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On the Invariance of Root Distribution of Interval Polynomials

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Abstract: In this paper, an extension of Kharitonov's theorem is studied. The problem considered here is the invariance of the numbers of stable and unstable roots of interval polynomials. A simple criterion is provided to test interval polynomials for the root distribution invariance.

1. Introduction

Robust stability of linear systems with uncertain parameters has been a very popular research topic. One of the great impetus so far given to this topic is Kharitonov's theorem[1] on the stability of interval polynomials. The theorem says that robust stability is assured only by knowing that four specially constructed "extreme polynomials", called the Kharitonov polynomials, are Hurwitz. Hence, if information on the lower and upper bounds of the coefficients of the characteristic polynomial is available, it is very easy to check the robust stability.

His result has got attention of many researchers. Anderson et al.[2] has shown that fewer polynomials are sufficient when low order polynomials are concerned. Similar results in the case of discrete-time systems have been rigorously sought[3]-[6]. Kharitonov's Theorem has been extended to the case of complex polynomials[7][8], where only eight extreme polynomials are sufficient for checking the robust Hurwitz stability. An extension to time-delay systems has been obtained by the authors[9]. Recently Chapellat and Bhattacharyya[10] have shown "Box theorem" for linear combinations of interval polynomials with known polynomial coefficients.

This paper considers Kharitonov's original situation, but in a wider view: The target of the paper is the invariance of root distribution of interval polynomials. Precisely speaking, the invariance concerned here is that of the numbers of stable and unstable roots over interval polynomials, where the roots in the open left-half plane are said to be stable and those in the open right-half plane are said to be unstable.

Apparently there seem to hold similar results to Kharitonov's theorem. That is, the root distribution invariance seems to hold if the four Kharitonov polynomials, or if all the extreme polynomials, have the same root distribution. This conjecture however has been negated by the following fourth order polynomials

$$f_a(s) = s^4 + 2s^3 + 5s^2 + as + 4, \quad a \in [0,9]. \qquad (1)$$

The interval polynomials have two extreme polynomials, $f_0(s)$ and $f_9(s)$. The first column of the Routh table for these polynomials is {1, 2, 5, -1.6, 4} and {1, 2, 0.5, -7, 4}, respectively. Hence both the extreme polynomials have two stable roots and two unstable roots. Notice that they have a same root distribution. Here we check the root distribution of $f_5(s)$, a member of (1). The first column of the Routh table is {1, 2, 2.5, 1.8, 4}, and we know that $f_5(s)$ has only stable roots. Thus the aforementioned conjecture is proved too optimistic.

The goal of the present paper is to show that it suffices to check the Kharitonov edge polynomials, defined later, for the root distribution invariance. A sufficient condition is also provided which enables us to verify the property with computational efficiency.

2. Notation and Definition

Let C be the complex plane, and let C_- and C_+ be the open left-half and right-half plane, respectively. For k=0,1,..,n, let H_k denote the set of the n-th order real polynomials that have k roots in C_- and n-k roots in C_+. Note that H_n represents the set of Hurwitz polynomials. A family of polynomials is said to be **H_k-invariant** if all the member polynomials are included in H_k.

Let F be the family of n-th order real polynomials

$$f(s) = s^n + f_1 s^{n-1} + f_2 s^{n-2} + \dots + f_n \qquad (2)$$

where

$$\alpha_i \le f_i \le \beta_i, \quad i=1,\dots,n. \qquad (3)$$

The polynomials (2) and (3) are called interval polynomials. By separating even and odd order parts, they can be written as

$$f(s) = h(s^2) + sg(s^2). \qquad (4)$$

Define four extreme polynomials of h(s) and g(s):

$$h_m(s) = \alpha_n + \beta_{n-2}s + \alpha_{n-4}s^2 + \beta_{n-6}s^3 + \dots$$
$$h_M(s) = \beta_n + \alpha_{n-2}s + \beta_{n-4}s^2 + \alpha_{n-6}s^3 + \dots$$
$$g_m(s) = \alpha_{n-1} + \beta_{n-3}s + \alpha_{n-5}s^2 + \beta_{n-7}s^3 + \dots$$
$$g_M(s) = \beta_{n-1} + \alpha_{n-3}s + \beta_{n-5}s^2 + \alpha_{n-7}s^3 + \dots \quad (5)$$

