On the Improvement of the Guaranteed Stability Margins for the Discrete Time LQ Regulator

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In this paper, the selection method of weighting matrices in the discrete-time LQ problem are suggested in order to improve the guaranteed stability margins, i.e. the gain and phase margins. The asymptotic properties of the solution of the algebraic Riccati equations are investigated by using the closed form solution of the difference Riccati equations. It is shown that the solution of the algebraic Riccati equations monotonically increases as the state weighting matrix Q or the control weighting matrix R increase. The increasing rate of the solution is shown to be much less than that of R for large R. It is also proven that the guaranteed stability margins increases as the ratio between Q and R decreases.

1. Introduction

It is well known that the LQ regulator for the single-input single-output (SISO) continuous time system has excellent guaranteed stability margins, i.e., the gain margin of [0.5, ∞) and the phase margin over $\pm 60^\circ$ for arbitrary weighting matrices. For the multivariable continuous time system, it is also known that the LQ regulator can posses the above guaranteed stability margins when the control weighting matrix is diagonal [1]. It is known that the guaranteed stability margins in the discrete time LQ regulator can be equal to those of the continuous time LQ regulator for a specific weighting matrices [2]. For general weighting matrices, however, the discrete time LQ regulator does not have the same good stability margins as the continuous time LQ regulator. The fundamental reason for the above fact is not mentioned in detail. Also the general method to improve the guaranteed stability margins for the discrete time LQ regulator are not discussed in the literatures. This paper explains the basic reason why the discrete time LQ regulator does not have the guaranteed stability margins like the continuous time LQ regulator and suggests the general method to improve the guaranteed stability margins for the discrete time case.

This paper is organized as follow. In Section 2, the asymptotic properties of the solution of the algebraic Riccati equations (ARE) and the property of the feedback gain in the discrete time LQ regulator are investigated by using the closed form solution of the discrete time difference Riccati equation (DRE) [3] when the state weighting matrix tends to zero or the control weighting matrix approaches infinity. In Section 3, some comments for the basic reason why the discrete time LQ regulator can not have the same guaranteed stability margins as the continuous time LQ regulator are mentioned. The method to improve the guaranteed stability margins in the discrete time case are suggested. Conclusions are given in Section 4.

2. Asymptotic Properties of the ARE solution

We consider the discrete time, linear, time invariant and controllable system $S(A,B)$ whose state space description is given by the following representation :

$$x_{k+1}=Ax_k+Bu_k \tag{2.1}$$

where $x_k \varepsilon\ R^n$, $u_k \varepsilon\ R^m$, $x_0 \neq 0$ and A and B are constant matrices with the appropriate dimensions. It is desired to obtain the control law for u(k) that minimize the following index of performance :

$$J=\sum_{i=1}^{\infty} [x^T_i Q x_i + u^T_i R u_i] \tag{2.2}$$

where $Q \varepsilon\ R^{nxn}$ is positive definite and $R \varepsilon\ R^{mxm}$ is positive definite and diagonal. It is well known that the solution to the above optimal regulator problem is given by the following control law [4] :

$$u_k = -Kx_k \tag{2.3}$$

where K is a constant feedback gain matrix that is defined by

$$K=(R+B^TPB)^{-1}B^TPA \tag{2.4}$$

The constant matrix P is the positive definite solution of the following ARE :

$$P=A^TPA+Q-A^TPB(R+B^TPB)^{-1}B^TPA \tag{2.5}$$

The solution of the ARE (2.5) can be obtained from the following DRE :

$$P_{i+1}=A^TP_iA+Q-A^TP_iB(R+B^TP_iB)^{-1}B^TP_iA \quad (2.6)$$

That is, the DRE solution P_i converges to the ARE solution P as i tends to infinity. Thus we can know the properties of P by using the properties of P_i. The following closed form solution of DRE is recently derived in [3].

