비선형 이산 시간 시스템 $x_{k+1} = G_{u_k} \circ F(x_k)$ 의 선형화에 관하여

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On the Linearization of the Discrete-time Nonlinear Systems, $x_{k+1} = G_{u_k} \circ F(x_k)$

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Abstract

We investigate the feedback linearizability of nonlinear discrete-time systems of a specific form, $x_k = G_{u_k} \circ F(x_k)$, where F is a diffeomorphism and $\{G_{u_k}\}$ forms an one parameter group of diffeomorphisms. This structure represents a class of systems which are state equivalent to linear ones and approximates the sampled data model of a continuous-time system. It is also considered a relationship between linearizability and discretization.

1. Introduction

Linearization of nonlinear systems has been one of the most interesting research topic in the nonlinear systems analysis. Recently, there has been a considerable development in the study of linearization of discrete-time systems [3,6,7], as well as of continuous-time systems [2,4,5,8]. Grizzle [3] and Lee & Marcus [6] derived in an implicit manner the necessary and sufficient conditions for the feedback linearizability of the general discrete model,

$$x_{k+1} = H(x_k, u_k). \qquad (1.1)$$

In the case of continuous-time affine systems, the characterization of linearizability is given by in terms of vector fields. Correspondingly, if the dynamics of a discrete-time system are represented by a composition of diffeomorphisms, then the linearizability may be characterized by the properties of diffeomorphisms. Hence, it may be interesting to study the linerizability of the system (1.1) when the map $H : M \times R \to M$ has a certain structure. In this work, we investigate the linearizability of the discrete model

$$x_{k+1} = G_{u_k} \circ F(x_k), \quad x_k \in M, \quad u_k \in R, \qquad (1.2)$$

where F is a diffeomorphism from M to M and $\{G_u\}_{u \in R}$ is an one-parameter group of diffeomorphisms of M.

In practice, most discrete-time systems arise by sampling continuous-time systems. Hence, in the aspect of applicability, the validity of a discrete model may be determined by its proximity to the sampled data model of a continuous-time system. It will be shown later that the discrete model (1.2) is reasonably good in the sense that its trajectory converges to that of a continuous-time affine system. Notice that the system structure like (1.2) is more general than that of the form $x_{k+1} = f(x_k) + u_k g(x_k)$, since this system belongs to the class of systems of the form (1.2). Finally, we also investigate the effects of sampling on the linearizability.

2. Preliminaries and Definitions

Let M be a smooth (C^∞*-differentiable*) n-dimensional manifold. $C^r(M, R)$ denotes the space of r-times continuously differentiable functions from M to R. Let $T(M)$, $T^*(M)$ denote the tangent bundle of M and its dual space, respectively. Let $T_x M$ denote a tangent space at $x \in M$. Given $A(t, x) \in T_x M$ for each $t \in R$, we denote by $\Phi(A; t, t_0)p_0$ the solution $\phi(t) \in M$ of

$$\frac{d\phi}{dt} = A(t, \phi), \quad \phi(t_0) = p_0$$

and we often write $\Phi(A\,;t,0)p_0$ as $\Phi_t^A(p_0)$. We denote by $x^{(i)}$ the i-th component of a coordinate map x and by DF the jacobian of a map F.

Definition 2.1: Let $\sigma : U_1 \to U_2$ be a local homeomorphism of class at least C^1. For a vector field X over an open set V of M, we define $Ad_\sigma X$ to be a vector field on $\sigma(U_1 \cap V)$ such that

$$Ad_\sigma X(p) = D\sigma\big|_{\sigma^{-1}(p)} X(\sigma^{-1}(p)).$$

Note that for a vector field X on M,

$$\Phi_t^{Ad_\sigma X}(p) = \sigma \circ \Phi_t^X \circ \sigma^{-1}(p),$$

$$\left[Ad_\sigma^i X, Ad_\sigma^j X\right] = Ad_\sigma^i \left[X, Ad_\sigma^{j-i} X\right] \quad \text{for } j > i.$$

We utilize the following theorem later.

Theorem 2.1:[9] A C^∞ distribution Δ on M is involutive if and only if the ideal $I(\Delta)$ is a differential ideal, where

$$I(\Delta) = \{\omega \in T^*(M) : \omega \text{ annihilates } \Delta\}.$$

Consider a nonlinear discrete system

$$x_{k+1} = H(x_k, u_k) \tag{2.1}$$

where $x_k \in M$, $u_k \in R$, $H(\cdot,\cdot) : M \times R \to M$ is a smooth function of both arguments, and $H(x_e, 0) = x_e$.

