입력에 시간지연이 있는 시스템에 대한 LQG/LTR 기법

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LQG/LTR Methods for Systems with Input Delay

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Abstract

This paper presents robustness properties of LQ regulators for input-delayed systems. Using frequency-domain representations, the Kalman inequality concerning the return-difference matrix is derived. The stability margins of LQ regulators are investigated using the Kalman inequality when the open-loop system is stable. In order to obtain stability margins a upper bound of the solution of LQ Riccati equations is derived. Finally, the LQG/LTR method to improve the robustness of LQG regulators is obtained and illustrated with an example.

I. Introduction

The importance of the robust feedback control has long been recognized. In classical frequency-domain techniques for single-input single-output system designs, the robustness problem is naturally handled by various graphical methods (e.g., Bode, Nyquist, Nichols plots). Commonly used measures of the robustness of single-input single-output feedback systems are gain and phase margins. Recently, great attention has been devoted to the multivariable robust controller designs [1]-[5]. In particular, it is well-known that the LQ regulator for ordinary systems has good robustness properties [3], [6], [9]. The LQG/LTR method [1], [8] has received special attention as a robust controller design method for the output feedback systems. To the authors' knowledge, however, the robustness properties of LQ regulators for input-delayed systems are not reported yet. In this paper, the stability margins of LQ regulators for input-delayed systems are analyzed when the systems are open-loop stable. For this purpose, the operator-type Riccati equations are transformed to algebraic ones in the frequency domain, and the Kalman inequality is derived based on these frequency-domain relations. A bound of the solution of Riccati equations is derived when the systems are open-loop stable.

This paper organized as follows. In Section 2, the stability margins of LQ regulators are analyzed. The LQG/LTR method, which is the same as for ordinary systems, is illustrated with an example in Section 3. Finally, some concluding remarks are given in Section 4.

II. Frequency-domain Characteristics of LQ Regulators

It is well-known that LQ regulators for ordinary systems possess guaranteed stability margins ; ($\frac{1}{2}, \infty$) gain margin and $\pm 60^{\circ}$ phase margin [6], [9]. In this section, a neccessary condition for optimality of LQ regulators for input-delayed systems is derived in the frequency domain, and stability margins are investigated from this condition.

Consider the linear time-invariant system with delay in control :

$$\dot{x}(t) = A\,x(t) + B_0\,u(t) + B_1\,u(t-h) \qquad (2.1)$$

with the initial condition

$$u(\theta) = \phi(\theta), \quad \theta \in [-h,0), \qquad (2.2)$$

where $x(t) \in R^n$, $u(t) \in R^m$; A, B_0, and B_1 are constant matrices with appropriate dimensions. Also, consider the quadratic cost function

$$J(u) = \int_0^{\infty}(x'(t)Q\,x(t) + u'(t)R\,u(t))\,dt, \qquad (2.3)$$

where $Q = Q' \geq 0$ and $R = R' > 0$.

Under stabilizability condition the solution to this infinite-time regulator problem is [10]

$$u(t) = -R^{-1}(B_0'E_0+B_1'E_1'(0))x(t) - R^{-1}\int_{-h}^{0}(B_0'E_1(\theta)+B_1'E_2(0,\theta))B_1u(t+\theta)d\theta, \qquad (2.4)$$

where E_0, $E_1(\theta)$, and $E_2(\eta,\theta)$ are the solution of Riccati equations :

$$0 = A'E_0 + E_0A + Q - (E_0B_0 + E_1(0)B_1)R^{-1} \cdot (B_0'E_0 + B_1'E_1'(0)), \qquad (2.5)$$

$$\frac{dE_1(\theta)}{d\theta} = A'E_1(\theta) - (E_0B_0 + E_1(0)B_1)R^{-1} \cdot (B_0'E_1(\theta) + B_1'E_2(0,\theta)), \qquad (2.6)$$

$$\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\theta}\right)E_2(\eta,\theta) = -(E_1'(\eta)B_0 + E_2(\eta,0)B_1)R^{-1} \cdot(B_0'E_1(\theta) + B_1'E_2(0,\theta)) \qquad (2.7)$$

with the boundary conditions

$$E_1(-h) = E_0, \qquad (2.8)$$

$$E_2(-h,\theta) = E_1(\theta) \qquad (2.9)$$

with $E_0' = E_0$, $E_2'(\eta,\theta) = E_2(\theta,\eta)$ for $-h \leq \eta \leq 0$, $-h \leq \theta \leq 0$. Moreover, the optimal closed-loop system is asymptotically stable, and the optimal cost is

$$J(u^*) = x'(0)E_0x(0) + 2x'(0)\int_{-h}^{0}E_1(\theta)B_1\phi(\theta)d\theta + \int_{-h}^{0}\int_{-h}^{0}\phi'(\eta)B_1'E_2(\eta,\theta)B_1\phi(\theta)d\eta d\theta. \qquad (2.10)$$