LEMMA 1 : The closed form solution can be represented by the following equation :

$$P_i=A^T\Sigma_iA+Q \quad (2.7)$$

where

$$\Sigma_i = N_i^T[Q_i^{-1} + M_iR_i^{-1}M_i^T]^{-1}N_i \quad (2.8)$$

$$N_i=\begin{bmatrix} I_n \\ A \\ \cdot \\ \cdot \\ \cdot \\ A_{i-1} \end{bmatrix}, \quad M_i=\begin{bmatrix} B & 0 & \cdot & \cdot & \cdot & 0 \\ AB & B & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ A^{i-1}B & A^{i-2}B & \cdot & \cdot & \cdot & B \end{bmatrix},$$

$Q_i=diag[Q, \cdots, Q]$, $R_i=diag[R, \cdots, R]$, and $P_0 = Q$.

Since P_i converges to P as i tends to infinity, there always exists a finite constant n_ε for fixed Q and R such that for a given $\varepsilon>0$, $\sigma_1[P_i-P]\leq \varepsilon$ for $i\geq n_\varepsilon$ where $\sigma_1[\cdot]$ denotes the maximum singular value of $[\cdot]$. By using this fact, we can state the following theorem about the properties of P.

THEOREM 1 : Assume that $Q=\alpha Q_0$ and $R=\beta R_0$ where Q_0 is positive definite and R_0 is positive definite and diagonal. For the solution P of the ARE (2.5), the following statements hold.

(1) If $\sigma_1(A)<1$, P approaches zero as α tends to zero for a fixed β and P converges to a finite constant matrix as β goes to infinity for a fixed α.

(2) P is a monotonical increasing function of α and β and for a fixed α, the increasing rate of P with respect to β is much less than that of R for large β.

(3) The feedback gain K is same for all Q and R if the ratio α/β is equal.

(4) The feedback gain K approaches a finite constant matrix as α/β tends to infinity.

PROOF : (1) Since $\sigma_1(A)<1$, N_i and M_i are finite even though i tends to infinity. This implies that Σ_i tends to zero as α approaches zero for a fixed β. This fact also implies that P_i tends to zero as α goes to zero for a fixed β. Therefore P tends to zero as α approaches zero for a fixed β since $|P_i-P| \leq \varepsilon$. For a fixed α, since M_i is finite,

$$\Sigma_i \rightarrow \sum_{j=0}^{i-1}(A^T)^jQA^j \quad \text{as} \quad \beta\rightarrow\infty$$

$$\text{and } P_i \rightarrow \sum_{j=1}^{i}(A^T)^jQA^j + Q \quad \text{as} \quad \beta\rightarrow\infty.$$

This limit of P_i is finite since $\sigma_1(A)<1$. Therefore P approaches a finite constant matrix as β tends to infinity for a fixed α.

(2) For a fixed β and i, the derivative of P_i with respect to α is given by

$$dP_i/d\alpha = Q_0+A^TN_i^TH^T(I-\alpha X(\alpha))^2HN_iA$$

where $X(\alpha) = HM_i(\alpha M_i^TQ_1+R_i)^{-1}M_i^TH^T$ and $H^TH = Q_1 = Q_i/\alpha$. Since $dP_i/d\alpha$ is positive definite, P_i monotonically increases with respect to α. Therefore P also monotonically increases with respect to α. For a fixed α and i, the derivative of P_i with respect to β is given by

$$dP_i/d\beta = A^TN_i^TQ_iM_i(M_i^TQ_iM_i/\beta+R_1)^{-1}R_1$$

$$(M_i^TQ_iM_i/\beta+R_1)^{-1}M_i^TQ_iN_iA/\beta^2$$

where $R_1 = R_i/\beta$. Since $dP_i/d\beta$ is positive definite, P_i, consequently P, monotonically increases with respect to β. But since its derivative is proportional to $1/\beta^2$, its increasing rate becomes much less than that of R for large β.

