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A Canonical Structure for Nonlinear Observers

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Abstract: In order to construct a nonlinear observer, change of coordinate system is necessary. However, as in the case of feedback linearizable system it is not easy to obtain a coordinate transformation map. In this paper, a canonical structure is proposed for observable systems with an objective of finding a vector field which is necessary for the generation of a new coordinate system.

1 Intorduction

Following the work of Krener and Isidori[1] the theory of nonlinear observer has been developed. Bestle and Zeitz[2] dealt the problem of constructing an observer for nonlinear time-varying systems and Krener & Respondek[3] and Xia & Gao[5] extended their work to the design of an observer for multi-output systems. Zeitz[6] constructed an observer for a system with an input as an extended Luenberger observer type, while the comparison study was performed by by Walcott and Zak[4].

In this paper, we consider the conditions of transformability into an observer form for practicality. We propose a canonical structure which is useful as an intermediate step to a complete observer structure.

2 Problem Statement

Consider a single-output system

$$\dot{x} = f(x), \tag{1}$$

$$y = h(x), \tag{2}$$

where $x \in M$ and M is a smooth ($C^\infty - differentiable$) n-dimensional manifold. Let the equilibrium point of the system (1) be denoted by x_e. We say that the system (1),(2) is observable if there exists a coordinate transformation map $T : M \rightarrow R^n$ such that in the new coordinate $\xi = T(x)$ the system is represented by

$$\dot{\xi} = A\xi + \psi(y) \tag{3}$$

$$y = c\xi, \tag{4}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad c = [1, 0, \cdots, 0],$$

$\psi \equiv [\psi_1, \cdots, \psi_n]^T$ is a vector of scalar functions of y.

Once the system is transformed into the nonlinear observer form (3),(4) then an observer can be constructed in such a way that for a matrix D

$$\dot{z} = Az + \psi(y) + D(cz - y) \tag{5}$$

Then, the error $e(t) \equiv z(t) - \xi(t)$ vanishes according to

$$\dot{e} = (Dc + A)e,$$

if the matrix $Dc + A$ is stable. Thus, one can find the state x of the original system (1) through the inverse coordinate transformation map $x = T^{-1}(\xi)$.

The problems concerning nonlinear observers are how to characterize the observable nonlinear systems and how to obtain a coordinate transformation map into the observable form. For the first question, Krener and Isidori [1] obtained an answer as follows:

Theorem 1; The system (1),(2) is observable if and only if the vector field g defined by

$$L_g L_f^k(h) = \begin{cases} 0, & 0 \le k < n-1 \\ 1, & k = n-1 \end{cases} \tag{6}$$

satisfies

$$[g, ad_f^k g] = 0, \quad k = 1, 3, ..., 2n-3. \tag{7}$$

Remark: From the Jacobi identity, the equivalent statement to (7) is that

$$[ad_f^i g, ad_f^j g] = 0, \quad 0 \le i, j \le n-1. \tag{8}$$

It is worthwhile to repeat the proof of *Theorem 1* here.

Proof: (Suffiency) Since the n vector fields $\{g,\ ad_f g,\ ...,\ ad_f^{n-1}g\}$ are commutative, one can choose coordinate system ξ such that for $k = 0, ..., n-1$

$$\frac{\partial}{\partial \xi_{n-k}} = (-1)^k ad_f^k g. \qquad (9)$$

Then by (6),

$$\frac{\partial h}{\partial \xi_{n-k}} = (-1)^k L_{ad_f^k g} h = \sum_{i=1}^{k} (-1)^i \binom{k}{i} L_f^i L_g L_f^{k-i}(h)$$

$$= \begin{cases} 0, & 0 \le k \le n-2, \\ 1, & k = n-1. \end{cases}$$

Hence, we obtain

$$y = c\xi.$$

If we let

$$f = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial \xi_i},$$

then

$$\left[f,\ \frac{\partial}{\partial \xi_{n-k}}\right] = -\sum_{i=1}^{n} \frac{\partial f_i}{\partial \xi_{n-k}} \frac{\partial}{\partial \xi_i}. \qquad (10)$$

On the other hand, it follows from the definition that

$$\left[f,\ \frac{\partial}{\partial \xi_{n-k}}\right] = \left[f,\ (-1)^k ad_f^k g\right]$$

$$= (-1)^k ad_f^{k+1} g = -\frac{\partial}{\partial \xi_{n-k-1}}. \qquad (11)$$

Thus,from (10) and (11), we obtain for $0 \le k < n-1$,

$$\frac{\partial f_i}{\partial \xi_{n-k}} = \begin{cases} 1, & i = n-k-1, \\ 0, & otherwise. \end{cases} \qquad (12)$$

Since (12) holds for each of ξ in a neighborhood of 0, f_i , $1 \le i \le n$ are linear functions of ξ_i, $2 \le i \le n$, but may be nonlinear functions of ξ_1. Hence the form (3),(4) is obtained. Necessity is omitted. Q.E.D.

