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On the singularity of the matrix sign function algorithm

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Abstract: Some properties of a matrix containing at least one pair of purely imaginary eigenvalues in the matrix sign function algorithm are explicated. It is shown that such a nonsingular matrix can be end up a singular matrix in the matrix sign function algorithm independently of the matrix condition. The result can be utilized to identify and locate all the eigenvalues theoretically.

1. Introduction

Matrix sign function algorithm has been used widely in the various systems engineering fields[1-5]. The standard matrix sign function algorithm proposed by Roberts[1] is represented by the following recursive equation

$$S_{k+1} = \frac{1}{2}(S_k + S_k^{-1}) \tag{1}$$

with initial value $S_0=A$. Then this algorithm can compute $sign(A)$. Roberts[1] suggested that convergence of the standard algorithm (1) can be improved by using the recursive equation

$$S_{k+1} = \alpha_k S_k + \beta_k S_k^{-1} \tag{2}$$

with suitably selected scalars α_k and β_k. Balzer[6] generalized the selection method of α_k and β_k under the constraints that

$$\alpha_k + \beta_k = 1 \quad \text{and} \quad \lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = \frac{1}{2}$$

and proposed the optimal values.

It is easy to show that this algorithm is satisfactory for complex and repeated eigenvalues. However, the algorithm is undefined in the case where a matrix A has either zero or purely imaginary eigenvalues. Since a zero or a pair of purely imaginary eigenvalues may not guarantee the algorithm convergence, a property of the matrix sign function algorithm for the matrix having these eigenvalues should be clarified. It can be demonstrated that the matrix sign function algorithm results a singular matrix though an S_0 is not singular, if the matrix contains the purely imaginary eigenvalues. Such singularities are independent of the condition of the matrix A.

2. Main Result

A singular matrix can be easily identified in the matrix sign function algorithm because its inverse matrix does not exist. A matrix having at least one purely imaginary eigenvalues can also be easily identified in the matrix sign function algorithm due to the methods which will be stated in this section.

Lemma 2-1. If an $n \times n$ matrix A has k eigenvalue pairs $\pm jm$, then $\det(A^2+m^2I)=0$.

Proof. Assume that A has an eigenvalue pair $\pm jm$. Then its pseudo-Jordan canonical form $M^{-1}AM$ is given by

$$M^{-1}AM = \begin{bmatrix} A_c & 0 \\ 0 & J_c \end{bmatrix},$$

where J_c is an $(n-2)\times(n-2)$ Jordan canonical form and

$$A_c = \begin{bmatrix} 0 & m \\ -m & 0 \end{bmatrix}. \tag{3}$$

Now,

$$M^{-1}(A^2+m^2I)M = M^{-1}\begin{bmatrix} A_c^2+m^2I & 0 \\ 0 & J_c^2+m^2I \end{bmatrix}M,$$

where $A_c^2+m^2I = 0$. Thus, $\det(A^2+m^2I)=0$.

It equally holds for the matrix having k repeated eigenvalue pairs $\pm jm$. It completes the proof. Q.E.D.

Theorem 2-1. A nonsingular matrix A has at least one eigenvalue pair $\pm jm$ if and only if $(A + m^2 A^{-1})$ is singular.

Proof. (*If part*) Let Λ be a pseudo-Jordan canonical form of A. Assume that A has at least one real eigenvalue a. Then, $(\Lambda+m^2\Lambda^{-1})$ has an entry $(a^2+m^2)/a$, which cannot be zero unless $a=0$. Since A is nonsingular by the assumption, $a\neq 0$.

Assume that A has an eigenvalue pair $a\pm jb$. Then Λ has a following pseudo-Jordan block of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Thus, $(\Lambda+m^2\Lambda^{-1})$ has a following pseudo-Jordan block

$$\begin{bmatrix} a+m^2\dfrac{a}{a^2+b^2} & b-m^2\dfrac{b}{a^2+b^2} \\ -b+m^2\dfrac{b}{a^2+b^2} & a+m^2\dfrac{a}{a^2+b^2} \end{bmatrix}$$

The above pseudo-Jordan block becomes singular if the following relation

$$(a+m^2\frac{a}{a^2+b^2})^2+(b-m^2\frac{b}{a^2+b^2})^2=0$$

holds. The pseudo-Jordan block of $(\Lambda+m^2\Lambda^{-1})$ becomes singular only when $a=0$ and $b=m$ since A is nonsingular. It equally holds for the matrix having repeated eigenvalue pairs $\pm jm$, too.