(3) From Lemma 1,

$$\Sigma_i = N_i^T[Q_1^{-1}/\alpha + M_iR_1^{-1}M_i^T/\beta]^{-1}N_i$$

$$= \alpha N_i^T[Q_1^{-1}+(\alpha/\beta)M_iR_1^{-1}M_i^T]^{-1}N_i$$

$$P_i = \alpha [A^TN_i^T[Q_1^{-1}+(\alpha/\beta)M_iR_1^{-1}M_i^T]^{-1}N_iA + Q_0]$$

$$= \alpha S_i(\alpha/\beta) \quad (2.9)$$

$$K_i(\alpha/\beta) = [R+B^TP_iB]^{-1}B^TP_iA$$

$$= \alpha [\beta R_0+\alpha B^TS_i(\alpha/\beta)B]^{-1}B^TS_i(\alpha/\beta)A$$

$$= [(\beta/\alpha)R_0+B^TS_i(\alpha/\beta)B]^{-1}B^TS_i(\alpha/\beta)A \quad (2.10)$$

where $Q_1 = Q_i/\alpha$ and $R_1 = R_i/\beta$. Since S_i does not vary if α/β does not change, K_i also does not vary. Therefore K, the limit of K_i, is same for all α and β with the same ratio.

(4) It is noted that $S_i(0) = A^TN_i^TQ_1N_iA+ Q_0$ from (2.9). Let $\gamma = \alpha/\beta$. Then

$$dS_i(\gamma)/d\gamma = -A^TN_i^T[Q_1^{-1}+\gamma M_iR_1^{-1}M_i^T]^{-1}M_iR_1^{-1}M_i^T$$

$$[Q_1^{-1}+\gamma M_iR_1^{-1}M_i^T]^{-1}N_iA \leq 0$$

So $S_i(\gamma)$ is a nonincreasing function of γ and it tends to $Q_0 + M_0$ for a finite constant matrix as γ goes to infinity. Therefore

$$\lim_{\gamma \to \infty} K_i(\gamma) \rightarrow [B^TQ_oB+B^TM_oB]^{-1}B^T(Q_o+M_o)A$$

Since Q_o is positive definite, the limit value of $K_i(\gamma)$ is finite even though γ tends to infinity. **[Q.E.D]**

The facts proven in Theorem 1 are illustrated by the following examples.

EXAMPLE 1 : We consider a system and weighting matrices as follows.

$$A=\begin{bmatrix} .999 & -.025 & -.162 \\ .094 & .867 & -.239 \\ .126 & .231 & .636 \end{bmatrix} \quad B=\begin{bmatrix} .0048 & .0939 \\ -.0129 & .0052 \\ .1065 & .0682 \end{bmatrix}$$

$$Q_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\sigma_1(A)$ = 1.0398. P and K for each α and each β are given as follows.

				α
P	1.699E-3	-5.005E-4	-4.257E-4	
	-5.005E-4	8.260E-4	1.475E-4	
	-4.257E-4	1.475E-4	3.298E-4	1.E-4
K	-3.003E-5	1.025E-5	2.419E-5	
	1.244E-4	-3.533E-5	-2.357E-5	
P	1.683E-1	-4.923E-2	-4.230E-2	
	-4.923E-2	8.218E-2	1.463E-2	
	-4.230E-2	1.463E-2	3.292E-2	1.E-2
K	-2.992E-3	1.020E-3	2.416E-3	
	1.230E-2	-3.464E-3	-2.335E-3	
P	1.045E01	-2.009E00	-3.000E00	
	-2.009E00	6.690E00	9.688E-1	
	-3.000E00	9.688E-1	2.955E00	1.E00
K	-2.298E-1	7.609E-2	2.143E-1	
	7.083E-1	-1.052E-1	-1.410E-1	
P	1.857E02	-4.096E00	-3.373E01	
	-4.096E00	4.643E02	4.499E01	
	-3.373E01	4.499E01	1.416E02	1.E02
K	-2.7579	0.38549	4.1269	
	6.0780	1.8375	-0.81898	
P	1.028E04	-1.981E03	2.204E02	
	-1.981E03	4.417E04	4.919E03	
	2.204E02	4.919E03	1.082E04	1.E04
K	-6.5414	-1.0656	7.7150	
	10.568	3.9064	-2.6572	
P	1.012E06	-2.028E05	2.998E04	
	-2.028E05	4.413E06	4.941E05	
	2.998E04	4.941E05	1.075E06	1.E06
K	-6.7170	-1.1468	7.8511	
	10.738	3.9883	-2.7537	
P	1.014E08	-2.296E07	3.403E06	
	-2.296E07	4.603E08	4.660E07	
	3.403E06	4.660E07	1.080E08	1.E08
K	-6.6891	-1.6487	7.9268	
	10.736	4.0504	-2.7638	