Then the **Kharitonov polynomials** can be written as follows:

$$\begin{aligned} f_{mm}(s) &= h_m(s^2) + sg_m(s^2) \\ f_{mM}(s) &= h_m(s^2) + sg_M(s^2) \\ f_{Mm}(s) &= h_M(s^2) + sg_m(s^2) \\ f_{MM}(s) &= h_M(s^2) + sg_M(s^2). \end{aligned} \quad (6)$$

For two polynomials,

$$a(s) = a_0s^p + a_1s^{p-1} + \dots + a_p$$
$$b(s) = b_0s^q + b_1s^{q-1} + \dots + a_q,$$

Sylvester's resultant matrix is defined by

$$S(a,b) = \begin{bmatrix} a_0 & a_1 & \cdots & a_p & & \\ & a_0 & \cdots & a_{p-1} & a_p & \\ & & & a_0 & \cdots & a_p \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_0 & b_1 & \cdots & b_q & & \\ & b_0 & \cdots & b_{q-1} & b_q & \\ & & & b_0 & \cdots & b_q \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} q \\ \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} p \end{matrix} \quad (7)$$

the bezoutian matrix(see, e.g.[11]) is defined by

$$Z(a, b) = \{z_{ij}\}, \quad (8)$$

where the z_{ij}'s are the coefficients of the bezoutian

$$Z[a, b] = \frac{a(\lambda)b(\mu) - a(\mu)b(\lambda)}{\lambda - \mu}$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{m} z_{ij}\lambda^{i-1}\mu^{j-1} \quad (9)$$

where m is the maximum of p and q. The bezoutian matrix is a symmetric matrix.

3. Invariance Condition of Root Distribution

The problem is to find a small subclass of interval polynomials such that the H_k-invariance of the subclass guarantees that of the whole class. A solution in the following has been obtained by scrutinizing the subclass[10][12]:

$$F_p = \{f_{\lambda\mu}(s) = (1-\lambda)h_m(s^2) + \lambda h_M(s^2) + s[(1-\mu)g_m(s^2) + \mu g_M(s^2)] : \lambda, \mu \in [0, 1]\}. \quad (10)$$

F_p constitutes a polytope that has the Kharitonov polynomials as its vertices. It is called the Kharitonov plane since it lies on an affine plane. Chapellat and Bhattacharyya have shown in [12] that if F_p is H_n-invariant, then F is also H_n-invariant. An extension of this result is as follows.

[Lemma 1] If F_p is H_k-invariant, then F is H_k-invariant. ($k = 0,1,\dots, n.$)

Proof: Assume that there exists an $f \in F$ such that $f \notin H_k$. Since $F_p \subset H_k$, a continuity property assures the existence of f_e such that $f_e \in F$ and it has a root on the imaginary axis. Furthermore the argument in [12] can lead to the assertion that there exists a polynomial in F_p that has a root on the imaginary axis. To be self-contained, we give a rough sketch of proof. From the assumption on f_e, we can find a $w \ge 0$ such that

$$f_e(jw) = 0.$$

Since $f_e \in F$, the extremum property of the Kharitonov polynomials yields

$$h_m(-w^2) \le \mathrm{Re}\ f_e(jw) \le h_M(-w^2)$$
$$g_m(-w^2) \le \mathrm{Im}\ f_e(jw) \le g_M(-w^2).$$

Then we can find some $\lambda, \mu \in [0, 1]$ such that

$$\mathrm{Re}\ f_e(jw) = (1-\lambda)h_m(-w^2) + \lambda h_M(-w^2)$$
$$\mathrm{Im}\ f_e(jw) = (1-\mu)g_m(-w^2) + \mu g_M(-w^2).$$

Hence $f_{\lambda\mu}$ with this pair of λ and μ belongs to F_p and satisfies

$$f_{\lambda\mu}(jw) = 0.$$

This contradicts the assumption of Lemma 1.

