# A Method of Measuring Frequency Response Function by Use of Characteristic M-sequence

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**Abstract :** A simple method is proposed for determining the frequency response function $G(j\omega)$ of a system using a pair of characteristic M-sequences (maximum length linear feed back shift register sequence).

When a characteristic M-sequence is sampled with $q_1$ and $q_2$ both of which are coprime with $N$, where N is the period of the M-sequence, the obtained pair of sequences have conjugate complex frequency spectrum. Making use of this fact, two charcteristic M-sequences having conjugate complex frequency spectrum are applied to a system to be measured. Since the magnitude of spectrum of M-sequence is known, the gain of $G(j\omega)$ is directly obtained from the Fourier transform of the system output. The phase of $G(j\omega)$ is obtained simply by taking the average of the two phases of output spectrum.

**1. Introduction :** Efficient implementation of automatic control demands the knowledge of the dynamic characteristics of the control system. To meet this purpose, considerable work has been done and differnt system identification techniques have been proposed. Presently, these methods are being compared on the basis of computation time and accuracy. Also, it is well known that in order to carry out effective identification, application of an appropriate test signal to the system is necessary.

In this paper a new method for the measurement of the frequency response function of a system is proposed. The method uses a pair of characteristic M-sequences having special properties as input signal to the system. Then, the frequence response of the system can be easily measured by obtaining only the Fourier coefficients of the output sequence. The proposed system is time-efficient when compared to conventional correlation method. In the following sections we mainly discuss the principles and the procedure of the proposed measurement method and give the results of simulation.

**2. Fourier Coefficients of Characteristic M-sequence :** An M-sequence $\{a_i\}$ of order n (n is a positive integer) is a periodic sequence of maximum period, $N = 2^n - 1$, with zero or one as its elements. The Fourier series expansion of M-sequence is

$$C_k = \frac{1}{N}\sum_{i=0}^{N-1} a_i\, e^{(-j2\pi ki)/N} \qquad k = 0, 1, 2, \cdots, N-1 \tag{1}$$

If we consider M-sequence $\{a_i\}$ as a train of unit impulses, then the magnitude of the Fourier coefficients can be expressed as

$$|C_k| = \begin{cases} \frac{(N+1)}{2N}, & k = 0, N, 2N, \cdots; \\ \frac{\sqrt{(N+1)}}{2N}, & k = 0, 1, 2, \cdots, (N-1) \end{cases} \tag{2}$$

Excluding the multiples of the period, the magnitude of $C_k$ for $k = 1, 2, \cdot, (N-1)$ takes a constant value. However, the phase of $C_k$ depends on the initial values of $\{a_i\}$. Let $\{a_{2i}\}$ be a sequence obtained by alternately sampling $\{a_i\}$. For a particular initial condition of $\{a_i\}$, the relation $\{a_{2i}\} = \{a_i\}$ holds and an M-sequence with such an initial condition is called characteristic M-sequence. The Fourier coefficients of a characteristic M-sequence possess the following properties.

1) If higher harmonic components $k_1$ and $k_2$ (modulo $N$) belong to the same cyclotomic residue coset, then the Fourier coefficients $C_{k_1}$ and $C_{k_2}$ are equal both in magnitude and phase. That is,

$$C_{k_1} = C_{k_2} \tag{3}$$

2) If $k_1$ and $k_2$ belong to mutually complementary cyclotomic residue cosets, then the Fourier coefficients $C_{k_1}$ and $C_{k_2}$ are complex conjugates of each other.

$$C_{k_1} = \overline{C}_{k_2} \tag{4}$$

The cyclotomic residue cosets are formed by classifying the integers $1 \sim 2^n - 1$ into residue cosets as indicated in (5)

$$C_l = \{\, l, l\cdot 2, l\cdot 2^2, \cdots, l\cdot 2^{m-1} (\bmod N)\} \tag{5}$$

We now discuss the sampling properties when every $q^{th}$ element of the characteristic M-sequence is sampled.

- If $q$ and $N$ are relatively prime, then the sampled sequence is again a characteristic M-sequence. The characteristic polynomial of the resulting M-sequence is determined by the cyclotomic residue coset to which $q$ belongs.