(β = 1)

				β
P	16.992	-5.0046	-4.2572	
	-5.0046	8.2604	1.4753	
	-4.2572	1.4753	3.2986	1.E04
K	-3.003E-5	1.025E-5	2.419E-5	
	1.244E-4	-3.533E-5	-2.357E-5	
P	16.826	-4.9230	-4.2301	
	-4.9230	8.2179	1.4630	
	-4.2301	1.4630	3.2922	1.E02
K	-2.992E-3	1.020E-3	2.416E-3	
	1.230E-2	-3.464E-3	-2.335E-3	
P	10.454	-2.0089	-3.0004	
	-2.0089	6.6897	0.9688	
	-3.0004	0.9688	2.9585	1.E00
K	-2.298E-1	7.609E-2	2.143E-1	
	7.083E-1	-1.052E-1	-1.410E-1	
P	1.8568	-4.096E-2	-3.373E-1	
	-4.096E-2	4.6433	4.499E-1	
	-3.373E-1	4.499E-1	1.4163	1.E-2
K	-2.7579	0.38549	4.1269	
	6.0780	1.8375	-0.81898	
P	1.0278	-1.981E-1	2.204E-2	
	-1.981E-1	4.4173	4.919E-1	
	2.204E-2	4.919E-1	1.0818	1.E-4
K	-6.5414	-1.0656	7.7150	
	10.568	3.9064	-2.6572	
P	1.0121	-2.028E-1	2.998E-2	
	-2.028E-1	4.4133	4.941E-1	
	2.998E-2	4.941E-1	1.0750	1.E-6
K	-6.7170	-1.1468	7.8511	
	10.738	3.9883	-2.7537	
P	1.0121	-2.028E-1	3.006E-2	
	-2.028E-1	4.4133	4.941E-1	
	3.006E-2	4.941E-1	1.075	1.E-8
K	-6.6891	-1.6487	7.9268	
	10.736	4.0504	-2.7638	

(α = 1)

From above two tables, it is noted that P monotonically increases as α and β increase and the feedback gain depends on only α/β.

3. Improvement of the Guaranteed Stability Margins for LQR

It is known that the discrete LQ regulator can not have the same guaranteed stability margins as the continuous time LQ regulator. This fact also applies to the discrete time feedback system with a constant gain feedback. The return difference matrix F(z) of the closed loop discrete time system with a constant gain feedback K is given by

$$F(z) = I_m + K(zI_n - A)^{-1}B \quad . \qquad (3.1)$$

From [5], it is noted that the discrete time feedback system can have the same guaranteed stability margins as the continuous time LQ regulator if $\sigma_n[F(z)] \geq 1$, $z=e^{jw}$ for any positive w where $\sigma_n[\cdot]$ denotes the minimum singular value of $[\cdot]$. Since $\det(e^{jw}I_n-A)$ is finite for all positive w, $\sigma_n[F(e^{jw})]=1$ only if $K(zI_n-A)^{-1}B$ has at least one zero on the unit circle. In general, the discrete time system has no zeros on the unit circle and the constant gain feedback can not change the zero position of the open loop system. Therefore any constant gain feedback discrete time system without zeros on the unit circle can not have the same guaranteed stability margins as the continuous time LQ regulator. But we can enlarge the guaranteed stability margins of the discrete time LQ regulator in some cases. We obtain the following equation from ARE (2.4) and the return difference matrix (3.1) [2].

$$F^*(z)(R+B^TPB)F(z) = R+B^T(z^{-1}I_n-A^T)^{-1}Q(zI_n-A)^{-1}B \quad (3.2)$$

where $[\cdot]^*$ denotes the conjugate transpose of $[\cdot]$. Since $B^T(z^{-1}I_n-A^T)^{-1}Q(zI_n-A)^{-1}B \geq 0$, we can obtain the following inequality :

$$F^*(z)(R+B^TPB)F(z) \geq R \quad (3.3)$$

If $\sigma_1(B^TPB)$ is very small compared with $\sigma_n(R)$, $\sigma_n[F(z)]$ becomes nearly 1. In consequence, the guaranteed stability margins become large. Therefore we only need to investigate the method to make $\sigma_1(B^TPB)/\sigma_n(R)$ as small as possible. By using the results of Theorem 1, we obtain the following theorem.

