Modeling of an Elastomer Constitutive Relation

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Abstract

This study is concerned with modeling an elastomer constitutive relation by utilizing the truncated Volterra series. Actual experimental data from the Instron Tester are obtained for combined inputs, i.e. constant strain rate followed by a constant strain input. These data are then estimated for step inputs and utilized for the truncated Volterra series models. One second order and one third order truncated Volterra series models have been employed to estimated the force-displacement relation which is one of the prominent properities to characterize the viscoelastic material. The third order Volterra series model has better results, compared with those of the second order Volterra series model.

1. Introduction

Since Vito Volterra[1] introduced an infinite functional series around 1910, which is now referred to as the Volterra series, to represent functionals which are analytic, the Volterra series has been applied in the wide range of nonlinear/bilinear systems such as, communications[2], circuits[3,4,5,6], viscoelastic material[7,8], and identification[9].

The input-output relations for nonlinear analytic systems can be explicitly represented by the Volterra series and they can be characterized by the Volterra kernels. If the system is not strongly nonlinear, the nonlinear system can be approximated by the truncated Volterra series. This study is concerned with modeling an elastomer constitutive relation by utilizing the truncated Volterra series. In section 2, stress relaxation phenomena are represented by the one-dimensional Volterra series model. In section 3, the force-displacement relations are modeled by both one second order truncated Volterra series model and one third order truncated Volterra series model, and they are analyzed for the step inputs.

2. Representation of an Elastomer Constitutive Relation

Consider an isotropic, homogeneous, and nonaging elastomer material under isothermal conditions. The linear constitutive equation for the material can be written as [10]

$$\sigma(t)=\int_{-\infty}^{t} G_1(t-t_1)\dot{\varepsilon}(t_1)dt_1, \qquad (1)$$

where the function $G_1(t-t_1)$ is called the stress relaxation function and $\dot{\varepsilon}(t)$ is the strain history. The principle of superposition is valid in this linear model.

The general constitutive equation for nonlinear viscoelastic material is originally proposed by Green and Rivlin[11]. For the case of one-dimensional deformation the constitutive equation is given as [10]

$$\sigma(t)=\int_{-\infty}^{t} G_1(t-t_1)\dot{\varepsilon}(t_1)dt_1$$
$$+\int_{-\infty}^{t}\int_{-\infty}^{t} G_2(t-t_1,t-t_2)\dot{\varepsilon}(t_1)\dot{\varepsilon}(t_2)dt_1dt_2$$
$$+\int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} G_3(t-t_1,t-t_2,t-t_3)\dot{\varepsilon}(t_1)\dot{\varepsilon}(t_2)\dot{\varepsilon}(t_3)dt_1dt_2dt_3+\cdots \qquad (2)$$

where the integrating functions $G_1(t-t_1)$, $G_2(t-t_1,t-t_2)$ and $G_3(t-t_1,t-t_2,t-t_3)$ are called the stress relaxation functions. This is a Volterra series representation between the stress and the strain rate. If there exits only one stress relaxation function G_1 , then the material exhibits linear behavior. Thus, the above representation may be a general model of nonlinear viscoelastic behavior.

3. Modeling and Analysis of the Elastomer Constitutive Relation

We now consider an estimation of stress relaxation function due to the step inputs by using the experimental data from the Instron Tester on the UPJOHN'S Urethane Elastomer sample. Suppose that we apply a combined input, i.e. constant strain rate followed by a constant strain input shown in Fig. 1. The basic assumption is that the estimated stress, $\sigma(t)$, due to a stress applied stepwise at zero time converge to the response due to a constant strain rate and constant strain input for $t > qt^*$ where q is constant[12,13].

Since experimental data from the Instron Tester are obtained in terms of force-displacement, we use the force-displacement relation instead of stress-strain. Eqn(2) can be rewritten as

$$f(t)=\int_{0}^{t} l_1(t-t_1)\dot{x}(t_1)dt_1$$
$$+\int_{0}^{t}\int_{0}^{t} l_2(t-t_1,t-t_2)\dot{x}(t_1)\dot{x}(t_2)dt_1dt_2$$
$$+\int_{0}^{t}\int_{0}^{t}\int_{0}^{t} l_3(t-t_1,t-t_2,t-t_3)\dot{x}(t_1)\dot{x}(t_2)\dot{x}(t_3)+\cdots \qquad (3)$$

where all initial times are assumed to be 0(zero) and l_1,l_2, and l_3 are Volterra kernels characterizing the intput-output relation.