For the convenience of the frequency-domain representation, we define the following variables :

$$\hat{E}_1(s) = \int_{-h}^{0}E_1(\theta)e^{s\theta}\,d\theta,$$

$$\hat{E}_2(s) = \int_{-h}^{0}E_2(0,\theta)e^{s\theta}\,d\theta,$$

$$\hat{E}_3(s) = \int_{-h}^{0}\int_{-h}^{0}E_2(\eta,\theta)e^{s(\theta-\eta)}\,d\eta d\theta, \qquad (2.11)$$

$$F_0 = R^{-1}(B_0'E_0 + B_1'E_1'(0)),$$

$$F_1(s) = R^{-1}(B_0'\hat{E}_1(s) + B_1'\hat{E}_2(s))B_1,$$

$$F(s) = (\ I + F_1(s)\ )^{-1}\ F_0.$$

Using these variables, the differential equations (2.6) and (2.7) can be transformed to algebraic ones in the frequency domain.

Lemma 2.1 : Differential equations (2.6) and (2.7) with the boundary conditions (2.8) and (2.9) satisfy the following relations in the frequency domain :

$$\Delta'(-s)\hat{E}_1(s) + (E_0B_0 + E_1(0)B_1)R^{-1}(B_0'\hat{E}_1(s) + B_1'\hat{E}_2(s)) + E_1(0) - E_0e^{-sh} = 0, \qquad (2.12)$$

$$\hat{E}_2(s) + \hat{E}_2'(-s) - \hat{E}_1(s)e^{sh} - \hat{E}_1'(-s)e^{-sh} + (\hat{E}_1'(-s)B_0+\hat{E}_2'(-s)B_1)R^{-1}(B_0'\hat{E}_1(s)+B_1'\hat{E}_2(s)), \qquad (2.13)$$

where $\Delta(s) = sI - A$.

Proof : Multiplying both sides of the equation (2.6) by $e^{s\theta}$ and integrating from $-h$ to 0 with respect to θ yield

$$\int_{-h}^{0}\frac{dE_1(\theta)}{d\theta}e^{s\theta}d\theta = A'\hat{E}_1(s) - (E_0B_0 + E_1(0)B_1)R^{-1} \cdot(B_0'\hat{E}_1(s) + B_1'\hat{E}_2(s)). \qquad (2.14)$$

Integrating the left-hand side of the equation (2.14) by parts with the boundary condition (2.8), we obtain the equation (2.12).

Similarly, we can obtain the following equation from the equation (2.7) :

$$\int_{-h}^{0}\int_{-h}^{0}\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\theta}\right)E_2(\eta,\theta)e^{s(\theta-\eta)}d\eta d\theta = -(\hat{E}_1'(-s)B_0 + \hat{E}_2'(-s)B_1)R^{-1}(B_0'\hat{E}_1(s) + B_1'\hat{E}_2(s)). \qquad (2.15)$$

Integrating the left-hand side of the equation (2.15) by parts with the boundary condition (2.9) yields the equation (2.13). This completes the proof.

The return difference matrix has played important roles in the robustness analysis for ordinary systems [4], [9].

Similarly as for ordinary systems, the return difference matrix $T_F(s)$ for the closed-loop input-delayed system (2.1) and (2.4) can be defined by

$$T_F(s) = I + R^{\frac{1}{2}}F(s)\Phi(s)\ (B_0 + B_1e^{-sh})R^{-\frac{1}{2}}, \qquad (2.16)$$

where

$$\Phi(s) = (\ sI - A\ )^{-1} \qquad (2.17)$$

Using this return difference matrix, a neccessary condition for optimality of LQ regulators can be given in the frequency domain, and the stability margins can be analyzed from that condition.

Theorem 2.1 : LQ regulators for input-delayed system (2.1) with the cost (2.3) satisfy the following relation in the frequency domain :

$$T_F^*(j\omega)(I+R^{\frac{1}{2}}F_1(j\omega)R^{-\frac{1}{2}})^*(I+R^{\frac{1}{2}}F_1(j\omega)R^{-\frac{1}{2}})T_F(j\omega) \geq I, \qquad (2.18)$$

for $0 \leq \omega < \infty$, where * denotes the conjugate transpose.