Q.E.D.

Chapellat and Bhattacharyya have shown that if the Kharitonov polynomials are in H_n, F_p is H_n-invariant. Although parallelism does not hold for the general H_k-invariance, we obtain an alternative by utilizing **"Kharitonov edges"**, i.e., edge polynomials of the Kharitonov plane:

$$\begin{aligned} F_1 &= \{(1 - \lambda)f_{mm}(s) + \lambda f_{Mm}(s): \lambda \in [0, 1]\} \\ F_2 &= \{(1 - \lambda)f_{Mm}(s) + \lambda f_{MM}(s): \lambda \in [0, 1]\} \\ F_3 &= \{(1 - \lambda)f_{MM}(s) + \lambda f_{mM}(s): \lambda \in [0, 1]\} \\ F_4 &= \{(1 - \lambda)f_{mM}(s) + \lambda f_{mm}(s): \lambda \in [0, 1]\}. \end{aligned} \quad (11)$$

[Lemma 2] If $F_1 \cup F_2 \cup F_3 \cup F_4 \subset H_k$, (12) then

$$F_p \subset H_k \quad (k=0,1,\dots, n),$$

i.e., the Kharitonov plane is H_k-invariant.

Proof: The key of proof is application of the edge theorem [13][14]. In the cases when $k \neq 0$ and $k \neq n$, the polynomials in H_k have their roots in the disconnected region $D = C_+ \cup C_-$. However, since the complement of D is pathwise connected, the extended edge theorem[14] is applicable. By noting that the exposed edges of F_p are (11), we have the conclusion.

Q.E.D.

These are summarized as follows.

[Theorem 1] The family of real interval polynomials F is H_k-invariant if and only if

$$F_1 \cup F_2 \cup F_3 \cup F_4 \subset H_k.$$
$$(k = 0,1,\dots, n.)$$

4. One shot Approach

To implement Theorem 1, we need a method of testing the Kharitonov edges for the H_k-invariance. The method proposed is a kind of "one shot test", considered by Bialas[15] and Fu and Barmish[16], for checking the Hurwitz invariance. The following lemma is essential for our method.

[Lemma 3] Let f(s) be a real polynomial written by

$$f(s) = h(s^2) + sg(s^2).$$

If $f(0)=h(0)\neq 0$ and the bezoutian matrix Z(h, g) (or the resultant matrix S(h, g)) is nonsingular, then f(s) has no roots on the imaginary axis.

Proof: Suppose f(s) has a root $s=jw$. Then $f(jw)=h(-w^2)+jwg(-w^2)=0$. Since w cannot be 0, $h(-w^2)=g(-w^2)=0$. Thus h(s) and g(s) are not relatively prime, contradicting the assumption that Z(h, g)(or S(h, g)) is nonsingular.

Q.E.D.

Note that the nonsingularity condition is sufficient but not necessary. We give a simple example:

$$d(s) = s^4 + s^3 - 3s^2 - s + 2.$$

Since the real part of d(jw) is always positive, d(s) has no roots on the imaginary axis. However Sylvester's resultant S for the case of d(s) is

$$\det\begin{bmatrix} 1 & -3 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = 0.$$

Due to Lemma 3, we have a simple criterion for the H_k-invariance.

[Theorem 2] Under the condition

$$0 \notin [\alpha_n, \beta_n], \qquad (13)$$

$F \subset H_k$ if

(a) at least one of the Kharitonov polynomials (6) is in H_k, and

(b) the following matrices have no real eigenvalues in $(-\infty, 0]$:

$$\begin{aligned} V_1 &= Z(h_M, g_m)Z(h_m, g_m)^{-1} \\ V_2 &= Z(h_M, g_M)Z(h_M, g_m)^{-1} \\ V_3 &= Z(h_m, g_M)Z(h_M, g_M)^{-1} \\ V_4 &= Z(h_m, g_m)Z(h_m, g_M)^{-1}. \end{aligned} \qquad (14)$$

