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ESTIMATION OF FREQUENCIES FROM MODIFIED LINEAR PREDICTION METHODS

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ABSTRACT

The problem of estimating the frequencies of multiple sinusoids from noisy measurements by using the modified linear prediction methods - Modified Forward-Backward Linear Prediction(MFBLP) and Model Reduction(MR) methods is addressed in this paper. The MFBLP and MR methods are derived by singular value decomposition and approximation of linear system, respectively. Monte Carlo simulations are done and the performances compared with linear prediction and forward-backward linear prediction. Simulations show a great promise for MFBLP and MR.

I. Introduction

Estimation of spectrum from finite noisy measurements is a very interesting and practical problem. It has been studied and used in many fields, such as communications, control systems, geophysics and econometrics. With the rapid development of modern technology, the need for estimation of spectra becomes ever increasing, and therfore motivates more and more researchers on this issue.

In this paper we are mainly interested in estimation of multiple sinusoidal frequencies, or narrow-banded spectral estimation in its broad sense, from finite noisy data, using the linear prediction.

Unlike the Yule-Walker method which involves the covariance function, the linear prediction method uses the measured data directly. Without taking approximation of covariance function for finite data, the linear prediction method may be expected to give better results for very short data cases.

Especially, we derive the modified linear prediction methods from the forward-backward linear prediction and linear prediction by using the singular value decomposition and approximation of linear system, respectively. Monte carlo simulations are done and the performances compared with linear prediction and forward-backward linear prediction.

2. Problem Formulation

Consider the following sinusoidal signal

$$x(t) = \sum_{t=1}^{m} \alpha_i \sin(\omega_i t + \varphi_i) \qquad (2.1)$$

where $\alpha_i, \varphi_i \in R$, $\omega_i \in (0,\pi)$ and $\omega_i \neq \omega_j$ for $i \neq j$.

Let y(t) denote the noise-corrupted measurements of x(t)

$$y(t) = x(t) + e(t) \qquad (2.2)$$

where e(t) is a sequence of independent and identically distributed random variable of zero mean and variance σ^2. It is assumed that x(t) and e(s) are uncorrelated for any t and s.

As is well-known, x(t) obeys the following

autoregressive(AR) process

$$A(q^{-1})x(t) = 0 \qquad (2.3a)$$

where q^{-} denotes the unit delay operator and $A(q^{-1})$ is a polynomial of degree 2m defined by (2.3b)

$$A(q^{-1}) = 1+a_1q^{-1}+\ldots+a_{2m}q^{-2m} = \prod_{i=1}^{m}(1-2\cos\omega_i q^{-1}+q^{-2})$$

It follows from (2.2) and (2.3) that y(t) obeys the following autoregressive moving average(ARMA) process

$$A(q^{-1})y(t) = A(q^{-1})e(t) \qquad (2.4)$$

It is easy to show that the roots of A(z) appear on the unit circle at $e^{\pm j\omega_i}$, i=1,2..m. Next multiplying both sides of equation (2.4) by a nonzero polynomial in q^{-1}, say $B(q^{-1})$, we obtain

$$C(q^{-1})y(t) = C(q^{-1})e(t) \qquad (2.5a)$$

where

$$C(q^{-1}) = B(q^{-1})A(q^{-1}) \qquad (2.5b)$$

Throughout the paper it will be assumed that $C(q^{-1})$ is a polynomial degree L(L>2m) given by

$$C(q^{-1}) = c_0+c_1q^{-1}+\ldots+c_Lq^{-L} \qquad (2.5c)$$

The problem is to estimate the augular frequencies (W_i) from the available data y(1), y(2),...,y(N).
The frequency estimates are usually obtained from the following two-step procedure:
1. Estimate the coefficients $[c_i]$.
2. Find the frequency estimates $[\hat{w}_i]$ either from the angular positions of 2m largest modulus roots,

$$\rho_i e^{\pm j\hat{\omega}_i} \quad (i=1,2,\ldots,m)$$

of C(z) or from those 2m values at which the spectrum $1/\|C(e^{\ })\|^2$ reaches its largest peaks.