Definition 2.2: The discrete system (2.1) is said to be *state equivalent to a linear system* if there exists a C^∞-diffeomorphism $T : M \to R^n$ which transforms the system (2.1) to a linear controllable system,

$$z_{k+1} = Az_k + bu_k \tag{2.2}$$

in $Im(T)$. If T is onto R^n, we say that (2.1) is *globally state equivalent to a linear system.*

Definition 2.3: The discrete system (2.1) is said to be *(locally) feedback equivalent to a linear system* if there exists a function $\alpha(\cdot,\cdot) \in C^\infty(M \times R, R)$, $\dfrac{\partial \alpha(x,v)}{\partial v} \neq 0$ for all x in a neighborhood U_{x_e} of x_e such that the closed loop system with $u_k = \alpha(x_k, v_k)$ is state equivalent to a linear system.

If a system is feedback equivalent to a linear system, then we call it a feedback linearizable system. According to the above definitions, a system is feedback linearizable if there exists a nonlinear feedback such that the closed loop system is linearizable by a coordinate transformation map. In other words, if the closed loop system for a certain feedback is differentially equivalent to a controllable linear system, then the system is feedback linearizable.

3. A Discrete-time System Model

Considering the linear system (2.2), one may observe that it consists of noncommuting two maps, $\psi(z) = Az$ and $\phi_{u_k}(z) = z + bu_k$. Therefore, one may express (2.2) equivalently as

$$z_{k+1} = \phi_{u_k} \circ \psi(z_k). \tag{3.1}$$

Note that the collection $\{\phi_u\}_{u \in R}$ forms an one-parameter group of diffeomorphisms of R^n, i.e.,

$$\phi_{u_1+u_2}(z) = \phi_{u_1} \circ \phi_{u_2}(z) \quad \text{and} \quad \phi_0(z) = z. \tag{3.2}$$

Suppose that the system (2.1) is state equivalent to the linear system (2.2) and $z_k = T(x_k)$ is the linearizing coordinate transformation map. Transforming the linear system (3.1) back by $x_k = T^{-1}(z_k)$, we obtain

$$x_{k+1} = T^{-1}(\phi_{u_k} \circ \psi(z_k)) = \tilde{\phi}_{u_k} \circ \tilde{\psi}(x_k), \tag{3.3}$$

where $\tilde{\phi}_{u_k} = T^{-1} \circ \phi_{u_k} \circ T$ and $\tilde{\psi} = T^{-1} \circ \psi \circ T$. Therefore, it follows from (3.1) and (3.3) that

$$H(x_k, u_k) = \tilde{\phi}_{u_k} \circ \tilde{\psi}(x_k). \tag{3.4}$$

Here, one can easily verify that if a diffeomorphism ϕ_{u_k} satisfying the group property (3.2) is equivalent to a diffeomorphism $\tilde{\phi}_{u_k}$, then $\tilde{\phi}_{u_k}$ also satisfies the group property. Thus, we may conclude that if a system is state equivalent to a linear system, then it consists of two diffeomorphisms: One of which depends on input, while the other does not. Further, the former forms an one-parameter group of diffeomorphisms for different values of the input. Based upon the previous observation, we restrict our concern to the systems of the form (1.3) in the study of linearizability.

The validity of the discrete model (1.3) also can be justified from the fact that the discrete-time system (1.3) approximates the sampled data model of a continuous-time system

$$\dot{x} = f(x) + ug(x), \quad x(t_0) = x_0. \tag{3.5}$$

The exact solution of (3.5) is given by

$$x(t) = \Phi(f\,; t, t_0) \circ \Phi(uAd_{\Phi(f;-\tau,t_0)}g\,; t, t_0)(x_0) \tag{3.6}$$