Proof : From the definition (2.16) of the return difference matrix and the definition (2.11), it follows that

$$T_F'(-s)(I+R^{-\frac{1}{2}}F_1'(-s)R^{\frac{1}{2}})(I+R^{\frac{1}{2}}F_1(s)R^{-\frac{1}{2}})T_F(s)$$

$$= I+R^{-\frac{1}{2}}[B_0'\hat{E}_1(s)B_1+B_1'\hat{E}_2(s)B_1+B_1'\hat{E}_1'(-s)B_0+B_1'\hat{E}_2'(-s)B_1$$
$$+ (B_0'E_0+B_1'E_1'(0))\Phi(s)(B_0+B_1e^{-sh})$$
$$+ (B_0'+B_1'e^{sh})\Phi'(-s)(E_0B_0+E_1(0)B_1)$$
$$+ B_1'(\hat{E}_1'(-s)B_0+\hat{E}_2'(-s)B_1)R^{-1}(B_0'\hat{E}_1(s)+B_1'\hat{E}_2(s))B_1$$
$$+ B_1'(\hat{E}_1'(-s)B_0+\hat{E}_2'(-s)B_1)R^{-1}(B_0'\hat{E}_0+B_1'\hat{E}_1'(0))$$
$$\cdot\Phi(s)(B_0+B_1e^{-sh})+(B_0'+B_1'e^{sh})\Phi'(-s)$$
$$\cdot(E_0B_0+E_1(0)B_1)R^{-1}(B_0'\hat{E}_1(s)+B_1'\hat{E}_2(s))B_1$$

$$+ (B_0'+B_1'e^{sh})\Phi'(-s)(E_0B_0+E_1(0)B_1)R^{-1}$$

$$\cdot (B_0'E_0+B_1'E_1'(0))\Phi(s)(B_0+B_1e^{-sh})]R^{\frac{1}{2}} \qquad (2.19)$$

Using the Lemma 2.1, the equation (2.19) becomes

$$T_F'(-s)(I+R^{-\frac{1}{2}}F_1'(-s)R^{\frac{1}{2}})(I+R^{\frac{1}{2}}F_1(s)R^{-\frac{1}{2}})T_F(s)$$

$$= I+R^{-\frac{1}{2}}(B_0'+B_1'e^{sh})[E_0\Phi(s)+\Phi'(-s)E_0+\Phi'(-s)$$

$$\cdot(E_0B_0+E_1(0)B_1)R^{-1}(B_0'E_0+B_1'E_1'(0))\Phi(s)]$$

$$\cdot(B_0+B_1e^{sh})R^{-\frac{1}{2}}$$

$$= I+R^{-\frac{1}{2}}(B_0'+B_1'e^{sh})\Phi'(-s)Q\Phi(s)(B_0+B_1e^{-sh})R^{-\frac{1}{2}}. \qquad (2.20)$$

The last equality in the equation (2.20) follows from the equation (2.5). With $s = j\omega$, the equation (2.20) becomes

$$T_F^*(j\omega)(I+R^{\frac{1}{2}}F_1(j\omega)R^{-\frac{1}{2}})^*(I+R^{\frac{1}{2}}F_1(j\omega)R^{-\frac{1}{2}})T_F(j\omega)$$

$$= I+R^{-\frac{1}{2}}(B_0+B_1e^{-j\omega h})^*\Phi^*(j\omega)Q\Phi(j\omega)(B_0+B_1e^{-j\omega h})R^{-\frac{1}{2}}. \qquad (2.21)$$

The left-hand side of the equation (2.21) is a Hermitian, while the right-hand side of the equation (2.21) is of the form $I+L^*(j\omega)Q\,L(j\omega)$ for the matrix $L(j\omega)=\Phi(j\omega)(B+B\,e^{-j\omega h})R^{-\frac{1}{2}}$. From the fact that $L^*(j\omega)Q\,L(j\omega) \geq 0$, the equation (2.21) implies the relation (2.18). This completes the proof.

From the inequality (2.18), it can be seen that guaranteed stability margins will be obtained if a upper bound of $F_1(j\omega)$ is given. In order to obtain a upper bound of $F_1(j\omega)$, we will derive a upper bound of $B_0'E_1(\theta)B_1+B_1'E_2(0,\theta)B_1$. In the following, $\sigma(\cdot)$ denotes the singular value which is defined by $\sigma(A)=\lambda^{\frac{1}{2}}(A^*A)$.