We use log f(t)-log t plot in order to fit the estimated f(t) to displacement applied stepwise at zero time, since it is easy to figure out the trend of curve [12,13]. One log f(t)-log t plot is shown in Fig. 2. From this plot, we can obtain the estimated relation.

$$\log f(t)=-0.089\log t+2.368 \; . \qquad (4)$$

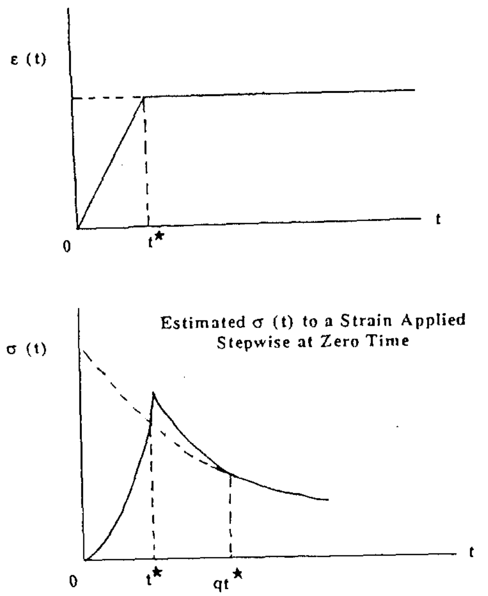

Fig. 1 Strain Input and Stress Output

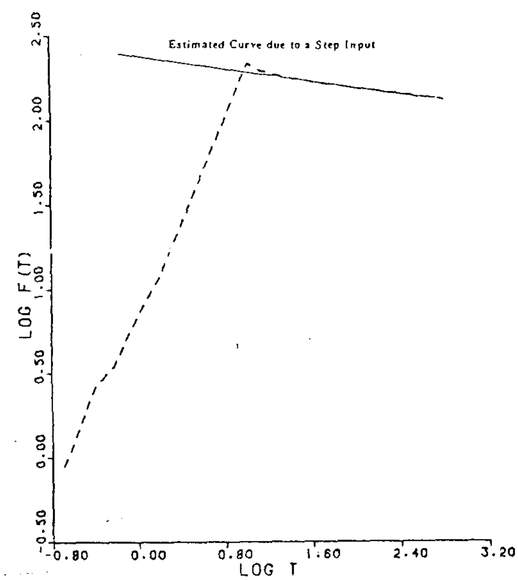

Fig. 2 log f(t) - log t Plot

The above equation is rewritten as

$$f(t)=233.44t^{-0.089}. \quad (5)$$

We now consider the second-order truncated Volterra series form.

$$f(t)=\int_0^t h_1(t-t_1)\dot{x}(t_1)dt_1 + \int_0^t\int_0^t h_2(t-t_1,t-t_2)\dot{x}(t_1)\dot{x}(t_2)dt_1dt_2, \quad (6)$$

where h_1 and h_2 are the first- and second-order Volterra kernels, respectively. Suppose that we have two input-output sets (force-displacement), $f_1(t)-x_1(t)$ and $f_2(t)-x_2(t)$

Let

$$x_1(t)=a_1u(t) \quad (7)$$

$$x_2(t)=a_2u(t), \quad (8)$$

where $u(t)$ is a step function. Substituting $x_1(t)$ and $x_2(t)$ into eqn(6), and solving two algebraic equations for $h_1(t)$ and $h_2(t,t)$, we can obtain two kernels

$$h_1(t)=\frac{a_2^2f_2(t)-a_1^2f_2(t)}{a_1a_2(a_2-a_1)} \quad (9)$$

$$h_2(t,t)=\frac{a_1f_2(t)-a_2f_1(t)}{a_1a_2(a_2-a_1)}. \quad (10)$$

Eight indentation test inputs for the urethane elastomer sample with a 0.9525*cm* radius penetrator are shown in Fig. 3 .

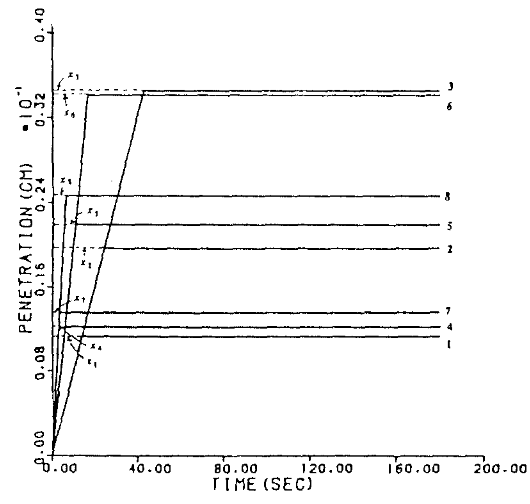

Fig.3 Indentation Test Inputs

we can estimate the corresponding force outputs due to assumed stepwise inputs as explained earlier. For the second order truncated Volterra series form, we use two data sets, i.e. $f_1(t)$ and $f_3(t)$, and estimate $h_1(t)$ and $h_2(t,t)$. In order to verify these two kernels, we apply two step inputs, $x_2(t)$ and $x_8(t)$ shown in Fig. 3, and estimated outputs, $\hat{f}_2(t)$ and $\hat{f}_8(t)$. These estimated outputs calculated using the kernels $h_1(t)$ and $h_2(t,t)$, and the corresponding kernels are shown in Fig.4 and Fig.5, respectively. The maximum error here is about 9.5%.