3.MFBLP and MR Methods

It is a simple matter and a natural way to apply the noisy data y(1), y(2),...,y(N) to (2.4) or (2.5). In the following (2.5) will be considered.
For convenience, we rewrite (2.5) into

$$C(q^{-1})y(t) = n_f(t) \qquad (3.1a)$$

or (3.1b)

$$y(t) = -c_1y(t-1)-c_2y(t-2)-\ldots-c_Ly(t-L)+n_f(t)$$

This is a high-order autoregressive model. One immediately thinks of the least squares (LS) method in order to estimate c_i. Equation (3.1) can also be explained as a linear prediction filter problem, hence the name linear prediction method. Let $\hat{y}(t)$ be one-step "forward" linear predictor at time t, given a set of past values of y(t). This gives for the model (3.1)

$$\hat{y}(t) = -c_1y(t-1)-c_2y(t-2)-\ldots-c_Ly(t-L) \qquad (3.2)$$

Thus $n_f(t)$ in (3.1) is intrepreted as the linear prediction error. For finite data it is straightforward to write (3.1) into the following form.

$$Y_f = -A_f\theta+N_f \qquad (3.3a)$$

where

$$Y_f = [y(L+1)\ y(L+2)\ \ldots\ y(N)]^T \qquad (3.3b)$$

$$A_f = \begin{bmatrix} y(L) & y(L-1) & \ldots & y(1) \\ y(L+1) & y(L) & \ldots & y(2) \\ \vdots & \vdots & & \vdots \\ y(N-1) & y(N-2) & & y(N-L) \end{bmatrix} \qquad (3.3c)$$

$$\theta = [c_1\ c_2\ \ldots\ c_L]^T \qquad (3.3d)$$

$$N_f = [n_f(L+1)\ n_f(L+2)\ldots n_f(N)]^T \qquad (3.3e)$$

The estimate of θ is then given by

$$\hat{\theta} = -A_f^+Y_f \qquad (3.4)$$

where A_f^+ is the Moore-Pseudoinverse of A_f. If A_f is of rank L, then $A_f^+=(A_f^TA_f)^{-1}A_f^T$. The filter $1/\hat{C}(q^{-1})$ so obtained is called the forward linear prediction filter. Equations (3.3) and (3.4) will be called the forward linear prediction method, or rather the linear prediction(LP) method.
One may notice that when we made a linear prediction estimator in (3.2), some information is lost. More clearly, the predictor $\hat{y}(t)$ in (3.2) cannot be made for $t\le L$ in the forward linear prediction. This information can be restored if the predictor is performed in the "backward" direction, namely, if backward linear prediction is used. This idea was first introduced in maximum entr-

opy method by Burg , who used it in a Levinson recursion algorithm. Ullrych and Clayton used the same idea in the linear prediction case. Under the condition that the covariance function is symmetric for a real-valued process, we can write the model in the "backward" form:

$$y(t) = -c_1 y(t+1) - c_2 y(t+2) - \ldots - c_L y(t+L) + n_b(t) \quad (3.5)$$

Comvining (3.2) with (3.5), we obtain the following forward-backward linear prediction(FBLP) method:

$$Y_{fb} = -A_{fb}\theta + N_{fb} \quad (3.6a)$$

where

$$Y_{fb} = [y(L+1) \ldots y(N)\ y(1) \ldots y(N-L)]^T \quad (3.6b)$$

$$A_{fb} = \begin{bmatrix} y(L) & y(L-1) & \ldots & y(1) \\ \vdots & \vdots & & \vdots \\ y(N-1) & y(N-2) & \ldots & y(N-L) \\ y(2) & y(3) & \ldots & y(L+1) \\ \vdots & \vdots & & \vdots \\ y(N-L+1) & y(N-L+2) & \ldots & y(N) \end{bmatrix} \quad (3.6c)$$

(3.6d)

$$N_{fb} = [n_f(L+1) \ldots n_f(N)\ n_b(1) \ldots n_b(N-L)]^T$$

Thus the estimate of θ is given by

$$\hat{\theta} = -A_{fb}^{+} Y_{fb} \quad (3.7)$$

Equation (3.7) is expected to provide better performance than (3.4) for short data sets, since more information is used.

3.1 MFBLP

Further improvement on the accuracy of the estimate $\hat{\theta}$ can be made by modifying A_{fb} in some way. To give a full analysis, we need some results from singular value decomposition(SVD).
Perform the SVD on A_{fb}:

$$A_{fb} = U\Sigma V^T \quad (3.8)$$

where U and V are 2(N-L)x2(N-L) and LxL orthogonal matrices, respectively. U is an eigenvector matrix of $A_{fb}A_{fb}^T$, while V is an eigenvector matrix of $A_{fb}^T A_{fb}$. Σ is a diagonal matrix of dimension 2(N-L)xL with non-negative diagonal elements arranged in descending order, i.e.

$$\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_q\} \qquad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_q \quad (3.9)$$

where q=min(2(N-L),L).
If rank $(A_{fb})=r$ $(r \leq q)$, we have r nonzero singular values. In this case the pseudoinverse A_{fb}^{+} is given by

$$A_{fb}^{+} = \sum_{i=1}^{r} v_i u_i^T / \sigma_i \quad (3.10)$$

where u_i and v_i denote the i:th column of U and V, respectively.