Lemma 2.2 : Assume the system matrix A is stable, then

$$\nu(E_1(\theta)B_1) \leq \bar{\sigma}(M_1(\theta)), \qquad (2.22)$$

$$\nu(B_1'E_2(0,\theta)B_1) \leq \bar{\sigma}(M_2(\theta)) \qquad (2.23)$$

for $-h \leq \theta \leq 0$, where $\nu(\cdot)$ denotes the consistent norm defined by $\nu(D) = \max_{i,j}|d_{ij}|$ for a n x m matrix D, $\bar{\sigma}(\cdot)$ denotes the maximum singular value, and

$$M_1(\theta) = \begin{bmatrix} K & K\Phi(-h-\theta)B_1 \\ B_1'\Phi'(-h-\theta)K & B_1'\Phi'(-h-\theta)K\Phi(-h-\theta)B_1 \end{bmatrix}, \qquad (2.24)$$

$$M_2(\theta) = \begin{bmatrix} B_1'KB_1 & B_1'\Phi'(-\theta)KB_1 \\ B_1'K\Phi(-\theta)B_1 & B_1'KB_1 \end{bmatrix} \qquad (2.25)$$

with

$$K = \int_0^\infty \Phi'(t)Q\Phi(t)dt. \qquad (2.26)$$

Proof : Let $V(x(0),\phi)$ be the value of the cost (2.3) for the initial control function $\phi(\theta)$, $-h \leq \theta < 0$, initial state $x(0)$, and zero control $(u(t) \equiv 0,\ t \geq 0)$. Then,

$$V(x(0),\phi) = \int_0^\infty x'(t)Qx(t)dt$$

$$= x'(0)\int_0^\infty \Phi'(t)Q\Phi(t)dt\ x(0)$$

$$+ x'(0)\int_0^h \int_{-h}^{t-h} \Phi'(t)Q\Phi(t-h-\theta)B_1\phi(\theta)d\theta dt$$

$$+ x'(0)\int_h^\infty \int_{-h}^0 \Phi'(t)Q\Phi(t-h-\theta)B_1\phi(\theta)d\theta dt$$

$$+ \int_0^h \int_{-h}^{t-h} \phi'(\theta)B_1'\Phi'(t-h-\theta)Q\Phi(t)d\theta dt\ x(0)$$

$$+ \int_h^\infty \int_{-h}^0 \phi(\theta)B_1'\Phi'(t-h-\theta)Q\Phi(t)d\theta dt\ x(0)$$

$$+ \int_0^h \int_{-h}^{t-h} \int_{-h}^{t-h} \phi(\theta)B_1'\Phi'(t-h-\theta)Q\Phi(t-h-\theta)$$

$$\cdot B_1\phi(\eta)d\eta d\theta dt + \int_h^\infty \int_{-h}^0 \int_{-h}^0 \phi'(\theta)$$

$$\cdot B_1'\Phi'(t-h-\theta)Q\Phi(t-h-\eta)B_1\phi(\eta)d\eta d\theta dt, \qquad (2.27)$$

since with the zero control $(u(t) \equiv 0,\ t \geq 0)$

$$x(t) = \begin{cases} \Phi(t)x(0) + \int_{-h}^{t-h}\Phi(t-h-\theta)B_1\phi(\theta)d\theta, & 0 \leq t < h \\ \Phi(t)x(0) + \int_{-h}^{0}\Phi(t-h-\theta)B_1\phi(\theta)d\theta, & t \geq h. \end{cases} \qquad (2.28)$$

Let $x(0) = \varepsilon\xi_1$, $\phi(\theta) = \xi_2$ for $\theta_0 \leq \theta \leq \theta_0+\varepsilon$, $-h \leq \theta_0 < -\varepsilon$, and $\phi(\theta) = 0$ elsewhere. Then, it follows that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}V(x(0),\phi) = \xi_1'\int_0^\infty \Phi'(t)Q\Phi(t)dt\ \xi_1 + \xi_1'\int_0^\infty \Phi'(t)Q$$

$$\cdot \Phi(t-h-\theta_0)dtB_1\xi_2$$

$$+ \xi_2'B_1'\int_0^\infty \Phi'(t-h-\theta_0)Q\Phi(t)dt\xi_1$$

$$+ \xi_2'B_1'\int_0^\infty \Phi'(t-h-\theta_0)Q$$

$$\cdot \Phi(t-h-\theta_0)dt\ B_1\xi_2$$

$$= [\xi_1'\ \xi_2']$$

$$\cdot \begin{bmatrix} K & K\Phi(-h-\theta_0)B_1 \\ B_1'\Phi'(-h-\theta_0)K & B_1'\Phi'(-h-\theta_0)K\Phi(-h-\theta_0)B_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \qquad (2.29)$$