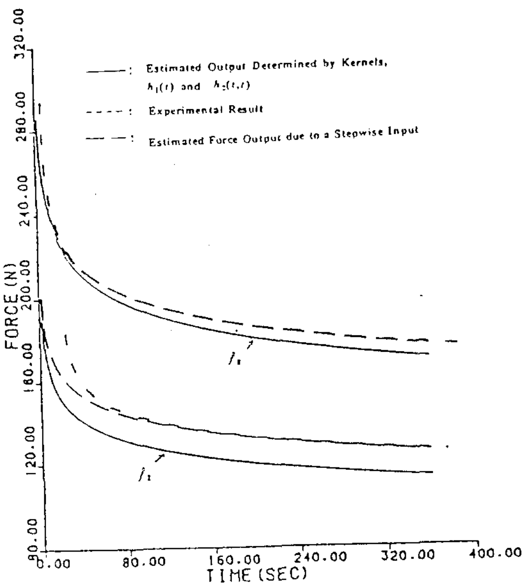

Fig. 4 Estimated Outputs of $f_2(t)$ and $f_8(t)$ from $f_1(t)-x_1(t)$ and $f_3(t)-x_3(t)$

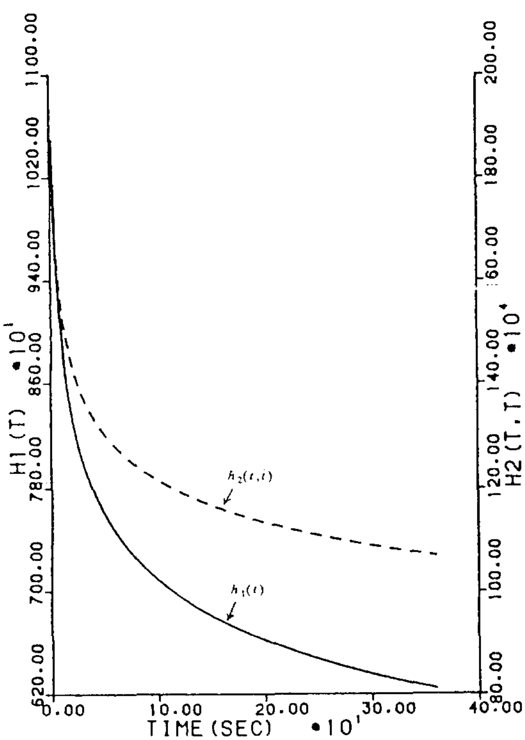

Fig. 5 Estimated Volterra Kernels

We now extend the previous model to the third order truncated Volterra series model.

$$f(t)=\int_0^t h_1(t-t_1)\dot{x}(t_1)dt_1$$
$$+\int_0^t\int_0^t h_2(t-t_1,t-t_2)\dot{x}(t_1)\dot{x}(t_2)dt_1dt_2$$
$$+\int_0^t\int_0^t\int_0^t h_3(t-t_1,t-t_2,t-t_3)\dot{x}(t_1)\dot{x}(t_2)\dot{x}(t_3)dt_1dt_2dt_3 \quad (11)$$

Let

$$x_1(t)=a_1u(t) \quad (12)$$
$$x_2(t)=a_2u(t) \quad (13)$$
$$x_3(t)=a_3u(t) \quad (14)$$

and the corresponding force responses be $f_1(t), f_2(t)$, and $f_3(t)$, respectively. Then, substituting $x_1(t), x_2(t)$ and $x_3(t)$ into eqn(11), and solving three algebraic equations for three kernels, we obtain

$$h_1(t)=[f_1(t)(a_2^2a_3^3-a_2^3a_3^2)-a_1^2(f_2(t)a_3^3-f_3(t)a_2^3)$$
$$+a_1^3(f_2(t)a_3^2-f_3(t)a_2^2)]/\det \quad (15)$$
$$h_2(t,t)=[a_1(f_2(t)a_3^3-a_2^3f_3(t))-f_1(t)(a_2a_3^3-a_2^3a_3)$$
$$+a_1^3(a_2f_3(t)-a_3f_2(t))]/\det \quad (16)$$
$$h_3(t,t,t)=[a_1(a_2^2f_3(t)-a_3^2f_2(t))-a_1^2(a_2f_3(t)-a_3f_2(t))$$
$$+f_1(t)(a_2a_3^2-a_2^2a_3)]/det, \quad (17)$$