(see [1]). Notice that $uAd_{\Phi(f;-\tau,t_0)}g$ is a time-dependent vector fields on M. If $\bar{t}(= t - t_0)$ is sufficiently small, then fixing $\tau = t$, we can approximate $uAd_{\Phi(f;-\tau,t_0)}g$ for $\tau \in [t_0, t)$ by $u(t)Ad_{\Phi(f;-t,t_0)}g$ which is time-independent over the interval $[t_0, t)$. However, since

$$\begin{aligned}&\Phi(u(t)Ad_{\Phi(f;-t,t_0)}g\,; t, t_0)\\ &= \Phi(f\,; -t, t_0) \circ \Phi(u(t)g\,; t, t_0) \circ \Phi(f\,; t, t_0),\end{aligned} \tag{3.7}$$

the solution (3.6) can be approximated by

$$x(t) = \Phi(u(t)g; t, t_0)\circ\Phi(f; t, t_0)(x_0). \quad (3.8)$$

Thus, if the sampling period $\bar{t}$ is sufficiently small, the states $\{x_k\}_{k\in Z^+}$ of the system

$$x_{k+1} = \Phi_{\bar{t}}^{u_k g}\circ\Phi_{\bar{t}}^{f}(x_k), \quad (3.9)$$

where $u_k = u((k+1)\bar{t})$ for $k \in Z^+$, approximates the sampled points of the system (3.5). In other words, for any $t_f > t_0$ and piecewise continuous input $u(t)$,

$$\lim_{\bar{t}\to 0}\prod_{k=1}^{[t_f/\bar{t}]}\{\Phi_{\bar{t}}^{u_k g}\circ\Phi_{\bar{t}}^{f}\}(x_0)$$
$$= \Phi(f; t_f, t_0)\circ\Phi(uAd_{\Phi(f; -\tau, t_0)}g; t_f, t_0)(x_0), \quad (3.10)$$

where $[a]$ implies the largest integer less than a.

4. Main results

Consider the following system

$$x_{k+1} = G_{u_k}\circ F(x_k), \quad x_k \in M, \quad u_k \in R, \quad (4.1)$$

where $F \in C^\infty(M, M)$ is a diffeomorphism with x_e being a fixed point of the map F, and the collection $\{G_u\}_{u\in R}$ is an one-parameter group of diffeomorphisms of M. Then, locally we can express $G_{u_k}(x)$ as

$$G_{u_k}(x) = \Phi_{u_k}^{g}(x) = \Phi_1^{u_k g}(x), \quad (4.2)$$

where g is the infinitesimal generator of G_u.

Theorem 4.1 The system (4.1) is state equivalent to a linear system if and only if

i) $\{g, Ad_F g, \cdots, Ad_F^{n-1}g\}$ are linearly independent in a neighborhood of x_e,

ii) $\left[g, Ad_F^i g\right] = 0, \quad i = 1, 2, \cdots, n-1.$

Proof. Necessity is straightforward since a linear controllable system on R^n satisfies i) and ii).

(Sufficiency) We define a map $\Psi : R^n \times M \to M$ by

$$\Psi((\xi^{(1)},\dots,\xi^{(n)}), x) = \Phi_{\xi^{(n)}}^{Ad_F^{n-1}g}\circ\Phi_{\xi^{(n-1)}}^{Ad_F^{n-2}g}\circ\cdots\circ\Phi_{\xi^{(1)}}^{g}(x)$$

Let $\chi((\xi^{(1)},\dots,\xi^{(n)})) = \Psi((\xi^{(1)},\dots,\xi^{(n)}), x_e)$. It follows from ii) that for $1 \le j \le i \le n-1$,

$$\left[Ad_F^j g, Ad_F^i g\right] = Ad_F^j\left[g, Ad_F^{i-j}g\right] = 0. \quad (4.3)$$

Therefore, for a p in a neighborhood of 0 in R^n,

$$\frac{\partial\chi}{\partial\xi_i}(p) = Ad_F^{i-1}g(\chi(p)), \quad 1 \le i \le n. \quad (4.4)$$

From i), $rank D\chi(0) = n$. Thus, χ is a local diffeomorphism mapping 0 to x_e. Choosing χ^{-1} as a coordinate change, we obtain that

$$\xi_{k+1} = \chi^{-1}(x_{k+1}) = \chi^{-1}\circ G_{u_k}\circ\chi\circ\chi^{-1}\circ F\circ\chi(\xi_k). \quad (4.5)$$