  For example, in Table 1, cyclotomic residue cosets are shown for which period $N (N = 255)$ and $q$ are relatively prime. It can be noted that if the characteristic M-sequence is sampled with any one of the following 16 values of $q$, then the resulting series is again a characteristic M-sequence.

$$q = \{1, 7, 11, 13, 19, 23, 29, 31, 37, 43, 47, 52, 59, 61, 91, 127\}$$

In the above case,

$$(1,127),\ (11,61),\ (13,47),\ (19,59),\ (23,29),\ (37,91)$$

are complementary pairs.

Table 1 Cyclotomic Coset N = 255.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| C0 | =( | 0 | | | | | | | |
| C1 | =( | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 ) |
| C7 | =( | 7 | 14 | 28 | 56 | 112 | 224 | 193 | 131 ) |
| C11 | =( | 11 | 22 | 44 | 88 | 176 | 97 | 194 | 133 ) |
| C13 | =( | 13 | 26 | 52 | 104 | 208 | 161 | 67 | 134 ) |
| C19 | =( | 19 | 38 | 76 | 152 | 49 | 98 | 196 | 137 ) |
| C23 | =( | 23 | 46 | 92 | 184 | 113 | 226 | 197 | 139 ) |
| C29 | =( | 29 | 58 | 116 | 232 | 209 | 163 | 71 | 141 ) |
| C31 | =( | 31 | 62 | 124 | 248 | 241 | 227 | 199 | 143 ) |
| C37 | =( | 37 | 74 | 148 | 41 | 82 | 164 | 73 | 146 ) |
| C43 | =( | 43 | 86 | 172 | 89 | 178 | 101 | 202 | 149 ) |
| C47 | =( | 47 | 94 | 188 | 121 | 242 | 229 | 203 | 151 ) |
| C53 | =( | 53 | 106 | 212 | 169 | 83 | 166 | 77 | 154 ) |
| C59 | =( | 59 | 118 | 235 | 217 | 179 | 103 | 206 | 157 ) |
| C61 | =( | 61 | 122 | 244 | 233 | 211 | 167 | 79 | 158 ) |
| C91 | =( | 91 | 182 | 109 | 218 | 181 | 107 | 214 | 173 ) |
| C127 | =( | 127 | 254 | 253 | 251 | 247 | 239 | 223 | 191 ) |

- If $q$ and $N$ are not relatively prime then the resulting sequence may not be an M-sequence.

Thus by choosing $q$ to be relatively prime to $N$ a characteristic M-sequence $\{a_i\}$ can be sampled to obtain another characteristic M-sequence $\{b_i\} = \{a_{qi}\}$. The Fourier coefficients of such a sequence can be obtained from (1) as follows.

$$\begin{aligned} C_k{}^a &= \frac{1}{N} \sum_{i=0}^{N-1} a_{qi}\, e^{(-j2\pi kqi)/N} \\ &= \frac{1}{N} \sum_{i=0}^{N-1} b_i\, e^{(-j2\pi kqi)/N} = C_{kq}{}^b \end{aligned} \tag{6}$$

Here $C_k{}^a$ represents the $k^{th}$ Fourier coefficient of $\{a_i\}$. Hence if $\{b_i\}$ is obtained by sampling every $q^{th}$ element of $\{a_i\}$, then the Fourier coefficient of $kq^{th}$ harmonic of $\{b_i\}$ and that of $k^{th}$ harmonic of $\{a_i\}$ are equal both in magnitude and phase.

Next, we consider two characteristic M-sequences $\{b_i\}$ and $\{d_i\}$ obtained by sampling every $q_1^{th}$ and $q_2^{th}$ element of $\{a_i\}$. That is $\{b_i\} = \{a_{q_1 i}\}$, $\{d_i\} = \{a_{q_2 i}\}$ and $q_1$, $q_2$ both coprime with $N$. In that case the Fourier coefficients of $k_{q_1}^{th}$ harmonic of $\{b_i\}$, $k_{q_2}^{th}$ harmonic of $\{d_i\}$, and $k^{th}$ harmonic of $\{a_i\}$ are all equal.

$$C_k{}^a = C_{kq_1}{}^b = C_{kq_2}{}^d \tag{7}$$

$$C_{kq_2}{}^d = \overline{C_{kq_1}{}^d} \tag{8}$$

therefore,

$$C_{kq_1}{}^b = \overline{C_{kq_1}{}^d}$$

Note that the Fourier coefficients of $\{b_i\}$ and $\{d_i\}$ are complex conjugates.

