THE DELAY MARGIN OF THE LQG REGULATOR

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ABSTRACT In this paper, the delay margins of the LQ and the LQG regulators are obtained in the time domain. These margins are represented in terms of the singular values of system matrices and the solutions of a Riccati equation and a Lyapunov one. And their asymptotical properties when gains tend to infinity are investigated.

1. INTRODUCTION

In recent year, great attention has been paid to the research into stability of the time delay system and a series of effective stability criteria and stability test methods have been presented [1 - 7]. This is natural because delay elements always exist in practical control systems, the stability problem of the delay systems is very important in practice. The cases that control systems have time delay are roughly divided into four. The first case is that the original system has delay elements, so its model is expressed by a delay differential equation and the designed compensator has delay elements or memoryless ones. The second one is that the high order plant is modeled as a lower order model and a delay term for easy manipulation and the delay term is neglected in many cases. The major part of the research about delay systems is concerned with the systems of above two type. The third one is that the plant is presented by an ordinary differential equation but in the compensator the delayed information is used to improve the performance of the closed loop system. Research about this system is started in recent by a few researcher[8]. The last one is that both the plant and the compensator are described by ordinary differential equations but the computation time is considered. Research about this case is rarely found in literatures but it must be performed because the computation time is unavoidable whenever compensators are practically implemented and the stability of the real control systems may be affected by it. In this paper, we are concerned with the effects of the neglected delay term and the computation time delay in the stability of the LQ and the LQG regulators.

In all cases, the closed loop systems can be described by a linear differential equation

$$\dot{x}(t) = A_1 x(t) + B_1 x(t-h) \qquad (1.1)$$

with $h > 0$ and $x(t) \in R^n$. To analyze the stability properties of this equation, there are three methods : graphical techniques[5,9], the second method of Lyapunov[10,11], and solving a transcendental equation or its transformed one using the approximation of exp(-sh)[1,2,3,4,6, 7,10]. Each of these methods have serious limitation : the graphical methods are rather timeconsuming and typically require access to digital computers with program for plotting; the Lyapunov approach gives only sufficient conditions for stability, and for systems with large dimension the transcendental characteristic equation and its transformation are difficult to be solved except in a few special case[3]. The obtained results using above methods are roughly classified into three groups : stability conditions independent of delay h [1,2,5,6,7,9], stability test methods for given h [4,11], and the estimation of maximum h not to destabilize a closed loop system [1,3,9,10]. In this paper, we are concerned with the results of third group, to say in detail, the computation of the maximum bound (delay margin) of computation time delay or neglected delay for model reduction to guarantee the stability of the closed loop LQ and LQG regulators. Using the Nyquist plot or the characteristic root loci or Thowsen's method, we can obtain this bound exactly if design parameters (weighting and/or covariance matrices) are given. But the calculation is difficult and the obtained bound is not explicitly related to design parameters. Thus we use a Lyapunov approach to obtain a delay margin and to investigate its properties when gains tend to infinity. Halanay had obtained a delay margin using a Lyapunov approach but the delay margin is conservative to some extent[11]. In this paper, using the Razumikhin-type theorems [10] we obtain a less conservative delay margin than Halanay's one.

This paper is organized as follows. In section 2, the delay margins of the LQ and the LQG regulators are obtained in terms of the singular values of system matrices and the solutions of a Riccati equation and a Lyapunov one. Their asymptotical properties are investigated when the regulator gain or the Kalman filter gain approach infinity.

2. Main Results

We consider a linear time invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

$$y(t) = Cx(t) \tag{2.2}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ the control input vector, and $y(t) \in R^p$ the output vector, and A, B, and C are constant matrices with appropriate dimensions. We assume that (A,B) is a controllable pair, (C,A) an observable pair, and, without loss of generality, the number of control inputs is equal to that of outputs.

The control of the LQ regulator for system (2.1) is given by

$$u(t) = -R^{-1}B'Px(t) = -Kx(t) \tag{2.3}$$

where P satisfies the algebraic Riccati equation

$$A'P + PA + Q - PBR^{-1}B'P = 0 \tag{2.4}$$

and Q is a positive semi-definite matrix and R a positive definite diagonal matrix. We assume that it takes h second(s) to compute the control law (2.3). Then the real control law is represented by

$$u(t) = -Kx(t-h) \tag{2.5}$$

and this control is applied to the plant in practice. The following theorem states the stability of the new LQ regulator to take account into the computation time h.