By (4.3), there exist real constants $a_1, a_2, \dots, a_n$ such that $Ad_F^n g = \sum_{i=1}^{n} a_i Ad_F^{i-1}g$. Therefore,

$$D\chi^{-1}DFD\chi = (D\chi)^{-1}DFD\chi \quad (4.6)$$
$$= \left[Ad_F^{n-1}g, \dots, g\right]^{-1}\left[Ad_F^n g, \dots, Ad_F g\right] = A,$$

where

$$A = \begin{bmatrix} a_n & 1 & 0 & \cdots & 0 \\ a_{n-1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & 0 & 0 & \cdots & 1 \\ a_1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Hence, $\chi^{-1}\circ F\circ\chi(\xi_k) = A\xi_k$. Similarly,

$$\chi^{-1}\circ G_{u_k}\circ\chi(p) = \Phi^{u_k Ad_{\chi^{-1}}g}(p).$$

Note that

$$Ad_{\chi^{-1}}g(p) = D\chi^{-1}g(\chi(p))$$
$$= \left[Ad_F^{n-1}g, \dots, g\right]^{-1}g(\chi(p)) = b,$$

where $b = [0, 0, \dots, 0, 1]^T$. Hence, rewriting (4.4), we obtain

$$\xi_{k+1} = A\xi_k + bu_k. \quad \blacksquare$$

Corollary 4.1 The system (4.1) is globally state equivalent to a linear system if and only if

i) $\{g, Ad_F g, \cdots, Ad_F^{n-1}g\}$ are linearly independent,

ii) $\left[g, Ad_F^i g\right] = 0, \quad i = 1, 2, \cdots, n-1,$

iii) the vector fields $\{g, Ad_F g, \cdots, Ad_F^{n-1}g\}$ are complete,

iv) M is simply-connected.

Proof. It suffices to show that the linearizing coordinate transformation map T is onto R^n. Since this argument is similar to the one in [2], it is omitted.

Theorem 4.2 The system (4.1) is locally feedback linearizable if and only if

i) $\{g, Ad_F g, \cdots, Ad_F^{n-1}g\}$ are linearly independent in a neighborhood of x_e,

ii) $\{g, Ad_F g, \cdots, Ad_F^{n-2}g\}$ are involutive.

Proof. Omitted

5. Linearizability and Sampling

In this section, we consider the relationship between linearizations and discretization. We are concerned with the problem of whether a discrete-time system is linearizable, if it is obtained through sampling of a linearizable continuous-time system.

It is shown in [3] that the feedback linearizability is not preserved under sampling. That is, the discrete system obtained through sampling of a feedback linearizable system is not, in general, feedback linearizable. This may be illustrated by the following observation: It is involved in the process of feedback linearization the cancellation of nonlinear term which is is a function of time. However, since the input u to the discretized system must be constant over each sample period, it is impossible to cancel out the time-varying nonlinear term with such a piecewise constant input u_k.

But, we claim that the linearizability via state coordinate change is preserved under discretization. That is, if a continuous-time system is state equivalent to a linear system, then the discrete-time system obtained by periodic sampling is also state equivalent to a linear system. For the proof, consider the following:

Specifically, suppose that the system

$$\dot{x} = f(x) + ug(x) \tag{5.1}$$

state equivalent to a linear system and $T: M \to R^n$ is a linearizing coordinate transformation map. Then,

$$z_{k+1} = T\Phi_1^{f+u_k g}T^{-1}(z_k) = \Phi_1^{Ad_T(f+u_k g)}(z_k). \tag{5.2}$$

By assumption, we obtain for some controllable pair (A, b),

$$\begin{aligned} Ad_T(f + u_k g)(z_k) &= DT|_{x_k}(f + u_k g)(x_k) \\ &= Az_k + bu_k \end{aligned} \tag{5.3}$$

Hence, it follows from (5.2) and (5.3) that

$$\begin{aligned} z_{k+1} &= \Phi_1^{Az_k + bu_k}(z_k) \\ &= \tilde{A}z_k + \tilde{b}u_k, \end{aligned} \tag{5.4}$$

where $\tilde{A} = e^A$ and $\tilde{b} = \int_0^1 e^{A(1-\tau)} b \, d\tau$.

6. Conclusion

In the study of feedback linearization, we consider the systems of the form $x_k = G_{u_k} \circ F(x_k)$, where F is a diffeomorphism and $\{G_{u_k}\}_{u_k \in R}$ forms an one parameter group of diffeomorphisms. This model not only represents the class of linearizable systems via state coordinate change, but also can be obtained from a continuous time system under a reasonably good approximating assumption. The linearizability conditions obtained here are similar to the continuous ones. We also have shown that the state equivalence to a linear system is preserved under discretization.

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