}\sin\frac{r\pi}{\ell}x)\dot{x}$$
$$=\frac{r\pi}{\ell}(\cos\frac{r\pi}{\ell}x)(Ax+f(x))=g_{2r-1}(x) \qquad (2.6)$$

$$\dot{\phi}_{2r}(x)=\frac{d}{dt}\cos\frac{r\pi}{\ell}x=(\frac{d}{dx}\cos\frac{r\pi}{\ell}x)\dot{x}$$
$$=-\frac{r\pi}{\ell}(\sin\frac{r\pi}{\ell}x)(Ax+f(x))=g_{2r}(x). \qquad (2.7)$$

Expand each g_r ($r=0,1,2,\cdots,2N$) in Fourier series:

$$g_r(x)=\sum_{k=1}^{\infty}\alpha_{rk}\phi_k(x)+\alpha_{r0} \qquad (2.8)$$

where

$$\alpha_{r0}=\frac{1}{2\ell}\int_0^{2\ell}g_r(x)dx \qquad (2.9)$$

$$\alpha_{rk}=\frac{1}{\ell}\int_0^{2\ell}g_r(x)\phi_k(x)dx. \qquad (2.10)$$

By truncation at $r=2N$, we have the N-th order system with respect to ϕ

$$\dot{\phi}(x)=\begin{bmatrix}\dot{x}\\ \frac{d}{dt}\sin\frac{\pi}{\ell}x\\ \frac{d}{dt}\cos\frac{\pi}{\ell}x\\ \vdots\\ \frac{d}{dt}\sin\frac{N\pi}{\ell}x\\ \frac{d}{dt}\cos\frac{N\pi}{\ell}x\end{bmatrix}=\begin{bmatrix}Ax+\sum_{k=1}^{2N}\alpha_{0k}\phi_k(x)+\alpha_{00}\\ \sum_{k=1}^{2N}\alpha_{1k}\phi_k(x)+\alpha_{10}\\ \sum_{k=1}^{2N}\alpha_{2k}\phi_k(x)+\alpha_{20}\\ \vdots\\ \sum_{k=1}^{2N}\alpha_{2N-1\,k}\phi_k(x)+\alpha_{2N-1\,0}\\ \sum_{k=1}^{2N}\alpha_{2N\,k}\phi_k(x)+\alpha_{2N\,0}\end{bmatrix}$$
$$=B\phi(x)+C \qquad (2.11)$$

where

$$B=\begin{bmatrix}A & \alpha_{01} & \cdots & \alpha_{0\,2N}\\ 0 & \alpha_{11} & \cdots & \alpha_{1\,2N}\\ \vdots & \vdots & & \vdots\\ 0 & \alpha_{2N\,1} & \cdots & \alpha_{2N\,2N}\end{bmatrix} \quad C=\begin{bmatrix}\alpha_{00}\\ \alpha_{10}\\ \vdots\\ \alpha_{2N\,0}\end{bmatrix}.$$

Thus the linear system Eq.(2.11), namely Eq.(2.4), is obtained.

In the next, we study the error trajectory by this truncation.

3. Error Bound

Rewrite the approximated equation of (2.11) as

$$\dot{\hat{\phi}}(x)=B\hat{\phi}(x)+C. \qquad (3.1)$$

In case of no truncation, it follows that

$$\dot{\phi}(x)=B\phi(x)+C+R_{N+1}(x) \qquad (3.2)$$

where

$$R_{N+1}(x)=\sum_{k=2N+1}^{\infty}\alpha_k\phi_k(x) \qquad (3.3)$$

$$\alpha_k=[\alpha_{0\,k}\ \alpha_{1\,k}\ \cdots\ \alpha_{2N\,k}]'.$$

From Eqs.(3.1) and (3.2), the dynamics of the error $(\phi-\hat{\phi})$ is

$$\frac{d}{dt}(\phi(x)-\hat{\phi}(x))=B(\phi(x)-\hat{\phi}(x))+R_{N+1}(x). \quad (3.4)$$