Proof: Due to symmetry of (a) and (b), it is sufficient to consider the case when $f_{mm}(s)$ is in H_k. First we show that $F_1 \subset H_k$. F_1 is rewritten as

$$F_1 = \{f_\lambda(s)=(1-\lambda)h_m(s^2) + \lambda h_M(s^2) + sg_m(s^2): \lambda \in [0, 1]\}. \qquad (15)$$

Note that $f_0(s)=f_{mm}(s)$ and $f_1(s)=f_{Mm}(s)$. As is seen from the definition (9), the bezoutian is bilinear. Hence the bezoutian in Lemma 3 applied to the case of $f_\lambda(s)$ in (15) can be written as

$$Z((1-\lambda)h_m + \lambda h_M, g_m) = (1-\lambda)Z(h_m, g_m) + \lambda Z(h_M, g_m). \qquad (16)$$

Now the assumption (b) of V_1 leads to the following inequality:

$$\det Z((1-\lambda)h_m + \lambda h_M, g_m) \neq 0, \qquad \lambda \in [0, 1]. \qquad (17)$$

The proof of (17) is quite similar to [16] and thus will be omitted.

Since (13) implies $f_\lambda(0)\neq 0$, (17) with Lemma 3 yields that $f_\lambda(s)$, $\lambda \in [0, 1]$, has no roots on the imaginary axis. Hence by a continuity argument, $F_1 \subset H_k$, especially $f_{Mm}(s) \in H_k$. Repeating the same argument with V_2, V_3 and V_4, we have $F_2 \subset H_k$, $F_3 \subset H_k$ and $F_4 \subset H_k$ sequentially. Thus by Theorem 1, $F \subset H_k$.

Q.E.D.

The following is a special case of Theorem 2.

[Corollary 1] Assume that the interval polynomials (2) have the common odd-order part (even-order part). Then under the condition (13), $F \subset H_k$ if

(a) $f_{mm}(s) \in H_k$ ($f_{Mm}(s) \in H_k$), and

(b) the matrix V_1 (V_2) in (14) has no real eigenvalues in $(-\infty, 0]$.

Remarks on Theorem 2 and Corollary 1 are in order.

(i) The condition (13) assures that the interval polynomials have no roots on the origin. Then it is only a necessary condition for the H_k-invariance.

(ii) The inverse matrices in (14) are assumed to exist implicitly. However, it is unnecessary to check the existence for all the matrices $Z(h_m, g_m)$, $Z(h_M, g_m)$, $Z(h_M, g_M)$ and $Z(h_m, g_M)$ beforehand. In fact, the proof of Theorem 2 have shown that if one of them is nonsingular and if the corresponding part of condition (b) holds, then the the succeeding matrix is also nonsingular.

(iii) It is to be noticed that the condition (b) in itself gurantees that all the interval polynomials have the same number of stable and unstable roots. The condition (a) only declares the numbers.

(iv) It has been found[15][16] that convex combinations of n-th order Hurwitz polynomials $f_1(s)$ and $f_2(s)$ are Hurwitz invariant if and only if

$$\mathrm{Hur}(f_1)\mathrm{Hur}(f_2)^{-1} \tag{18}$$

is nonsingular, where Hur(f) is the Hurwitz discriminant matrix for the stability of f(s). Under the assumption of the nonzero constant term like (13), the Hurwitz matrix can be replaced by its (n-1)x(n-1) leading principal submatrix H(f). The submatrix H(f) is written as

$$H(f) = \begin{bmatrix} a_1 & a_3 & \cdots & a_n & 0 & \cdots & 0 \\ a_0 & a_2 & \cdots & a_{n-1} & 0 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & a_n & & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & a_0 & a_2 & \cdots & a_{n-1} \end{bmatrix} \tag{19}$$

where n is assumed to be odd and

$$f(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_n.$$

Note that H(f) is transformed to S(h, g) by interchanging rows, where h and g are obtained from even and odd order decomposition of f: if n is odd,

$$h(s) = a_1 s^m + a_3 s^{m-1} + \ldots$$
$$g(s) = a_0 s^m + a_2 s^{m-1} + \ldots$$

where m=(n-1)/2, and if n is even,

$$h(s) = a_0 s^m + a_2 s^{m-1} + \ldots$$
$$g(s) = a_1 s^{m-1} + a_3 s^{m-2} + \ldots$$

where m=n/2.