And using the optimal cost (2.10), it can be seen that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}J(u^*) = [\xi_1'\xi_2'] \begin{bmatrix} E_0 & E_1(\theta_0)B_1 \\ B_1'E_1'(\theta_0) & B_1'E_2(\theta_0,\theta_0)B_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad (2.30)$$

The equations (2.29) and (2.30) yield

$$0 \leq [\xi_1' \xi_2'] \begin{bmatrix} E_0 & E_1(\theta_0)B_1 \\ B_1'E_1'(\theta_0) & B_1'E_2(\theta_0,\theta_0)B_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

$$\leq [\xi_1' \xi_2'] \begin{bmatrix} K & K\Phi(-h-\theta_0)B_1 \\ B_1'\Phi'(-h-\theta_0)K & B_1'\Phi'(-h-\theta_0)K\Phi(-h-\theta_0)B_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \quad \text{------ (2.31)}$$

Note that for any symmetric positive semidefinite matrix D, $\nu(D) \leq k$ if $D \leq \mathrm{diag}(k)$. And for any matrix D, $D \leq \mathrm{diag}\,[\bar{\sigma}(D)]$. Thus, the inequality (2.31) gives

$$\nu(E_1(\theta_0)B_1) \leq \bar{\sigma}(M_1(\theta_0)), \qquad (2.32)$$

since for any block matrix $Z = \begin{bmatrix} Z1 & Z2 \\ Z3 & Z4 \end{bmatrix}$, $\nu(Z_i) \leq \nu(Z)$, $i = 1,2,3,4$. Taking all possible θ_0, we obtain $\nu(E_1(\theta)B_1) \leq \bar{\sigma}(M_1(\theta))$ for $-h \leq \theta < 0$ and by continuity, this holds for $-h \leq \theta \leq 0$. This completes the proof of (2.22). The proof of boundedness of $B_1'E_2(0,\theta)B_1$ is similar to the above. Let $x(0) = 0$ and $\phi(\theta) = \xi_1$ for $\theta_1 \leq \theta \leq \theta_1+\varepsilon$, $\phi(\theta) = \xi_2$ for $\theta_2 \leq \theta \leq \theta_2+\varepsilon$ with $-h+\varepsilon \leq \theta_1+\varepsilon \leq \theta_2 \leq -\varepsilon$, and $\phi(\theta) = 0$ elsewhere. Then, it can be easily seen that

$$0 \leq \begin{bmatrix} B_1'E_2(\theta_1,\theta_1)B_1 & B_1'E_2(\theta_1,\theta_2)B_1 \\ B_1'E_2(\theta_2,\theta_1)B_1 & B_1'E_2(\theta_2,\theta_2)B_1 \end{bmatrix}$$

$$\leq \begin{bmatrix} B_1'KB_1 & B_1'\Phi'(\theta_2-\theta_1)KB_1 \\ B_1'K\Phi(\theta_2-\theta_1)B_1 & B_1'KB_1 \end{bmatrix}. \qquad (2.33)$$

From this relation, we can obtain the relation (2.23). This completes the proof.

Now, we can present stability margins of LQ regulators for input delayed systems if the systems are open-loop stable.

Theorem 2.2 : Assume the system matrix A is stable. Then,

$$\underline{\sigma}(T_F(j\omega)) \geq \frac{1}{1+\alpha}, \quad 0 \leq \omega < \infty, \qquad (2.34)$$

where $\underline{\sigma}(\cdot)$ denotes the minimum singular value and

$$\alpha = m\,\bar{\sigma}(R^{-1})\frac{\bar{\sigma}(R^{\frac{1}{2}})}{\underline{\sigma}(R^{\frac{1}{2}})}\int_{-h}^{0}\{\nu(B_0)\bar{\sigma}(M_1(\theta))+\bar{\sigma}(M_2(\theta))\}d\theta \quad \text{------- (2.35)}$$