Integration yields

$$x(t)-\hat{x}(t)=[1\ 0\ \cdots\ 0]\int_0^t e^{B(t-\tau)}R_{N+1}(x)d\tau \qquad (3.5)$$

with the initial condition $\phi(x_0)=\hat{\phi}(x_0)$.
Thus the error bound becomes

$$|x(t)-\hat{x}(t)|$$

$$\leq \|[1\ 0\ \cdots\ 0]\|\int_0^t e^{\|B\|(t-\tau)}\|R_{N+1}(x)\|d\tau$$

$$=\frac{\|R_{N+1}\|}{\|B\|}(e^{\|B\|t}-1) \qquad (3.6)$$

$$\fallingdotseq \|R_{N+1}\|t. \qquad (t\fallingdotseq 0) \qquad (3.7)$$

4. Numerical Examples

We illustrate the use of the method. Given a nonlinear system

$$\Sigma_1: \dot{x}(t)=-x(t)+x^2(t) \qquad (4.1)$$

$$x(0)=x_0\in[0,2\ell]\subset R.$$

From Eq.(2.11), when the order of ϕ is N=2, the coefficients of the linearization are as follows. The other coefficients are similarly obtained.

$$B=\begin{pmatrix} -1 & -\frac{4\ell^2}{\pi} & \frac{4\ell^2}{\pi^2}-\frac{4\ell^2}{\pi} & -\frac{2\ell^2}{\pi} & \frac{\ell^2}{\pi^2} \\ 0 & \frac{1}{2}-\ell & -\pi+\frac{4}{3}\pi\ell+\frac{\ell}{2\pi} & \frac{4}{3}-\frac{8}{3}\ell & \frac{20\ell}{9\pi} \\ 0 & \frac{\ell}{2\pi}+1-\frac{4}{3}\pi\ell & -\frac{1}{2}+\ell & -\frac{16\ell}{9\pi} & \frac{2}{3}-\frac{4}{3}\ell \\ 0 & -\frac{4}{3}+\frac{8}{3}\ell & \frac{9\pi}{40\ell} & \frac{1}{2}-\ell & \frac{\ell}{4\pi}-2\pi+\frac{8}{3}\pi\ell \\ 0 & -\frac{32\ell}{9\pi} & -\frac{8}{3}+\frac{16}{3}\ell & \frac{\ell}{4\pi}+2\pi-\frac{8}{3}\pi\ell & -\frac{1}{2}+\ell \end{pmatrix}$$

$$C=\begin{pmatrix} \frac{4}{3}\ell^2 \\ \frac{2\ell}{\pi} \\ -1+2\ell \\ \frac{\ell}{\pi} \\ -1+2\ell \end{pmatrix} \qquad \phi(x)=\begin{pmatrix} x \\ \sin\frac{\pi}{\ell}x \\ \cos\frac{\pi}{\ell}x \\ \sin\frac{2\pi}{\ell}x \\ \cos\frac{2\pi}{\ell}x \end{pmatrix} \qquad (4.2)$$

Fig.1 is the numerical results in case of $x_0=0.8$ and $\ell=0.42$ by computer. $\hat{x}(t)$ is the trajectory by the linearization of Eq.(2.11) when N is parameter (N=1∼5). x(t) is the true value which is the trajectory of the original equation (4.1). Fig.2 is the integration of the square error

$$J(t)=\int_0^t(x(\tau)-\hat{x}(\tau))^2d\tau \qquad (4.3)$$

For comparison, Fig.3 shows the trajectory $\hat{x}(t)$ by the linearization method based on Taylor expansion in which $\phi(x)=[x,x^2,\cdots,x^N]'$ [4][5]. Fig.4 is J(t) of Eq.(4.3) in case of Fig.3.

Consider another nonlinear system

$$\Sigma_1: \dot{x}(t)=2\sin(1.2x(t))$$

$$(A=0,\ x_0=0.3,\ \ell=1.28).$$

Fig.5 is the numerical results of x(t) and $\hat{x}(t)$ when N=1∼5. Fig.6 is J(t) in this case.