(*Only if part*) Since A has at least one eigenvalue pair $\pm jm$, its characteristic polynomial $p(s)$ is of the form

$$p(s) = (s^2 + m^2)\, q(s).$$

By Cayley-Hamilton Theorem,

$$p(A) = (A^2 + m^2 I)\, q(A) = 0.$$

From Lemma 2-1, $\det(A^2+m^2I)=0$. Since A is nonsingular, we have

$$\det(A+m^2A^{-1})=0.$$

It equally holds for the matrix having repeated eigenvalue pairs $\pm jm$, too. It completes the proof. Q.E.D.

The above Theorem 2-1 states that although a matrix A is not singular, S_1 can be singular. That is to say, a nonsingular matrix having at least one purely imaginary eigenvalue pair $\pm j1$ in the standard matrix sign function algorithm (1) or $\pm j\sqrt{\beta_0/\alpha_0}$ in the accelerated matrix sign function algorithm (2) generates a singular matrix S_1.

Assume that A_c has a pair of purely imaginary eigenvalue pair $\pm jm_k$ having the form (3) such that

$$A_c = \begin{bmatrix} 0 & m_0 \\ -m_0 & 0 \end{bmatrix}.$$

Then the standard matrix sign function algorithm (1) for the A_c is of the form

$$S_{k+1} = \frac{1}{2}\begin{bmatrix} 0 & m_k - \dfrac{1}{m_k} \\ -m_k + \dfrac{1}{m_k} & 0 \end{bmatrix}, \quad S_0 = A_c.$$

Thus,

$$m_{k+1} = \frac{1}{2}(m_k - m_k^{-1}). \tag{4}$$

If $m_k=\pm 1$, then $m_{k+1}=0$ from (4). That is to say, if a matrix A_c has an eigenvalue pair $\pm j1$, then S_1 of (1) becomes singular. A sequence w_k that drives m_k to be zero at the very $(k+1)$th step can be identified from the inverse mapping of (4) such that

$$w_{k+1} = w_k \pm \sqrt{u_k}, \quad w_0 = \pm 1 \tag{5}$$

with $u_k = w_k^2 + 1$. Thus S_k with $S_0 = A_c$ having at least one pair of purely imaginary eigenvalues $\pm jw_k$ obtained from the above recursive equation (5) becomes singular at the very $(k+1)$th step in the standard matrix sign function algorithm (1).

Similarly, the accelerated matrix sign function algorithm (2) for the A_c is of the form

$$S_{k+1} = \frac{1}{2}\begin{bmatrix} 0 & \alpha_k m_k - \dfrac{\beta_k}{m_k} \\ -\alpha_k m_k + \dfrac{\beta_k}{m_k} & 0 \end{bmatrix}, \quad S_0 = A_c.$$

Thus,

$$m_{k+1} = \frac{1}{2}(\alpha_k m_k - \beta_k m_k^{-1}). \tag{6}$$

If $m_k=\pm\sqrt{\beta_0/\alpha_0}$, then $m_{k+1}=0$ from (6). That is to say, if a matrix A_c has an eigenvalue pair $\pm j\sqrt{\beta_0/\alpha_0}$, then S_1 of (1) becomes singular. A sequence w_k that drives m_k to be zero at the very $(k+1)$th step can be identified from the inverse mapping of (4) such that

$$w_{k+1} = (w_k \pm \sqrt{u_k})\alpha_k^{-1}, \quad w_0 = \pm\sqrt{\beta_0/\alpha_0} \tag{7}$$

with $u_k = w_k^2 + \alpha_k\beta_k$. Thus S_k with $S_0 = A_c$ having at least one pair of purely imaginary eigenvalues $\pm jw_k$ obtained from the above recursive equation (7) becomes singular at the very $(k+1)$th step in the accelerated matrix sign function algorithm (2).

The A_c that makes S_k to be singular at the very $(k+1)$th step is not unique. Since the m_k's have two values in the equations (5) and (7), the number of m_k's that drives S_k into a singular matrix is 2^{k+1}. At any rate, the purely imaginary eigenvalue pair can be identified and located at the $(k+1)$th step in the matrix sign function algorithms (1) and (2) or from the Theorem 2-1. It can be applied to identify and locate the ordinary eigenvalue pair $\lambda \pm jm_k$ theoretically by shifting the original eigenvalue pair $\pm jm_k$ by λ such that

$$A_c + \lambda I$$

where λ is a known scalar.

3. Conclusion

Some properties concerning the purely imaginary eigenvalues in the matrix sign function algorithm have been explicated. It should be mentioned that a nonsingular matrix can generate a singular matrix in the matrix sign function algorithm independently of the matrix condition. That is to say, even a well-conditioned matrix can generate a singular matrix in the matrix sign function algorithms (1) and (2). These properties can be used to identify and locate all the eigenvalues theoretically.

References

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