Theorem 1 : The control (2.5) is applied to the system (2.1). The closed loop system is asymptotically stable if the following inequality holds.

$$h < \frac{1}{\sigma_1(P_cBKA) + \sigma_1(P_cBKBK)} = h_1 \tag{2.6}$$

where $\sigma_1(.)$ denotes the maximum singular value and P_c satisfies the following Lyapunov equation.

$$(A-BK)'P_c + P_c(A-BK) = -2I_n \tag{2.7}$$

Proof : When the control (2.5) is applied to the system (2.1), the closed loop system is given by

$$\dot{x}(t) = (A-BK)x(t) + BKx(t-h) \tag{2.8}$$

By Razumikhin-type theorems, we only consider initial data satisfying the following equation to confirm the stability of the system (2.8).

$$\|x(t+\theta)\| \leq \|x(t)\| \text{ for all } \theta \in [-2h,0] \tag{2.9}$$

where $\|.\|$ denotes the Euclidean norm. We consider a candidate for the Lyapunov function as

$$V(x(t)) = x'(t)P_cx(t) \tag{2.10}$$

Then

$$\begin{aligned}
\dot{V} &= -2x'(t)x(t)+2x'(t)P_cBK(x(t)-x(t-h)) \\
&= -2x'(t)x(t)+2x'(t)P_cBK\int_{t-h}^{t}\dot{x}(\sigma)d\sigma \\
&= -2x'(t)x(t)+2x'(t)P_cBKA\int_{t-h}^{t}x(\sigma)d\sigma \\
&\quad + 2x'(t)P_cBKBK\int_{t-h}^{t}x(\sigma-h)d\sigma \\
&= -2(\|x(t)\|^2+\sigma_1(P_cBKA)\|x(t)\|\int_{t-h}^{t}\|x(\sigma)\|d\sigma \\
&\quad +\sigma_1(P_cBKBK)\|x(t)\|\int_{t-h}^{t}\|x(\sigma-h)\|d\sigma \\
&\leq -2\|x(t)\|^2(1-\sigma_1(P_cBKA)h-\sigma_1(P_cBKBK)h)
\end{aligned}$$

Therefore $\dot{V}$ is negative if the inequality (2.6) holds.

Q.E.D.

From above results, we know that the larger is the gain, the smaller is h_1. In equation (2.7), P_cBK converges to a finite constant matrix as K approaches infinity if the condition number of A-BK is finite[12]. $\sigma_1(P_cBKA)$ is finite but $\sigma_1(P_cBKBK)$ tends to infinity. h_1 approaches zero. So it may be necessary to take account into the computation time of the control algorithm when the feedback gain is large.

We consider the LQG regulators where the controller consists of the Kalman filter and the LQ regulator with gain matrix K. The Kalman filter is found from the linear minimum variance state estimation problem for the system (2.1) and (2.2) with state and measurement noise signal vector w(t) and v(t) whose covariance matrices are Q_o and R_o respectively. It is expressed as follows.

$$\dot{\hat{x}}(t)=A\hat{x}(t)+Bu(t)+F(y(t)-C\hat{x}(t)) \quad (2.11)$$

$$F = \Sigma C' R_o^{-1} \quad (2.12)$$

$$A\Sigma + \Sigma A' + Q_o - \Sigma C' R_o^{-1} C\Sigma = 0 \quad (2.13)$$

where $(A, Q_o^{1/2})$ is controllable. The LQ regulator with gain matrix K is given by equation (2.3). If we consider the computation time of the control algorithm, the control is expressed by.

$$u(t) = -Kx(t-h) \quad (2.14)$$

We obtain a new equation by rearranging (2.1), (2.2), (2.11), and (2.14) and by defining error signal vector $e(t) = x(t) - \hat{x}(t)$.

$$\begin{bmatrix}\dot{x}(t))\\ \dot{e}(t))\end{bmatrix} = \begin{bmatrix}A-BK & BK\\ 0 & A-FC\end{bmatrix}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix} + \begin{bmatrix}BK & -BK\\ BK & -bK\end{bmatrix}\begin{bmatrix}x(t)-x(t-h)\\ e(t)-e(t-h)\end{bmatrix} \quad (2.15)$$

We define a candidate for the Lyapunov function as

$$V(x(t),e(t)) = \begin{bmatrix}x(t)\\ e(t)\end{bmatrix}'\begin{bmatrix}\alpha P_c & 0\\ 0 & P_o\end{bmatrix}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix} \quad (2.16)$$

where P_c satisfies the Lyapunov equation (2.7) and P_o does following one

$$(A-FC)'P_o + P_o(A-FC) = -2I_n \quad (2.17)$$

and α is a positive real constant.