These results show that the accuracy of our linearization is improved as N increases and is effective in wider region than by Taylor expansion.

5. Observer

As an application of the linearization, we synthesize a nonlinear observer.
Assume that a nonlinear system with measurement is given:
Dynamical equation is the same as Eq.(2.1)

$$\dot{x}(t)=Ax(t)+f(x(t)) \quad (x(0)\in[0,2\ell]). \qquad (5.1)$$

Measurement equation is

$$y(t)=Hx(t)+h(x(t)) \qquad (5.2)$$

where y is real-valued measurement datum, and h is a nonlinear square integrable function. f(x) is expanded in Fourier series by the way of Section 2 so that the linear equation of (2.4)

$$\dot{\phi}(x)=B\phi(x)+C \qquad (5.3)$$

is derived. In a similar way, h(x) is also expanded in Fourier series so that we have

$$y(t)=D\phi(x)+e \qquad (5.4)$$

where

$$D=[H,\beta_1,\cdots,\beta_k,\cdots,\beta_{2N}]$$

$$e=\frac{1}{2\ell}\int_0^{2\ell}h(x)dx$$

$$\beta_k=\frac{1}{\ell}\int_0^{2\ell}h(x)\phi_k(x)dx.$$

The well-known linear observer theory[6] is applied to the linear system of Eqs.(5.3) and (5.4). Identity observer, for example, is

$$\dot{\hat{\hat{\phi}}}(t)=B\hat{\hat{\phi}}(t)+C+K(y(t)-D\hat{\hat{\phi}}(t)-e)$$

$$\hat{\hat{x}}(t)=[1\ 0\ 0\ \cdots\ 0]\hat{\hat{\phi}}(t) \quad (x(0)\in[0,2\ell]).$$

K is appropriately chosen so that all eigenvalues of the matrix (B-KD) have negative real parts.

6. Compensation

The linearization of Section 2 is an approximation. As shown in Section 3, the approximation error may diverge as t→∞. We here propose a compensation approach of this linearization. In this compensation, the linear equation is the same as Eq.(2.11) at t=0 but $\dot{x}=0$, at t=∞.
Assume that the system is described by

$$\dot{x}(t)=A(x(t)-w)+f(x(t)) \qquad (6.1)$$

where w is a steady state value:

$$\dot{x}(\infty)=f(w)=0$$

From Eq.(2.11) we have

$$\dot{\phi}(x)=B\phi(x)+C \qquad (6.2)$$

where

$$B=\begin{pmatrix} A & \alpha_{0\,1} & \cdots & \alpha_{0\,2N} \\ 0 & \alpha_{1\,1} & \cdots & \alpha_{1\,2N} \\ \vdots & \vdots & & \vdots \\ 0 & \alpha_{2N\,1} & \cdots & \alpha_{2N\,2N} \end{pmatrix} \qquad C=\begin{pmatrix} \alpha_{0\,0}-Aw \\ \alpha_{1\,0} \\ \vdots \\ \alpha_{2N\,0} \end{pmatrix}$$

Define that

$$\psi(x)=[\phi_1(x)\ \phi_2(x)\ \phi_3(x)\ \phi_4(x)\ \cdots\ \phi_{2N-1}(x)\ \phi_{2N}(x)]^T$$

$$\dot{\psi}(x)=\begin{bmatrix}\alpha_{1\,1} & \cdots & \alpha_{1\,2N}\\ \vdots & & \vdots\\ \alpha_{2N\,1} & \cdots & \alpha_{2N\,2N}\end{bmatrix}\psi(x)+\begin{bmatrix}\alpha_{1\,0}\\ \vdots\\ \alpha_{2N\,0}\end{bmatrix}$$

$$=\bar{B}\psi(x)+\bar{C}$$

$$\bar{B}'=\bar{B}-\varepsilon I \qquad (\varepsilon \geqq 0,\ I\text{:identity matrix}) \qquad (6.3)$$