Proof : From the Kalman inequality (2.18), it follows that

$$\underline{\sigma}(T_F(j\omega)) \geq \frac{1}{1+\dfrac{\bar{\sigma}(R^{\frac{1}{2}})}{\underline{\sigma}(R^{\frac{1}{2}})}\bar{\sigma}(F_1(j\omega))}. \qquad (2.36)$$

Using Lemma 2.2, we obtain

$$\begin{aligned} \bar{\sigma}(F_1(j\omega)) &\leq \bar{\sigma}(R^{-1})\bar{\sigma}(B_0'\hat{E}_1(j\omega)B_1+B_1'\hat{E}_2(j\omega)B_1) \\ &\leq m\bar{\sigma}(R^{-1})\int_{-h}^{0}\nu(B_0'E_1(\theta)B_1+B_1'E_2(0,\theta)B_1)d\theta \\ &\leq m\bar{\sigma}(R^{-1})\int_{-h}^{0}[\nu(B_0)\nu(E_1(\theta)B_1) \\ &\quad + \nu(B_1'E_2(0,\theta)B_1)]d\theta \\ &\leq m\bar{\sigma}(R^{-1})\int_{-h}^{0}[\nu(B_0)\bar{\sigma}(M_1(\theta)) \\ &\quad + \bar{\sigma}(M_2(\theta))]d\theta, \end{aligned}$$

where we have used the fact that $\bar{\sigma}(D) \leq m\cdot\nu(D)$ for any m x m matrix. This completes the proof.

From Theorem 2.2, we can see that LQ regulators for input-delayed systems have a guaranteed gain margin denoted by GM :

$$GM = (\frac{1+\alpha}{2+\alpha}, \frac{1+\alpha}{\alpha}), \qquad (2.37)$$

and a guaranteed phase margin denoted by PM :

$$PM = \pm\cos^{-1}[1 - \frac{1}{2(1+\alpha)^2}], \qquad (2.38)$$

simultaneously in each loop of the feedback system of Fig.1 when the open-loop system is stable. Note that α is a function of A, B_0, B_1, Q, R, and delay h. Hence, the guaranteed margins differ from system to system. However, we can say that the guaranteed margins decrease with increasing h, and increase with increasing $\underline{\sigma}(R)$. Since the design parameters are the weighting matrices Q and R in LQ regulator designs, we can achieve better guaranteed margins of closed-loop systems by selecting larger weighting matrix R, where "large" matrix means it has large minimum singular value. However, the transient response becomes slower when R becomes larger. Hence, the trade-off between the guaranteed stability margins and the transient response is necessary.

In single input systems, it is possible to derive stronger results than in multi-input systems.

Corollary 2.1 : Assume the open-loop system is stable and m = 1, then

$$|T_F(j\omega)| \geq \frac{1}{1+\beta} \quad \text{for } 0 \leq \omega < \infty, \qquad (2.39)$$

where

$$\beta = \frac{1}{r}\int_{-h}^{0}[\nu(b_0)\bar{\sigma}(M_1(\theta)) + b_1'Kb_1]d\theta, \qquad (2.40)$$

and r, b_0, and b_1 denote the corresponding variables to R, B_0, and B_1, respectively.

Proof : Using the relation (2.33), we obtain

$$\nu(b_1'E_2(0,\theta)b_1) = |b_1'E_2(0,\theta)b_1| \leq b_1'Kb_1, \quad -h \leq \theta \leq 0 \qquad (2.41)$$

for single input systems. Hence, it follows that

$$\bar{\sigma}(F_1(j\omega)) \leq \frac{1}{r}\int_{-h}^{0} [\nu(b_0)\nu(E_1(\theta)b_1)$$

$$+ \nu(b_1'E_2(0,\theta)b_1)]d\theta \leq \beta .$$

This completes the proof.

Note that the maximum singular value of $M_2(\theta)$ is greater than or equal to b'Kb for single-input systems. Therefore, the result in Corollary 2.1 is less conservative than that in Theorem 2.2 for single-input systems.

Example 2.1 : Consider the LQ problem specified by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t-h), \quad (2,42)$$

$$J(u) = \int_0^\infty \{ x'(t)Q\, x(t) + u^2(t)\}\, dt \quad (2.43)$$

with

$$Q = \begin{bmatrix} 2800 & 80\sqrt{35} \\ 80\sqrt{35} & 80 \end{bmatrix}, \quad h = 0.5.$$

This example is the same as that in [8] except for the delay h = 0.5. A Nyquist diagram for the LQ regulator design is given in Fig.2. Gain margin is (0.37, 2.0) and there is about 11.0° phase margin. According to Corollary 2.1, it can be seen that

$$|T_F(j\omega)| \geq 0.016$$

and the calculated margins are such that (0.98, 1.02) gain margin and 1.0° phase margin, which are too strict ones. The result, however, is the first which represents stability margins in terms of system parameters explicitly.