where

$$\det=a_1a_2a_3(a_2a_3^2-a_2^2a_3-a_1a_3^2+a_1a_2^2+a_1^2a_3-a_1^2a_2).$$

For the third order truncated Volterra series form, we need three input-output data sets to estimate $h_1(t), h_2(t,t)$ and $h_3(t,t,t)$. We assume that three data sets are $f_1(t)-x_1(t), f_3(t)-x_3(t)$, and $f_5(t)-x_5(t)$. We now want to estimate $f_2(t), f_6(t)$ and $f_8(t)$ due to step inputs, $x_2(t), x_6(t), x_7(t)$ and $x_8(t)$,respectively. Compared with the results for the second order truncated Volterra series model, the estimated outputs have better results as expected. The estimated outputs and kernels are shown in Fig.6 and Fig.7, respectively. the maximum error here is about 6.4%. The third order Volterra series model has better results, compared with those of the second order Volterra series model.

4. Conclusion

The input-output relations for nonlinear system can be explicity represented by the Volterra series and they can be characterized by the volterra kernels. If the system is not strongly nonlinear, the nonlinear solution can be approximated by the truncated Volterra series solution with only a few low order terms. The second order and third order truncated Volterra series models have been employed to estimate the force-displacement relation which is one of the prominent properites to characterize the viscoelastic material. Actual experimental data from the Instron Tester are obtained for combined inputs, i.e. constant strain rate followed by a constant strain input. These data are then estimated for step inputs and utilized for the truncated Volterra series models. The third order Volterra series model has better results, compared with those of the second order Volterra series model.

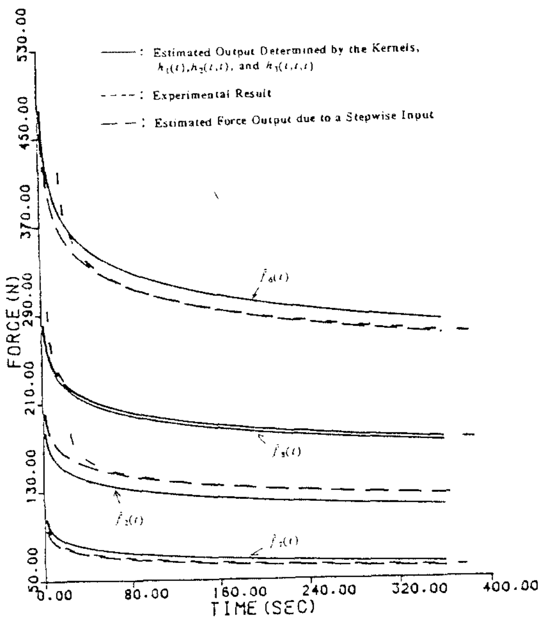

Fig. 6 Estimated Outputs from $f_1(t)-x_1(t)$, $f_3(t)-x_3(t)$, and $f_5(t)-x_5(t)$

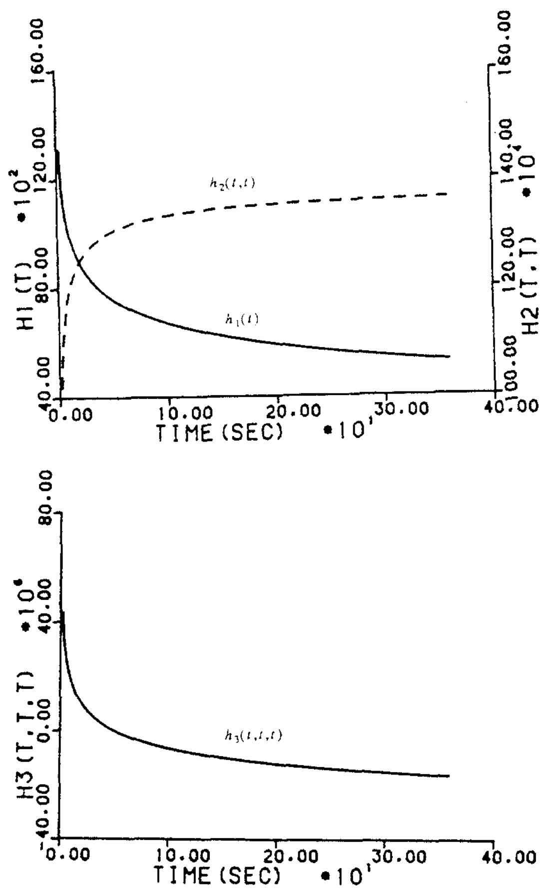

Fig. 7 Estimated Volterra Kernels

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