**3.Measurement of Frequency Response :** Let $x(t)$ be an arbitrary input and $y(t)$ be the corresponding output of the system. The frequency response of the system can be obtained by the Fourier transforms of the input $X(j\omega)$ and the output $Y(j\omega)$ as in (9).

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \tag{9}$$

Generally in frequency response measurement methods, it is necessary to compute the Fourier transforms of the input $x(t)$ and the corresponding output $y(t)$, each time the measurement is carried out. The proposed measurement method uses a pair of characteristic M-sequences, having complex conjugate spectrum, as input signal. The magnitude of the Fourier transform of the characteristic M-sequence is a constant [from Eq.(2)], and since the Fourier coefficients of the pair of sequences are complex conjugates of each other the computation of both magnitude and phase of the input is not necessary in the proposed method.

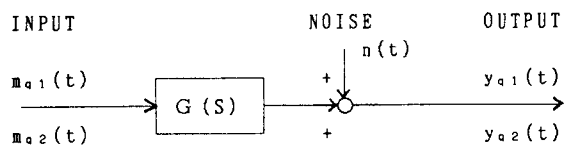


Fig.1. Block diagram of measurement system.

Fig.1 represents the block diagram of the measurement system, where $n(t)$ represents noise. $m_{q_1}(i)$ and $m_{q_2}(i)$ are the characteristic M-sequences obtained by sampling every $q_1^{th}$ and $q_2^{th}$ element of another characteristic M-sequence as explained in the previous section. Let $y_{q_1}(i)$ and $y_{q_2}(i)$ be the corresponding resopnse of the system to $m_{q_1}(i)$ and $m_{q_2}(i)$. Let $M_{q_1}(j\omega)$, $M_{q_2}(j\omega)$, $Y_{q_1}(j\omega)$, $Y_{q_2}(j\omega)$ represent the Fourier transforms of $m_{q_1}(i)$, $m_{q_2}(i)$, $y_{q_1}(i)$, $y_{q_2}(i)$ respectively. We have

$$\begin{aligned} Y_{q_1}(j\omega) &= G(j\omega) M_{q_1}(j\omega) + N_{q_1}(j\omega) \\ Y_{q_2}(j\omega) &= G(j\omega) M_{q_2}(j\omega) + N_{q_2}(j\omega) \end{aligned} \tag{10}$$

where $N_{q_i}(j\omega)$, $(i = 1,2)$, represent the noise present when the corresponding inputs are $M_{q_i}(j\omega)$, $(i = 1,2)$. Note that $N_{q_i}(j\omega)$ $(i = 1,2)$, are mutually independent and have an average value of zero. Also, $N_{q_i}(j\omega)$, $(i = 1,2)$, are uncorrelated with $M_{q_i}(j\omega)$, $(i = 1,2)$. Multiplying $Y_{q_1}(j\omega)$, $Y_{q_2}(j\omega)$ gives

$$
\begin{aligned}
Y_{q_1}(j\,\omega)\,Y_{q_2}(j\,\omega) \;&= \{G(j\,\omega)\}^2\, M_{q_1}(j\,\omega)\, M_{q_2}(j\,\omega) \\
&\quad + G(j\,\omega)\{M_{q_1}(j\,\omega)\, N_{q_2}(j\,\omega) \\
&\qquad + M_{q_2}(j\,\omega)\, N_{q_1}(j\,\omega)\} \\
&\quad + N_{q_1}(j\,\omega)\, N_{q_2}(j\,\omega)
\end{aligned}
$$

The ensemble average is

$$
\overline{Y_{q_1}(j\,\omega)\,Y_{q_2}(j\,\omega)} = \{G(j\,\omega)\}^2\, M_{q_1}(j\,\omega)\, M_{q_2}(j\,\omega) \quad (11)
$$

Since $M_{q_1}(j\omega)$, $M_{q_2}(j\omega)$ are complex conjugates of each other we get from Eq.(2).