Choose ε so that all eigenvalues of $\bar{B}'$ have negative real parts. An approximation of Eq.(6.3)

$$\dot{\psi}(x)=\bar{B}'\psi(x)+\bar{C} \qquad (6.4)$$

goes to $-\bar{B}'^{-1}\bar{C}$ at $t=\infty$. As $t\rightarrow\infty$,

$$\dot{x}=A(x-w)+\sum_{k=1}^{2N}\alpha_{0\,k}\phi_k(x)+\alpha_{0\,0} \qquad (6.5)$$

approaches

$$\eta=[\alpha_{0\,1}\ \cdots\ \alpha_{0\,2N}](-\bar{B}'^{-1}\bar{C})+\alpha_{0\,0}$$

instead of $x(\infty)=w$. We therefore compensate it as follows:

$$\dot{x}=A(x-w)+\sum_{k=1}^{2N}\alpha_{0\,k}\phi_k(x)+\alpha_{0\,0}-\frac{\eta}{w-x_0}(x-x_0) \qquad (6.6)$$

which is the same as Eq.(6.5) at $t=0$ or $x(0)=x_0$, is $\dot{x}=0$ at $t=\infty$ or $x(\infty)=w$, and is used a proportional allotment in region between x_0 and w. Combining with Eqs.(6.4) and (6.6), we have the linear system compensated in asymptotically stable:

$$\dot{\phi}(x)=B\phi(x)+C \qquad (6.7)$$

where

$$B=\begin{bmatrix}A-\frac{\eta}{w-x_0} & \alpha_{0\,1} & \cdots & \alpha_{0\,2N}\\ 0 & \alpha_{1\,1}-\varepsilon & \cdots & \alpha_{1\,2N}\\ \vdots & \vdots & & \vdots\\ 0 & \alpha_{2N\,1} & \cdots & \alpha_{2N\,2N}-\varepsilon\end{bmatrix},\quad C=\begin{bmatrix}\alpha_{0\,0}-Aw+\frac{\eta x_0}{w-x_0}\\ \alpha_{1\,0}\\ \vdots\\ \alpha_{2N\,0}\end{bmatrix}$$

Fig.7 is the numerical results which is compensated the linearization system of Eq.(4.1) (or Fig.1) by Eq.(6.7), where $\varepsilon=1.0$. This shows that the compensated linear system goes to the steady state $w=0$ as $t\rightarrow\infty$.

7. Conclusions

In this paper, we have proposed a formal linearization considering easy inversion of nonlinear systems. Using the proposed method, the problem of nonlinear systems are reduced to the problem of linear systems.

To demonstrate the effectiveness, we have illustrated the simulation on nonlinear example systems. The accuracy of the method presented here is better than of the previous one based on Taylor expansion in wide region and is improved as N increases.

We have also proposed a compensation of the linearization method and constructed an observer of a nonlinear system for demonstrating how to apply the method.

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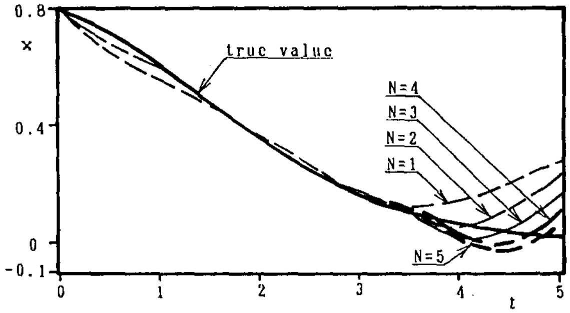