III. LQG/LTR Methods

In order to utilize LQ regulators for the feedback design, all states must be measured. Since full state feedback can be impossible or too expensive to realize, LQG regulators have been extensively used where the Kalman filter is used to provide state estimates for feedback. LQG regulators, however, do not possess the guaranteed stability margins of LQ regulators [7]. The LQG/LTR methods improve the stability margins by recovering the loop transfer function if the minimum phase assumption holds [8]. Note that it is not always the case that LQG regulators need to be robustified since in some cases LQG regulators may have better stability margins than LQ regulators.

In this section we consider the following system :

$$\dot{x}(t) = Ax(t) + B_1u(t-h) + \omega(t), \quad (3.1)$$

$$y(t) = Cx(t) + v(t), \quad (3.2)$$

with the initial condition

$$u(t) = \phi(t), \quad -h \leq t < 0, \quad (3.3)$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R$; A, B_1, and C are constant matrices with appropriate dimensions ; $\omega(t)$ and $v(t)$ are zero mean white noise with spectral intensity matrices Ξ and Θ, respectively. Under the assumption [A,C] is detectable, it is well-known that the state estimate is specified by

$$\dot{\hat{x}}(t)=A\hat{x}(t)+B_1u(t-h)+\Sigma C'\Theta^{-1}[y(t)-C\hat{x}(t)], \quad (3.4)$$

where

$$\Sigma A + \Sigma A' - \Sigma C'\Theta^{-1}C\Sigma + \Theta = 0, \quad \Sigma \geq 0. \quad (3.5)$$

With the quadratic cost

$$J(u) = E\int_0^\infty [x'(t)Qx(t)+u'(t)Ru(t)]dt, \quad (3.6)$$

the LQG regulator is given by

$$u(t) = -R^{-1}B_1'E_1'(0)\hat{x}(t)$$

$$-R^{-1}B_1'\int_{-h}^{0} E_2(0,\theta)B_1u(t+\theta)d\theta , \quad (3.7)$$

where $E_1'(0)$ and $E_2(0,\theta)$ is obtained from the Riccati equations (2.5)-(2.9). Since the Kalman filter for the system (3.1)-(3.3) is the same as for ordinary stystems, the same LQG/LTR method to improve the robustness of LQG regulators is applicable to the systems with delayed input only. To see this, consider the feedback control systems in Fig.1 and Fig.3. In Fig.1, the loop transfer function from u" to x is given by

$$x(s) = \Phi(s)B_1e^{-sh}\, u''(s). \quad (3.8)$$

In Fig.3, the loop transfer function from u" to $\hat{x}$ is given by

$$\hat{x}(s) = [I+\Phi(s)HC]^{-1}[\Phi B_1e^{-sh}u' + HC\Phi B_1e^{-sh}u''],$$

------- (3.9)

where $H = \Sigma C'\Theta^{-1}$. If the Doyle's condition [8]

$$H[I+C\Phi(s)H]^{-1} = B_1[C\Phi(s)B_1]^{-1} \quad (3.10)$$

is satisfied, then the equation (3.9) becomes

$$\hat{x}(s) = \Phi(s)B_1e^{-sh}\, u''(s) \quad (3.11)$$

which coincides with the equation (3.8). Thus, the same filter gain adjustment procedure as for the ordinary systems [8] will still recover the loop transfer function and improve the robustness of LQG regulators for the input-delayed systems.