THEOREM 2 : Assume that $Q=\alpha Q_0$ and $R=\beta I_m$ where $Q_0>0$, $\alpha>0$, and $\beta>0$. Then the following statements hold.
(1) The smaller is α, the larger the guaranteed stability margins. And the larger is β, the larger the guaranteed stability margins.
(2) When $\sigma_1(A)<1$, the guaranteed stability margins of the continuous time LQ regulator are asymptotically achieved in the discrete time LQ regulator as α tends to zero or β goes to infinity.
(3) The guaranteed stability margins are same if the ratio α/β is equal.

PROOF : (1) Form the (2) of Theorem 1, it is known that P monotonically decreases as $\alpha \rightarrow 0$ and the increasing rate of P becomes much less than that of R as β . This fact implies that $\sigma_1(B^TPB) \rightarrow 0$ as $\alpha \rightarrow 0$ and the increasing rate of $\sigma_1(B^TPB)$ becomes much lee than that of R as $\beta \rightarrow \infty$. Therefore the guaranteed stability margins monotonically increase as $\alpha \rightarrow 0$ or $\beta \rightarrow \infty$.

(2) From the (1) of Theorem 1, it is known that P monotonically tends to zero as α goes to zero and P monotonically increases and converges to a finite constant matrix as β approaches infinity. These facts imply that $\sigma_1(B^TPB)$ tends to zero as α goes to zero and $\sigma_1(B^TPB)/\beta$ approaches zero as β tends to infinity. Therefore the guaranteed stability margins of the continuous time LQ regulator can be asymptotically achieved in the discrete time LQ regulator.
(3) From the (3) of Theorem 1, it is noted that K is same if the ratio α/β is equal. Thus F(z) is same for all Q and R with the same ratio α/β. Therefore the guaranteed stability margins are also equal. **[Q.E.D.]**

From Theorem 2, it is noted that the guaranteed stability margins of the discrete time LQ regulator can be made large under some assumptions if Q becomes small or R becomes large. This fact is illustrated by the following example.

EXAMPLE 2 : We consider the same system and weighting matrices in Theorem 1. We obtain the following results about stability margins.

α/β	GM	PM(deg)	$\sigma_n[F(z)]$
1.E08	.6727, 1.9478	28.163	0.48661
1.E06	.6718, 1.9546	28.267	0.48838
1.E04	.6671, 1.9958	28.893	0.49891
1.E02	.5873, 3.4882	41.790	0.71331
1.E00	.5073, 34.483	58.090	0.97100
1.E-2	.5001, 1785.7	59.963	0.99944
1.E-4	.5000, 5882.4	59.989	0.99983

From the above table, it is noted that the gain and phase margins increase monotonically as α/β decreases and they are dependent on only α/β.

4. Conclusions

In this paper, the general method to improve the guaranteed stability margins for the discrete time LQ regulator has been suggested. By using the closed form solution of the difference Riccati equation, several properties of the solution of the algebraic Riccati equation has been easily proven. It was shown that if the maximum singular value of system matrix A is less than 1, the ARE solution monotonically tends to zero as the state weighting matrix Q approaches zero and to a finite constant matrix as the control weighting matrix R tends to infinity. For the general system matrix A, the ARE solution monotonically increases as Q or R increase and its increasing rate is much less than that of R for large R. The feedback gain depends on only the ratio between Q and R. The feedback gain tends to a finite constant matrix even though the ratio between Q and R goes to infinity. It was also shown that the guaranteed stability margins monotonically increases as the ratio between Q and R decreases. The good guaranteed stability margins of the continuous time LQ regulator can be achieved in the discrete time LQ regulator if the maximum singular value of the system matrix A is less than 1.

The method of this paper for the improvement of the guaranteed stability margins is believed to be useful for the design of the discrete time LQ regulator. The relationship between the guaranteed stability margins and the robustness against modeling errors are not clear. Thus this relationship needs to be investigated.

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