Fig.1 $x(t)$ and $\hat{x}(t)$ when $\dot{x}=-x+x^2$

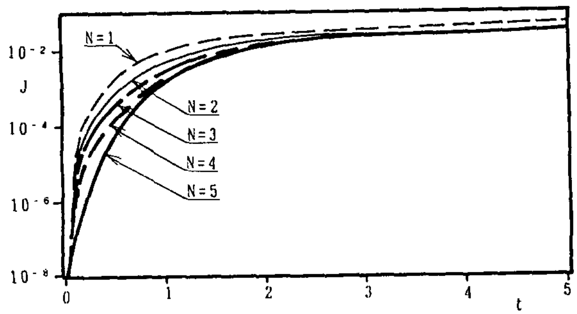

Fig.2 $J(t)$ when $\dot{x}=-x+x^2$

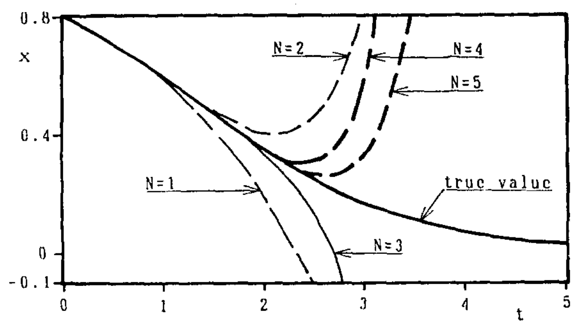

Fig.3 $x(t)$ and $\hat{x}(t)$ by Taylor expansion

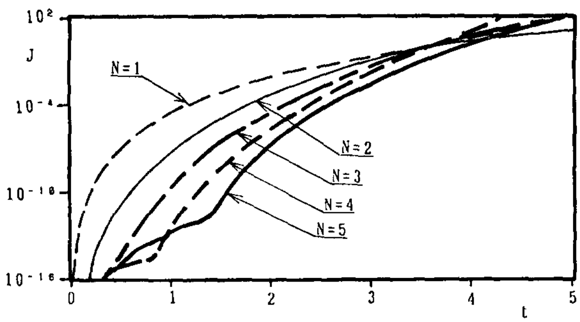

Fig.4 $J(t)$ by Taylor expansion

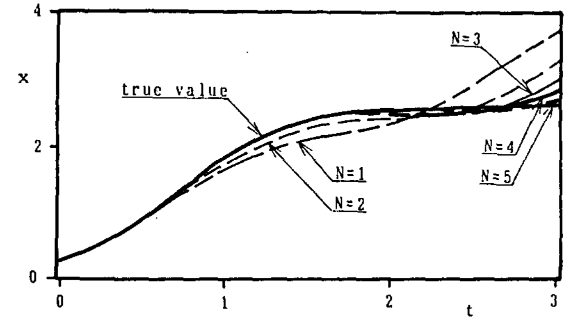

Fig.5 $x(t)$ and $\hat{x}(t)$ when $\dot{x}=2\sin(1.2x)$

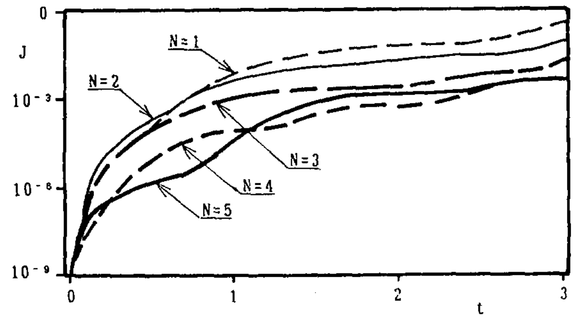

Fig.6 $J(t)$ when $\dot{x}=2\sin(1.2x)$

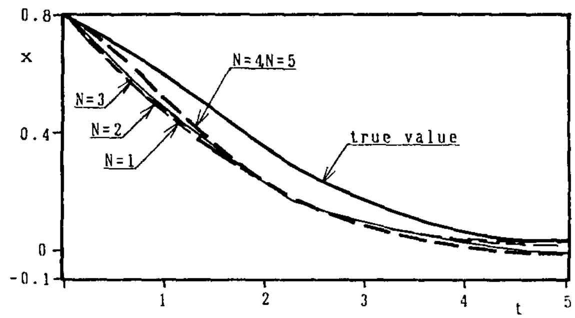

Fig.7 $\hat{x}(t)$ by Compensation Linearization when $\dot{x}=-x+x^2$