It is known that for noiseless data, i.e. y(t)=x(t) the rank of A_{fb} is equal to 2m (m being the number of sinusoids). This corresponds to 2m nonzero singular values. For noisy measurements, however, A_{fb} is generally of full rank. Thus we have $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_q > 0$. The last (q-2m) small nonzero singular values are caused by noise. It has been shown in that the first 2m singular values and associated eigenvectors are not much perturbed by the noise. Hence we may set the last (q-2m) "noise" singular values to zero. Consequently, we have

$$A_{fb} = \sum_{i=1}^{2m} \sigma_i u_i v_i^T \quad (3.11)$$

This is called the truncated singular value decomposition. It can be shown that (3.11) is the best approximation (in the sense of 2-norm) to the noise perturbed matrix A_{fb} in (3.6) among all matrices with rank 2m. With (3.11), the pseudoinverse is now approximated by

$$A_{fb}^{+} = \sum_{i=1}^{2m} v_i u_i^T / \sigma_i \quad (3.12)$$

This idea was first introduced by Tufts and Kumaresan , who called it a modified forward-backward linear prediction method. The important step in MFBLP is to use the truncated SVD. By setting the noise singular values to zero, part of the noise effect is cancelled. Furthermore, an ill-conditioned problem is avoided since we set small singular values to zero, cf (3.10) .

3.2 MR

Now we have got a high-order AR model from LP based methods.
This high order AR model may be reduced to a low-order model, which contains the significant information in the original model. The reduced order model can be achieved by *model reduction*(MR) techniques. Note that MR techniques are very important in the system design, for example, in simplifying the system models and high-order regulators. Here we are only interested in getting a minimal order(ARMA) model.We will briefly discuss the so-called balanced model reduction approach. The high- order AR model can be written in the state-space from:

$$x(t+1) = Fx(t) + Gn(t) \quad (3.13a)$$

$$y(t) = Hx(t) + Dn(t) \quad (3.13b)$$

where $F \in R^{L \times L}$, $G, H \in R^{L}$ and $D \in R$.
The main idea of this approach is to introduce measures of reachability and observability. The "most" reachable and observable part of the model is taken as the reduced order model.
The reachability and observability Gramians, P and Q, of an asymptotically stable model are defined, respectively, by

$$P = \sum_{k=0}^{\infty} F^k G G^T (F^T)^k \quad (3.14a)$$

$$Q = \sum_{k=0}^{\infty} (F^T)^k H^T H F^k \quad (3.14b)$$

P and Q satisfy the well-known Lyapunov equations

$$FPF^T - P + GG^T = 0, \quad F^T QF - Q + H^T H = 0 \quad (3.15)$$

Introduce a similarity transformation S(i.e. $\bar{x}=Sx$) such that the Gramians are diagonal:

$$\bar{P} = SPS^T = \Sigma_1, \quad \bar{Q} = S^{-T} Q S^{-1} = \Sigma_2 \quad (3.16)$$

Since P and Q are non-negative definite, the diagonal elements of Σ_1 and Σ_2 are non-negative. The i:th singular values of the model (3.13) are defined by

$$\sigma_i = \sqrt{eig(\bar{P}\bar{Q})} = \sqrt{eig(PQ)}, \qquad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_L \tag{3.17}$$

where eig(.) means taking eigenvalues of a matrix.
The order of the reduced model can be choson by examining the size of the singular values. Usually the first few singular values, which are associated with the signal, are much large than the rest.These largest singular values correspond to the most reachable and observable part of the high-order model. Therefore the gaps between the singular values give a good indication for determining the order of the reduced model.
once a minimal order (ARMA) model is obtained from model reduction technique, the problem of estimating frequencies of trivial, cf section 2.

4.Simulation Examples

In this section, simulation examples are given for frequency estimation performances of a sinusoids-in-noise process, using the methods developed in the previous sections. In all the examples, the signal was assumed to consist of two sinusoids. various cases were investigated.