Now we can state the following theorem with respect to the stability of the system (2.15).

Theorem 2 : The closed loop LQG system (2.15) is asymptotically stable if the following inequality holds.

$$h < \frac{\sigma_n(S)}{\sigma_1(M)} = h_2 \quad (2.18)$$

where $\sigma_n(\cdot)$ denotes the minimum singular value and

$$M = \begin{bmatrix}\alpha P_c Bk(A-BK) & -\alpha P_c BK(A-BK-FC)\\ P_o BK(A-BK) & -P_o BK(A-BK-FC)\end{bmatrix}$$

$$= \begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}, \quad S = \begin{bmatrix}\alpha I_n & -\alpha P_c BK/2\\ -\alpha(P_c BK)'/2 & I_n\end{bmatrix}$$

and $\alpha \in (0,\ 4/\sigma_1(P_c BKK'B'P_c))$ to be chosen to maximize h_2.

Proof :

$$\dot{V} = \begin{bmatrix}x(t)\\ e(t)\end{bmatrix}'\begin{bmatrix}-2\alpha I_n & \alpha P_c BK\\ \alpha(P_c BK)' & -2I_n\end{bmatrix}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}$$

$$+ \begin{bmatrix}x(t)\\ e(t)\end{bmatrix}'\begin{bmatrix}\alpha P_c & 0\\ 0 & P_o\end{bmatrix}\begin{bmatrix}BK & -BK\\ BK & -BK\end{bmatrix}\int_{t-h}^{t}\begin{bmatrix}\dot{x}(\sigma)\\ \dot{e}(\sigma)\end{bmatrix}d\sigma$$

$$= -2\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}' S\begin{bmatrix}x(t)\\ e(t)\end{bmatrix} + 2\begin{bmatrix}x(t)\\ e(t)\end{bmatrix} M\int_{t-h}^{t}\begin{bmatrix}x(\sigma)\\ e(\sigma)\end{bmatrix}d\sigma$$

We only consider initial data satisfying the following equation like theorem 1.

$$\left\|\begin{bmatrix}x(t-\theta)\\ e(t-\theta)\end{bmatrix}\right\| \le \left\|\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}\right\| \quad \text{for all } \theta \in [h,0] \quad (2.19)$$

Then

$$\dot{V} \le -2\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}^2\sigma_n(S) + 2\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}^2\sigma_1(M)h$$

and $\dot{V} < 0$ if $h < \sigma_n(S)/\sigma_1(M)$

Q.E.D.

Remark 1. We consider the case that K tends to infinity and assume that $P_c BK$ converges to a constant matrix. Then M_{11} and M_{22} converge, but M_{21} and M_{22} tends to infinite. The delay margin becomes very small since $\sigma_1(M)$ approaches infinite and $\sigma_n(s)$ is the order of α. Thus, in this case, the computation time of the algorithm may be a critical factor on the stability of the LQG regulator.

Remark 2. We consider the case that (A,B,C) is a minimum phase system and $Q_o = BB'$, $R_o = I/\mu$ and $\mu \to \infty$ (Loop Transfer recovery method adjusting filter gain). Then $F \to \sqrt{\mu}BW$ where W is an unitary matrix. If the condition number of the eigen vector matrix of (A-FC) is finite for all $\mu>0$, P_0B -> 0. So M becomes a finite matrix since M_{21} -> 0, M_{22} -> a finite matrix and M_{12} is made finite by adjusting α. From this fact, it is known that the delay margin of the LQG regulator converges to nonzero constant when the loop transfer recovery method adjusting filter gain is applied.

The obtained bounds h_1 and h_2 are conservative but they are used in selecting the design parameters. We can obtain the exact bounds using graphical methods (Nyquist plot and Characteristic root loci) and Thowsen's method but these methods are very complex and difficult and the obtained bounds do not show the effects of the design parameters.

3. Conclusion

In this paper, the delay margins of the LQ and the LQG regulator are obtained using the second method of Lyapunov and the Razumikhin-type theorems. It is shown that the LQ and the LQG regulator may become unstable due to the computation time of the control algorithm when the regulator gain is very large. It is also shown that the delay margin of the LQG regulator is not necessarily small when the filter gain become large under some assumptions.

Obtained bounds in this paper are conservative to some extent but they can be used in selecting design parameters since they are related to the regulator and the filter gains. The method to obtain less conservative bounds needs to be investigated.

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