Remark: Since $\{g,\ ad_f^g, \cdots, ad_f^{n-1}g\}$ form a commutative basis for a tangent space TM of a n-dimensional manifold M, there exists a set of scalar functions $a_i(\xi)$, $1 \le i \le n$ such that

$$ad_f^n g = \sum_{i=1}^{n} a_i(\xi) ad_f^{n-i} g. \qquad (13)$$

However, since

$$ad_f^n g = [f,\ ad_f^{n-1}g] \qquad (14)$$

$$= \left[\sum_{i=1}^{n} f_i \frac{\partial}{\partial \xi_i},\ \frac{\partial}{\partial \xi_1}\right] = \sum_{i=1}^{n} \frac{\partial f_i}{\partial \xi_1} \frac{\partial}{\partial \xi_i}, \qquad (15)$$

we obtain that

$$a_i(\xi) = \frac{\partial f_i}{\partial \xi_1}.$$

Furthermore, since f_i, $1 \le i \le n$ are linear functions of $\xi_2, \cdots, \xi_n$, a_i, $1 \le i \le n$ are functions of only ξ_1. Further, a_i is related to ψ_i in such a way that

$$\psi_i(y) = \int_{y_0}^{y} a_i(s) ds + y_0, \qquad for\ \ 1 \le i \le n, \qquad (16)$$

where the y_0 is an initial value of y.

3 Main Result

In this section we consider the problem of obtaining a coordinate transformation map that yields a nonlinear observer form. The first step must be to find the vector field g which satisfies the condition(6). However, for an arbitrary observable system it may be hard even to find such a vector field g. For this problem, we propose the following coordinate transformation map $T: M \rightarrow R^n$,

$$w = T(x) \equiv \begin{bmatrix} h \\ L_f h \\ \vdots \\ L_f^{n-1} h \end{bmatrix}. \qquad (17)$$

Lemma 1: The coordinate transformation map $w = T(x)$ transforms the system (1),(2) into a canonical form

$$\dot{w} = Aw + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \gamma(w) \end{bmatrix} \equiv \tilde{f}(w), \qquad (18)$$

$$y = cw \equiv \tilde{h}(w) \qquad (19)$$

where $\gamma(w) = L_f^n h(T^{-1}(w))$.

Proof:

$$\dot{w} = L_f w = Aw + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ L_f^n h(T^{-1}(w)) \end{bmatrix},$$

$$and \qquad dy = dh DT^{-1} = dh \begin{bmatrix} dh \\ \vdots \\ dL_f^{n-1} h \end{bmatrix}^{-1}$$

$$= [1, 0, \cdots, 0]. \qquad Q.E.D.$$

Remark: The necessary and sufficient condition for the system (1),(2) to be locally observable is that the rank of the observability matrix is equal to n at the equilibrium point x_e, i.e.,

$$rank \begin{bmatrix} dh \\ L_f dh \\ \vdots \\ L_f^{n-1} dh \end{bmatrix} (x_c) = n. \quad (20)$$

The existence of the diffeomorphism (17) follows from the full rank condition *(observability condition)* (20). Hence, each *(locally)* observable system, can be transformed into the canonical form (18),(19) through coordinate change (17).

The advantage of this coordinate change can be seen from the following observability matrix:

$$\begin{bmatrix} d\tilde{h} \\ L_{\tilde{f}} d\tilde{h} \\ \vdots \\ L_{\tilde{f}}^{n-1} d\tilde{h} \end{bmatrix} = I, \quad (21)$$

where I denotes the $n \times n$ dimensional identity matrix. Hence, the vector field g satisfying

$$L_g L_{\tilde{f}}^k \tilde{h} = \begin{cases} 0, & 0 \le k \le n-2 \\ 1, & k = n-1 \end{cases} \quad (22)$$

is directly obtainable, i.e., $g = [0, \cdots, 0, 1]^T$.

Summarizing the above statement, we can say as follows: through the change of coordinate (17), all the locally observable system can be transformed into a canonical form (18),(19), for which the vector field g satisfying (22) is given by $[0, \cdots, 0, 1]^T$. That is, through the available coordinate transformation map (17) each locally observable system can be transformed into the canonical form (18),(19), which looks useful in the sense that the vector field g is obtained straightforwardly.

Example: Consider the following Van der Pole oscillator with a nonlinear output equation:

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= 1.2x_1 + 10x_2 - x_2^3 \\ y &= e^{x_1} - 1 \end{aligned}$$

Then, through the coordinate transformation

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = T(x) = \begin{bmatrix} y \\ L_f y \end{bmatrix} = \begin{bmatrix} e^{x_1} - 1 \\ -x_2 e^{x_1} \end{bmatrix}, \quad (23)$$

we obtain the canonical form:

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ L_f^2 h \end{bmatrix}, \quad (24)$$

$$y = w_1, \quad (25)$$

where

$$L_f^2 h = e^{x_1}(x_2^2 - 1.2x_1 - 10x_2 + x_2^3) = \frac{w_2^2}{(w_1+1)} - 1.2(w_1+1)ln(w_1+1) + 10w_2 + \frac{w_2^3}{(w_1+1)^2}.$$

For the system (23), the vector field g satisfying (22) is $[0, 1]^T$. However, since the commutativity of the vector fields g and $ad_{\tilde{f}} g$ fails, the coordinate transformation map changing the system (23) into the nonlinear observer form does not exist.

4 Conclusion

In this work, a canonical form for an observable nonlinear system is postulated with an objective of obtaining the vector field g which is necessary for the coordinate change for an observer form. Further, one can obtain the corresponding coordinate transformation map directly from the vector field f and the output equation h.Hence, this canonical form looks useful as an intermediate step to a complete coordinate change to an observer form. The commutativity condition among the vector fields g, $ad_f g$, $\cdots$, $ad_f^{n-1} g$ looks very strong. Hence, weakening this condition may be an interesting topic for a future research.

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