$$
M_{q_1}\left(j\frac{k}{N}\right) M_{q_2}\left(j\frac{k}{N}\right) = \left(\frac{\sqrt{N+1}}{2N}\right)^2 \qquad k = 0, 1, 2, \cdots, N-1
$$

therefore,

$$
\begin{aligned}
\left\{G\left(j\frac{k}{N}\right)\right\}^2 &= \left(\frac{2N}{\sqrt{N+1}}\right)^2 \cdot \overline{Y_{q_1}\left(j\frac{k}{N}\right)} \cdot \overline{Y_{q_2}\left(j\frac{k}{N}\right)} \\
20\log_{10}\left|G\left(j\frac{k}{N}\right)\right| &= 20\log_{10}\frac{2N}{\sqrt{N+1}} \\
&\quad + \frac{1}{2}\left\{20\log_{10}\left|\overline{Y_{q_1}\left(j\frac{k}{N}\right)}\right| + 20\log_{10}\left|\overline{Y_{q_2}\left(j\frac{k}{N}\right)}\right|\right\} \\
\angle G\left(j\frac{k}{N}\right) &= \frac{1}{2}\left\{\angle\overline{Y_{q_1}\left(j\frac{k}{N}\right)} + \angle\overline{Y_{q_2}\left(j\frac{k}{N}\right)}\right\}
\end{aligned} \quad (12)
$$

From (12) it is clear that both magnitude and phase of the frequency response can be obtained by calculating the average of the expected values of the Fourier transforms of the output responses $y_{q_1}(i)$ and $y_{q_2}(i)$ of the system to the inputs $m_{q_1}(i)$ and $m_{q_2}(i)$.

The number of measurements to be carried out to obtain the average value can be arbitrary. However, for a charcteristic M-sequence of period $N = 255$ the number of possible sampling rates is 16 and therefore it is appropriate to repeate the measurement sixteen times, every time with a different $q$. Let the frequency response of the $r^{th}$ measurement be $G_r(j\omega)$, then the magnitude and phase of the average frequency response is given by

$$
\begin{aligned}
20\log_{10}\left|G\left(j\frac{k}{N}\right)\right| &= \frac{1}{16}\sum_{r=1}^{16} 20\log_{10}\left|G_r\left(j\frac{k}{N}\right)\right| \\
\angle G\left(j\frac{k}{N}\right) &= \frac{1}{16}\sum_{r=1}^{16} \angle G_r\left(j\frac{k}{N}\right)
\end{aligned} \quad (13)
$$

A second order system was considered for measurement purpose. For phase calculations, the principal values were taken from 0° to −360°. Accordingly, the sum of the phases of the Fourier coefficients of the two sequences of the input is −360°. The influence of these principal values of phase is shown in Table 2. The table shows $5^{th}$ harmonic of the observed signal in the abscence of noise.

Since the period of an M-sequence is of the form $N = 2^n - 1$ ($n$ a positive integer and $N = 255$), the advantage of Fast Fourier Transform (FFT) was taken by factorizing $N = r_1 r_2 = 15\times 17 = 255$. The influence of noise was investigated by carrying out simulation at various $S/N$(signal to noise ratio) values. The results of simulation are shown in Figs. 2(a), 2(b) and 2(c).

Table 2 The variation in the phase angle of the input and output signals.

| Degree of Harmonics | Phase Angle (deg) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Sampl: 1<br>127 | Sampl: 7<br>31 | Sampl:11<br>61 | Sampl:13<br>47 | Sampl:19<br>59 | Sampl:23<br>29 | Sampl:37<br>91 | Sampl:43<br>52 |
| $\angle$Mqi$(j\frac{5}{127})$ | -150.9<br>-209.1 | -224.3<br>-135.7 | -310.2<br>-49.8 | -17.3<br>-342.7 | -291.5<br>-69.5 | -276.8<br>-83.2 | -188.7<br>-171.3 | -338.7<br>-21.3 |
| sum | -360.0 | -360.0 | -360.0 | -360.0 | -360.0 | -360.0 | -360.0 | -360.0 |
| $\angle$Yqi$(j\frac{5}{127})$ | -180.6<br>-238.8 | -254.0<br>-165.4 | -339.9<br>-79.5 | -46.9<br>-12.5 | -321.1<br>-98.3 | -306.5<br>-112.9 | -218.4<br>-201.0 | -8.4<br>-51.0 |
| sum | -419.4 | -419.4 | -419.4 | -59.4 | -419.4 | -419.4 | -419.4 | -59.4 |
| corrected $\angle$Yqi$(j\frac{5}{127})$ | -419.4 | -419.4 | -419.4 | -419.4 | -419.4 | -419.4 | -419.4 | -419.4 |
| $\angle$G$(j\frac{5}{127})$ | -29.7 | -29.7 | -29.7 | -29.7 | -29.7 | -29.7 | -29.7 | -29.7 |