Example 3.1 : Consider the following LQG problem

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t-0.5) + \begin{bmatrix} 35 \\ -61 \end{bmatrix} w(t), \quad (3.12$$

$$y(t) = [\,2 \quad 1\,]\, x(t) + v(t) \quad (3.13)$$

with $E(w)=E(v)=0; E[w(t)w(\tau)]=E[v(t)v(\tau)]=\delta(t-\tau)$; performance index is

$$J(u) = E\int_0^\infty (x'H'Hx+u^2)dt \quad (3.14)$$

with

$$H = 4\sqrt{5}[\sqrt{35} \quad 1].$$

The Nyquist diagrams for the full-state design and the observer-based design are shown in Fig.2. The LQG regulator has about (0.99,1.2) gain margin and less than 2° phase margin, which are much decreased margins compared with those of LQ regulator ;(0.37, 2) gain margin and 11° phase margin. When the filter gain adjustment procedure is applied to this example, much more improved result is obtained. We let the process noise covariance be

$$\Xi = \begin{bmatrix} 35 \\ -61 \end{bmatrix} [35\ \ -61] + q^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0\ \ 1].$$

Fig.3 shows Nyquist diagrams for q =20,30,40,and 50. It is observed that margins are improved as the loop transfer function tends toward full-state design.

IV. Conclusion

In this paper, frequency-domain properties of LQ regulators for linear input-delayed systems have been investigated. In particular, the guaranteed stability margins of LQ regulators are presented using the sysytem matrices, the weighting matrices, and delay time when the open-loop systems are stable. To do this, the operator-type Riccati equations are transformed to algebraic ones, and based on these, the Kalman inequality concerning the return difference matrix is derived.

The LQG/LTR method to improve the robustness of LQG regulators is obtained and illustrated with an example when there exists delayed control only.

The results in this paper generalize the well-known ones for ordinary systems, and can be extended to more general input-delayed systems. The robustness properties and the LQG/LTR method for systems with delays both in input and in output remain for further investigations. Finally, it is remarked that the results in this paper are somewhat conservative, since we have assumed the open-loop stability and used the singular value as a robustness measure.

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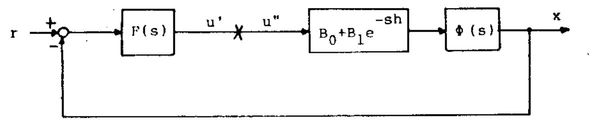

Fig.1 LQ Regulator

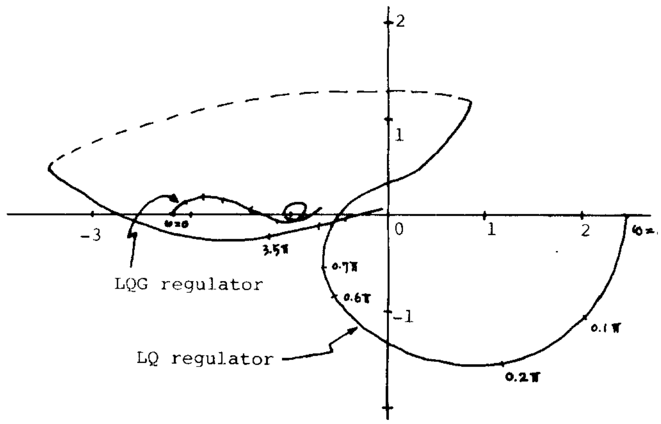

Fig.2 Nyquist Diagrams of LQ and LQG Regulator

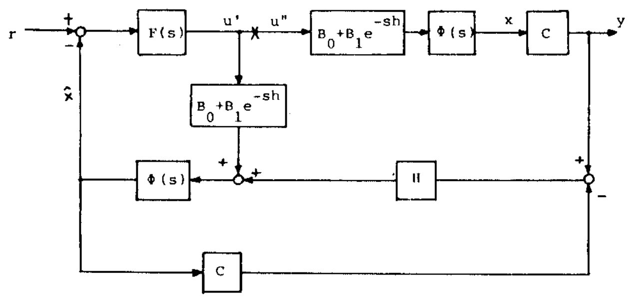

Fig.3 LQG Regulator

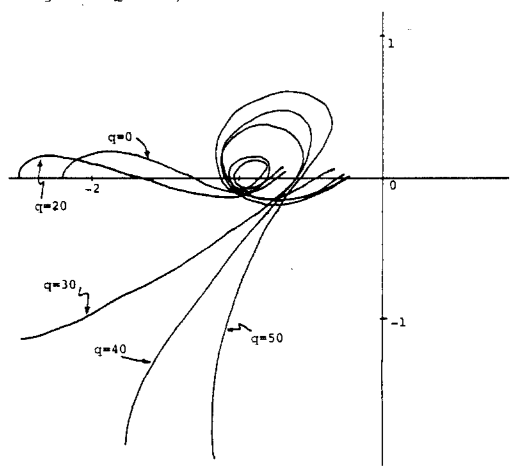

Fig.4 Nyquist Diagrams of Fictitious Noise Design Procedure