Example 4.1 (short data, widely separated frequencies)

The data we simulated is given by

$$y(t) = \sqrt{2}\sin(0.7226t)+\sqrt{2}\sin(1.0367t)+e(t) \quad (4.1)$$

where e(t) is a zero mean white Gaussian process. The data length is N=64, and the SNR is 10 dB, i.e. SNR =SNR =10dB. We have used 50 different noise realizations and computed the frequency estimates for a number of "design parameters" like L. The following quantities have been evaluated (for i=1,2) in all the examples

$$\bar{\omega}_i = \frac{1}{50}\sum_{j=1}^{50}\hat{\omega}_i^j \qquad \text{(mean value of } \hat{\omega}_i\text{)}$$

$$rms(\hat{\omega}_i) = \sqrt{mse(\hat{\omega}_i)} \qquad \text{(root mean square of } \hat{\omega}_i\text{)}$$

where

$$mse(\hat{\omega}_i) = \frac{1}{50}\sum_{j=1}^{50}(\hat{\omega}_i^j-\omega_i)^2 \qquad \text{(mean square error of } \hat{\omega}_i\text{)}$$

$$\delta(\hat{\omega}_i) = \frac{|\bar{\omega}_i - \omega_i|}{\omega_i} \qquad \text{(percentage bias of } \hat{\omega}_i\text{)}$$

Results for the approximately "best" design parameters are given in table4.1. For each method, the spectrum for one realization is shown in figure 4.1. The angular positions of frequency estimates for 50 realizations are illustrated in figure 4.2.

Example 4.2 (short data, closely spaced frequencies)

The data is taken from the following process:

$$y(t) = \sqrt{2}\sin(0.7226t)+\sqrt{2}\sin(0.8168t)+e(t) \quad (4.2)$$

where e(t) is Gaussian white noise of zero mean and variance σ^2=0.01, the data lenth is 64,and 50 noise realizations are used. the performances are summarized in table 4.2.

5. Conclusions

Estimation of frequencies from short data was studied. Simulation examples were presented for the data based methods. The following conclusions are drawn:
1).The MFBLP estimates are better than the LP and FBLP estimates.
2).By further increasing L in LP based methods, the "bad" estimates appear more often in the LP method than in the FBLP and MFBLP methods.
3).The MR method gets rid of the extraneous poles of the LP, FBLP and MFBLP.

References

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method	$\bar{\omega}_1$	$\bar{\omega}_2$	$rms(\hat{\omega}_1)$	$rms(\hat{\omega}_2)$	$\delta(\hat{\omega}_1)$	$\delta(\hat{\omega}_2)$	design parameter
LP	0.7222	1.0371	0.0026	0.0027	0.00055	0.00035	L=18
FBLP	0.7229	1.0364	0.0023	0.0027	0.00047	0.00028	L=25
MFBLP	0.7227	1.0367	0.0020	0.0025	0.00019	0.00010	L=35
MR	0.7221	1.0371	0.0026	0.0026	0.00060	0.00035	L=18

Table 4.1 Estimation performances.

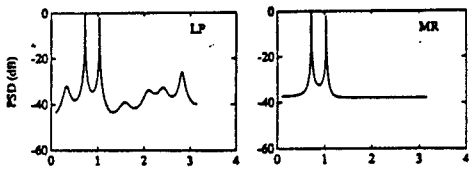

Figure 4.1 Normalized spectral densities

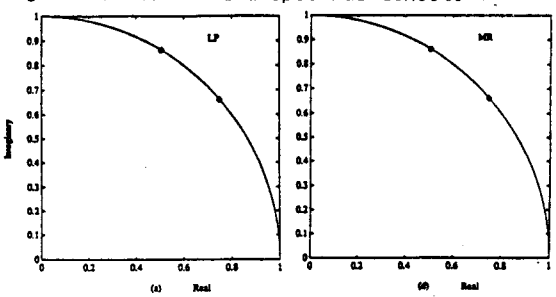

Figure 4.2 Angular positions pf estimates

method	$\bar{\omega}_1$	$\bar{\omega}_2$	$rms(\hat{\omega}_1)$	$rms(\hat{\omega}_2)$	$\delta(\hat{\omega}_1)$	$\delta(\hat{\omega}_2)$	design parameter
LP	0.7232	0.8186	0.0041	0.0041	0.00068	0.00216	L=16
FBLP	0.7227	0.8170	0.0028	0.0024	0.00015	0.00020	L=25
MFBLP	0.7227	0.8169	0.0020	0.0019	0.00013	0.00008	L=40
MR	0.7232	0.8186	0.0041	0.0041	0.00083	0.00217	L=16

Table 4.2 Estimation performances