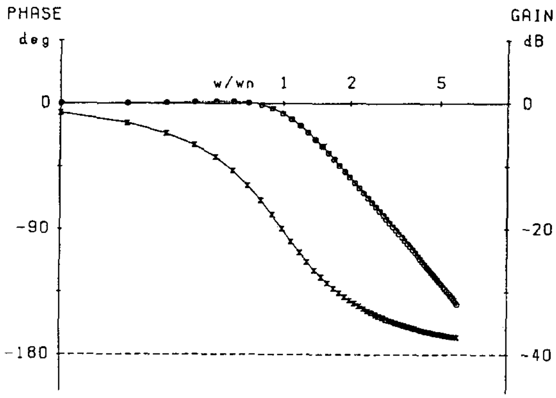


Fig.2(a). Frequency Response of a second order system in the absence of noise.

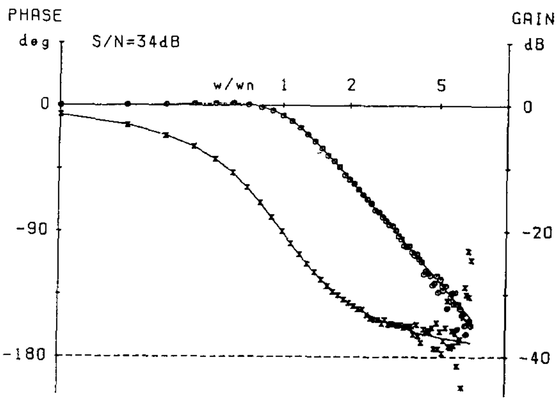


Fig.2(b). Frequency Response of a second order system with $S/N = 34\,\text{dB}$.

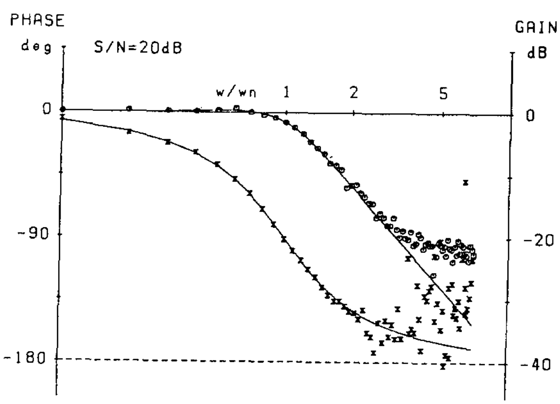


Fig.2(c). Frequency Response of a second order system with $S/N = 20\,\text{dB}$.

**4. Comparision with Correlation Method :** FFT alogorithm for an arbitrary sequence of length $N = r_1 r_2$ requires $N(r_1 + r_2)$ complex multiplications or $4N(r_1 + r_2)$ real multiplications. Since the computation time is directly proportional to the number of multiplications, for $N = 255 = r_1 r_2 = 17 \times 15$, a comparision of the number of multiplications required for FFT and that for cross-correlation gives

$$\frac{N^2}{4N(r_1 + r_2)} = \frac{255}{4 \times 32} \approx 2 \tag{14}$$

For $N = 255$, the number of multiplications for the proposed method is approximately half of that for cross-correlation method.

**5. Conclusion :** The phase of the Fourier coefficients of a characteristic M-sequence has a constant value that is determined by the characteristic polynomial. The sum of phases of a pair of M-sequences is zero, if the sequences are obtained by sampling every $q_1^{th}$ and $q_2^{th}$ element of another characteristic M-sequence, where $q_1$ and $q_2$ are both coprime with N.

A pair of M-sequences with the above mentioned properties was used to measure the frequency response function of a control system. Since it is not required to obtain the Fourier coefficients of the input sequence, the measurement is simplified in the proposed method and consequently the computation time is reduced. Finally, a comparasion with cross-correlation method is made on the basis of computation time.

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