Thus the Hurwitz invariance condition of [15][16] is equivalent to the following:

$$S(h_1, g_1)S(h_2, g_2)^{-1} \tag{20}$$

is nonsingular.

Compared to this criterion, Theorem 2 utilizes the bezoutian matrix. As noticed easily from Lemma 3 and the bilinearity of S(h, g), Theorem 2 remains valid even if $Z(h_*, g_*)$'s are replaced by $S(h_*, g_*)$'s.

Since the size of bezoutian matrices is m, approximately a half of that of S, we know that using the bezoutian matrices significantly alleviates the computational burden for the H_k-invariance test.

(v) We can obtain Kharitonov's theorem from Theorem 2. It suffices to see the validity of the conditions of Theorem 2, assuming that k=n and the Kharitonov polynomials (6) are in H_n. First, the condition (a) and (13) are immediate. From Lemma 4 in Appendix, $Z(h_m, g_m)$, $Z(h_m, g_M)$, $Z(h_M, g_m)$, and $Z(h_M, h_M)$ are positive-definite symmetric matrices. Thus all the eigenvalues of V_i's are real positive, leading to (b).

5. Example

We use the fourth-order example (1) in Introduction. Even- and odd-order parts of (1) are

$$h(s) = s^2 + 5s + 4$$
$$g_a(s) = 2s + a, \qquad a \in [0, 9],$$

and the bezoutian matrix is

$$Z(h, g_a) = \begin{bmatrix} 5a-8 & a \\ a & 2 \end{bmatrix}. \tag{21}$$

We can apply Corollary 1 since the even-order part is fixed. Recalling that the extreme polynomials $f_0(s)$ and $f_9(s)$ are in H_2 and $F \subset H_2$, V_2 should have a non-positive real eigenvalue. In fact, eigenvalues of

$$V_2 = \begin{bmatrix} 37 & 9 \\ 9 & 2 \end{bmatrix} \begin{bmatrix} -8 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = 1/8 \begin{bmatrix} -37 & 36 \\ -9 & 8 \end{bmatrix}$$

are -7/2 and -1/8.

If the interval of a is replaced by [0, 1], eigenvalues of

$$V_2 = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -8 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = 1/8 \begin{bmatrix} 3 & 4 \\ -1 & 8 \end{bmatrix}$$

are 1/2 and 7/8. Thus we know the H_2-invariance of $f_a(s)$ for $a \in [0, 1]$.

6. Conclusion

We have studied the problem concerned with the invariance of the numbers of stable and unstable roots over interval polynomials. In the first, the invariance desired has been shown to be equivalent to the invariance relation over the Kharitonov edge polynomials. Furthermore a simple criterion which can be viewed as an extension of Kharitonov's theorem was provided by using resultant matrices.

The results obtained would be useful to investigate robust stability of feedback systems using Nyquist plots. In this application, the H_k-invariance of open-loop systems is an indispensable requisite to be checked at first. Then robust stability will be discussed based on the distance between the point -1 and the Nyquist band described on the complex plane.

Further research is needed to reduce conservatism involved in the present criterion.

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Appendix

[Lemma 4] If an n-th order real polynomial

$$f(s) = h(s^2) + sg(s^2)$$

is Hurwitz, then the bezoutian matrix $Z(h, g)$ is positive-definite.

Proof: Since $h(s)$ and $g(s)$ constitute a "positive pair"(see [17]), the Cauchy index of $g(s)/h(s)$ on the interval $(-\infty, \infty)$ is m, where m is the order of $h(s)$. Recalling that the signature of $Z(h, g)$ is equal to the Cauchy index of $g(s)/h(s)$[11], $Z(h, g)$ is positive-definite.

Q.E.D.

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